PARTIAL DIFFERENTIAL EQUATIONS

V. P. MIKHAILOV

Pure Mathematical Physics

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ИЗДАТЕЛЬСТВО · МИР ·

В. П. Михайлов

ДИФФЕРЕНЦИАЛЬНЫЕ УРАВНЕНИЯ В ЧАСТНЫХ ПРОИЗВОДНЫХ

ИЗДАТЕЛЬСТВО «НАУКА» МОСКВА

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PREFACE

This book has developed from courses of lectures given by the author over a period of years to the students of the Moscow Physico-Technical Institute. It is intended for the students having basic knowledge of mathematical analysis, algebra and the theory of ordinary differential equations to the extent of a university course. All the necessary information can be found, for instance, in the following textbooks: S. M. Nikolsky, *A Course of Mathematical Analysis*, 2 vols., Mir Publishers, Moscow, 1977; A. I. Mal'cev, *Foundations of Linear Algebra*, W. H. Freeman, San Francisco, 1963; L. S. Pontryagin, *Ordinary Differential Equations*, Pergamon Press, Oxford, 1964.

Except Chapter I, where some general questions regarding partial differential equations have been examined, the material has been arranged so as to correspond to the basic types of equations. The central role in the book is played by Chapter IV, the largest of all, which discusses elliptic equations. Chapters V and VI are devoted to the hyperbolic and parabolic equations.

The method used in this book for investigating the boundary value problems and, partly, the Cauchy problem is based on the notion of generalized solution which enables us to examine equations with variable coefficients with the same ease as the simplest equations: Poisson's equation, wave equation and heat equation. Apart from discussing the questions of existence and uniqueness of solutions of the basic boundary value problems, considerable space has been devoted to the approximate methods of solving these equations:

Ritz's method in the case of elliptic equations and Galerkin's method for hyperbolic and parabolic equations.

Information regarding function spaces, in particular, S. L. Sobolev's embedding theorems, necessary for such arrangement of subject matter is contained in Chapter III. It is not assumed on the part of the reader that he is familiar with the required portions of the theory of functions and functional analysis; these have been treated in Chapter II which is of auxiliary nature.

Problems have been given on all the chapters except Chapter II. The majority of them are intended to deepen and broaden the contents of respective chapters; precisely with the same aim have the lists of suggested reading been supplied. For exercises we recommend the following books: V. S. Vladimirov *et al.*, *A Collection of Problems on Equations of Mathematical Physics*, Nauka, Moscow, 1974 (in Russian); B. M. Budak, A. A. Samarskii and A. N. Tikhonov, *A Collection of Problems on Mathematical Physics*, Pergamon Press, Oxford, 1964; M. M. Smirnov, *Problems on Equations of Mathematical Physics*, Pergamon Press, Oxford.

In conclusion the author expresses his sincere thanks to V. S. Vladimirov for his constant interest in this book, and to T. I. Zelenyak, I. A. Kipriyanov and S. L. Sobolev who, having gone through the manuscript, made a number of valuable comments. The author is especially indebted to his colleagues A. K. Gushchin and L. A. Muravei with whom he had fruitful discussions that led to considerable improvement in the book.

V. Mikhailov

July 1975

CHAPTER I

INTRODUCTION. CLASSIFICATION OF EQUATIONS. FORMULATION OF SOME PROBLEMS

Differential equations are those equations where the unknowns are functions of one or more variables and which contain not only these functions but their derivatives as well. If the unknowns are functions of several variables (not less than two), then the equations are called *partial differential equations.* We shall deal with only such equations, and shall consider a single partial differential equation in one unknown function.

A partial differential equation containing derivatives of the unknown function u with respect to the variables $x_1, \ldots, x_n$ is said to be of *Nth-order* if it contains at least one Nth-order derivative and does not contain derivatives of higher orders, that is, the equation

$$\Phi\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots, \frac{\partial^N u}{\partial x_n^N}\right) = 0. \qquad (1)$$

Eq. (1) is said to be *linear* if Φ, as a function of $u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial^N u}{\partial x_n^N}$, is linear. Henceforth we shall consider linear equation of the second order, that is, equation of the form

$$\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial u}{\partial x_i} + a(x) u = f(x); \qquad (2)$$

here $x = (x_1, \ldots, x_n)$. The functions $a_{ij}(x)$, $i, j = 1, \ldots, n$, $a_i(x)$, $i = 1, \ldots, n$ and $a(x)$ are called *coefficients* of Eq. (2) and $f(x)$ the *free term.*

Let R_n denote an n-dimensional Euclidean space, and let $x = (x_1, \ldots, x_n)$ be a point of R_n, $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. As usual, by a region in R_n or an n-dimensional region we mean an open connected (nonempty) set of points of R_n. In what follows, unless stated otherwise, all the regions are assumed to be bounded.

Let Q be an n-dimensional region. A set $E \subset Q$ is said to be *strictly interior* with respect to Q, $E \Subset Q$, if $\bar{E} \subset Q$, where $\bar{E}$ is the closure (in the sense of distance in R_n) of E.

The set of functions having in Q continuous partial derivatives up to order k, k being a nonnegative integer, is denoted by $C^k(Q)$, while its subset consisting of those functions whose all the partial derivatives up to order k are continuous in $\overline{Q}$ by $C^k(\overline{Q})$. For the sets $C^0(Q)$ and $C^0(\overline{Q})$ of functions which are continuous in Q and $\overline{Q}$ respectively, we shall also use the notation $C(Q)$ and $C(\overline{Q})$. We designate by $C^\infty(Q)$ the set of functions which belong to each of $C^k(Q)$, $k = 0, 1, \ldots$, that is, $C^\infty(Q) = \bigcap\limits_{k=0}^{\infty} C^k(Q)$, and by $C^\infty(\overline{Q})$ the set of functions belonging to each of $C^k(\overline{Q})$, $k = 0, 1, \ldots$, that is, $C^\infty(\overline{Q}) = \bigcap\limits_{k=0}^{\infty} C^k(\overline{Q})$.

A function $f(x)$ is said to *have compact support* in Q if there exists a subregion $Q' \Subset Q$ such that $f(x) = 0$ in $Q \setminus Q'$. The set $\dot{C}^k(\overline{Q})$ is composed of all the functions belonging to $C^k(\overline{Q})$ and having compact support, and the intersection of all these sets is denoted by $\dot{C}^\infty(\overline{Q})$: $\dot{C}^\infty(\overline{Q}) = \bigcap\limits_{k=0}^{\infty} \dot{C}^k(\overline{Q})$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector with nonnegative integer components, and put $|\alpha| = \alpha_1 + \ldots + \alpha_n$. If $f(x) \in C^k(Q)$, then the partial derivative $\dfrac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ will be often denoted for brevity by $D^\alpha f$. The first- and second-order derivatives will also be written as f_{x_i}, $f_{x_i x_j}$. For the gradient $(f_{x_1}, \ldots, f_{x_n})$ of a function $f \in C^1(Q)$ we use the notation $\nabla f(x)$.

By an $(n - 1)$-dimensional closed surface S we shall mean a bounded and closed $(n - 1)$-dimensional surface, without edge, of class C^k for a certain $k \geqslant 1$, that is, a connected, bounded and closed surface $(S = \overline{S})$ lying in R_n and having the following property: for any point $x^0 \in S$ there exist an (n-dimensional) neighbourhood U_{x^0} of it and a function $F_{x^0}(x) \in C^k(U_{x^0})$ such that $\nabla F_{x^0}(x^0) \neq 0$ so that the set $S \cap U_{x^0}$ is described by the equation $F_{x^0}(x) = 0$. (That is, all the points of the set $S \cap U_{x^0}$ satisfy the equation $F_{x^0}(x) = 0$ and any point of U_{x^0} satisfying the equation $F_{x^0}(x) = 0$ belongs to S.)

The boundary of Q will be denoted by ∂Q. In what follows, without further qualification, the boundaries of regions in question are assumed to consist of a finite number of disjoint closed $(n - 1)$-dimensional surfaces (of class C^1). By $|Q|$ we indicate the volume of Q.

We note that if an $(n-1)$-dimensional closed surface S belongs to the class C^k, then for a point x^0 on it there exists a small neighbourhood U'_{x^0} such that the intersection $S \cap U'_{x^0}$ is uniquely projected onto an $(n-1)$-dimensional region D_{x^0}, with boundary of class C^k, which lies in one of the coordinate planes, that is, for some i, $i = 1, \ldots, n$, the surface is described by the equation $x_i = \varphi_{x^0}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in D_{x^0}$, and $\varphi_{x^0} \in C^k(\overline{D}_{x^0})$. The intersection $S \cap U'_{x^0}$ will be called a *simple piece* (or *piece*) of the surface S.

Since S is bounded and closed, one may choose a finite subcover from the cover $\{U'_x, x \in S\}$ of S. The collection of simple pieces $S_1, \ldots, S_N$ corresponding to such a finite cover will be called the *cover of the surface S by simple pieces.*

By an $(n-1)$-dimensional surface S of class C^k, $k \geqslant 1$, we shall mean a connected surface which can be covered by a finite number of (n-dimensional) regions U_i, $i = 1, \ldots, N$, so that each of the sets $S_i = S \cap U_i$, $i = 1, \ldots, N$, is uniquely projected onto an $(n-1)$-dimensional region D_i having boundary of class C^k and lying in one of the coordinate planes, that is, for some $p = p(i)$, $p = 1, \ldots, n$, the surface is represented by the equation $x_p = \varphi_i(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n)$, $(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots) \in D_i$, and $\varphi_i \in C^k(\overline{D}_i)$. The collection of surfaces S_i, which are simple pieces of S, corresponding to such a cover $U_1, \ldots, U_N$ of S will be called a *cover of S by simple pieces*. Henceforth an $(n-1)$-dimensional surface will mean an $(n-1)$-dimensional surface of class C^k for a certain $k \geqslant 1$.

We denote by Q^δ, $\delta > 0$, the region obtained by taking the union over all $x^0 \in Q$ of balls $\{|x - x^0| < \delta\}$: $Q^\delta = \bigcup_{x^0 \in Q} \{|x - x^0| < \delta\}$; $Q \Subset Q^\delta$. Q_δ, $\delta > 0$, denotes the set containing all the points of Q whose distance from the boundary ∂Q exceeds δ; $Q_\delta \Subset Q$; for sufficiently small $\delta > 0$, Q_δ is a region. We shall show that for an arbitrarily small enough $\delta > 0$ there exists in R_n an infinitely differentiable function $\zeta_\delta(x)$ which equals unity in Q_δ and vanishes outside $Q_{\delta/2}$. The function $\zeta_\delta(x)$ will henceforth be called δ-*slicing function* (or, simply, *slicing function*) *for the region Q.* Before constructing the function $\zeta_\delta(x)$, we introduce an important notion of the averaging kernel.

Suppose that $\omega_1(t)$ is an infinitely differentiable and nonnegative even function of a single variable t $(-\infty < t < +\infty)$ which vanishes for $|t| \geqslant 1$, and is such that

$$\int_{R_n} \omega_1(|x|)\,dx = \int_{|x|<1} \omega_1(|x|)\,dx = 1. \tag{3}$$

For $\omega_1(t)$ we may take, for instance, the function

$$\omega_1(t)=\begin{cases} \dfrac{1}{C_n}e^{-\frac{1}{1-t^2}}, & 0\leqslant|t|<1,\\ 0, & |t|\geqslant 1,\end{cases}$$

where the constant C_n has been chosen so as to satisfy (3). Let h be any positive number. The function

$$\omega_h(|x|)=\frac{1}{h^n}\omega_1(|x|/h)$$

is called the *averaging kernel* (of radius h). The averaging kernel $\omega_h(|x|)$ has the following obvious properties:

(a) $\omega_h(|x|)\in C^\infty(R_n)$, $\omega_h(x)\geqslant 0$ in R_n,

(b) $\omega_h(|x|)\equiv 0$ for $|x|\geqslant h$,

(c) $\int\limits_{R_n}\omega_h(|x|)\,dx=1$,

(d) for any $\alpha=(\alpha_1, \ldots, \alpha_n)$, $|\alpha|\geqslant 0$, and for all $x\in R_n$

$$|D^\alpha\omega_h(|x|)|\leqslant C_\alpha/h^{n+|\alpha|},$$

where C_α is a positive constant independent of h.

Let $\omega_h(|x|)$ be an arbitrary averaging kernel. It is directly verified that for sufficiently small $\delta>0$ the function

$$\zeta_\delta(x)=\int\limits_{Q_{3\delta/4}}\omega_{\delta/4}(|x-y|)\,dy$$

is a slicing function for the region Q; moreover, $\zeta_\delta(x)$ satisfies in R_n the inequalities $0\leqslant\zeta_\delta(x)\leqslant 1$.

§ 1. THE CAUCHY PROBLEM. KOVALEVSKAYA'S THEOREM

1. Formulation of the Cauchy Problem. In a region Q of the n-dimensional space R_n (Q is not necessarily bounded, and, in particular, may coincide with the whole of R_n), we consider a linear differential equation of the second order

$$\mathcal{L}u\equiv\sum_{i,j=1}^{n}a_{ij}(x)\,u_{x_ix_j}+\sum_{i=1}^{n}a_i(x)\,u_{x_i}+a(x)\,u=f(x), \qquad (1)$$

where the coefficients and the free term are assumed to be sufficiently smooth complex-valued functions. We denote by $A(x)$ the matrix $\|a_{ij}(x)\|$, $i, j = 1, \ldots, n$, which is composed of the coefficients of the highest derivatives; at no point of Q, $A(x)$ is a null matrix.

When $n = 1$, Eq. (1) is an ordinary differential equation which can be written as ($a_{11} \neq 0$)

$$u'' + b(x)\,u' + c(x)\,u = g(x). \tag{2}$$

In this case the simplest problem is the Cauchy problem which consists in finding the solution of this equation that satisfies for some $x = x^0$ the initial conditions $u(x^0) = u_0$, $u'(x^0) = u_1$.

We shall now formulate an analogous problem for the partial differential equation (1). We take a sufficiently smooth (of class C^2) $(n - 1)$-dimensional surface S lying in Q that is given by the equation

$$F(x) = 0, \tag{3}$$

where $F(x)$ is a real-valued function and $|\nabla F| \neq 0$ for all $x \in S$.

Suppose that in Q there is given a vector field $l(x) = (l_1(x), \ldots, l_n(x))$, where $l_i(x)$, $i = 1, \ldots, n$, are real-valued functions belonging to $C^1(Q)$, $|l|^2 = l_1^2 + \ldots + l_n^2 > 0$, such that for no $x \in S$ the vector $l(x)$ is tangent to the surface S, that is,

$$\frac{\partial F}{\partial l}\bigg|_S = \frac{(l, \nabla F)}{|l|}\bigg|_S \neq 0.$$

(In what follows, we shall be interested in the values of the field $l(x)$ on S only.)

We take an arbitrary point $x^0 \in S$, and consider Eq. (1) in a sufficiently small neighbourhood U of this point. (Let U be a ball of sufficiently small radius with centre at x^0.) Let S_0 denote the intersection $S \cap U$.

Let u, $u \in C^2(U)$, be a solution of Eq. (1) in U, and let $u_0(x)$ be the value of u on S_0 and $u_1(x)$ the value of $\frac{\partial u}{\partial l}$ on S_0:

$$u|_{S_0} = u_0(x), \tag{4}$$

$$\frac{\partial u}{\partial l}\bigg|_{S_0} = u_1(x). \tag{5}$$

We shall show that for a partial differential equation, in contrast to an ordinary differential equation, u_0 and u_1 cannot be, generally speaking, arbitrary (smooth) functions.

Since $\nabla F(x^0) \neq 0$, one of the components of $\nabla F(x^0)$ is not zero; suppose, for example, $F_{x_n}(x^0) \neq 0$. We assume the neighbourhood U to be so small that $F_{x_n}(x) \neq 0$ in U and Eq. (3) may be written as

$$x_n = \varphi(x'),\quad x' = (x_1, \ldots, x_{n-1}),$$

where $\varphi(x')$ is a smooth function. We denote by $F_n(x)$ the function $F(x)$ and by $F_i(x)$ the functions $x_i - x_i^0$, $i = 1, \ldots, n - 1$, and consider one-to-one mapping

$$y_i = F_i(x), \qquad i = 1, \ldots, n, \tag{6}$$

of the region U into a neighbourhood V of the origin — the image of the point x^0. Let Σ denote the image of S_0 lying in the plane $y_n = 0$: $\Sigma = V \cap \{y' = (y_1, \ldots, y_{n-1}) \in R_{n-1}, y_n = 0\}$. The function $u(x(y))$ is denoted by $v(y)$. Since $u_{x_i} = \sum_{p=1}^{n} v_{y_p} F_{px_i}$ and $u_{x_i x_j} = \sum_{p,q=1}^{n} v_{y_p y_q} F_{px_i} \cdot F_{qx_j} + \sum_{p=1}^{n} v_{y_p} F_{px_i x_j}$, in new variables Eq. (1) has the form

$$\sum_{i,j=1}^{n} \beta_{ij}(y)\, v_{y_i y_j} + \sum_{i=1}^{n} \beta_i(y)\, v_{y_i} + \beta(y)\, v = f_1(y), \tag{1'}$$

where $\beta_{ij}(y)$ are elements of the square matrix $\|(A(x(y)) \times \nabla F_i(x(y)), \nabla F_j(x(y)))\|$, in particular,

$$\beta_{nn}(y(x)) = (A(x)\, \nabla F(x), \nabla F(x)). \tag{7}$$

Conditions (4) and (5) respectively become

$$v|_\Sigma = v_0(y') \tag{8}$$

and

$$(\nabla_y v, \lambda(y))|_\Sigma = v_1'(y'), \tag{5'}$$

where $v_0(y') = u_0(y', \varphi(y'))$, $v_1'(y') = u_1(y', \varphi(y'))$, and the vector $\lambda(y(x)) = \left(\frac{\partial F_1}{\partial l}, \ldots, \frac{\partial F_n}{\partial l}\right)$, $x \in S_0$; moreover,

$$\frac{\partial F_n}{\partial l} = \frac{\partial F}{\partial l} \neq 0 \text{ on } S_0.$$

We shall first show that the value of the vector ∇v on the surface Σ is uniquely determined by v_0 and v_1'. In fact, the derivatives $v_{y_i}|_\Sigma$, $i = 1, \ldots, n-1$, are computed from (8): $v_{y_i}|_\Sigma = v_{0y_i}$, $i = 1, \ldots, n-1$, and according to (5')

$$v_{y_n}|_\Sigma = v_1(y'), \tag{9}$$

where $v_1(y') = \dfrac{1}{\frac{\partial F}{\partial l}} \left(v_1'(y') - \sum_{i=1}^{n-1} v_{0y_i} \frac{\partial F_i}{\partial l}\right)$.

Clearly, the conditions (8), (5') and (8), (9) are equivalent.

We now consider the values on Σ of the second derivatives of $v(y)$. First, we note that by (8) and (9) the values on Σ of all the second derivatives, except $v_{y_n y_n}$, of $v(y)$ are uniquely determined by the functions v_0 and v_1. To find the value of $v_{y_n y_n}$ on Σ, we use

Eq. (1′). Noting (7), by (1′) we obtain

$$(A(x(y))\nabla_x F(x(y)),\ \nabla_x F(x(y)))\,v_{y_ny_n}$$

$$= f_1(y) - \sum_{i,j=1}^{n-1}\beta_{ij}v_{y_iy_j} - \sum_{i=1}^{n-1}\beta_{in}v_{y_iy_n} - \sum_{i=1}^{n}\beta_i v_{y_i} - \beta v. \qquad (1'')$$

If the function $(A(x)\nabla F, \nabla F) \neq 0$ on the surface S_0, then the function $(A(x(y))\nabla_x F(x(y)), \nabla_x F(x(y)))$ does not vanish on Σ, and therefore also in V (the neighbourhood U is assumed to be small). In this case Eq. (1″) in V can be written as

$$v_{y_ny_n} = \sum_{i,j=1}^{n-1}\gamma_{ij}v_{y_iy_j} + \sum_{i=1}^{n-1}\gamma_{in}v_{y_iy_n} + \sum_{i=1}^{n}\gamma_i v_{y_i} + \gamma v + h. \qquad (10)$$

Setting $y_n = 0$ in (10), we obtain the value of $v_{y_ny_n}$ on Σ.

Hence, if $(A(x)\ \nabla F, \nabla F) \neq 0$ on S_0, all the derivatives of $u(x)$ up to second order are uniquely determined on S_0.

However, if at some point $\tilde{x} \in S_0$ $(A(\tilde{x})\nabla F(\tilde{x}), \nabla F(\tilde{x})) = 0$, then at the corresponding point $\tilde{y}$ $(A(x(\tilde{y}))\nabla_x F(x(\tilde{y})), \nabla_x F(x(\tilde{y}))) = 0$. Then at the point $\tilde{y}$ the equality (1″) connects the known quantities $v(\tilde{y})$, $v_{y_i}(\tilde{y})$, $v_{y_iy_j}(\tilde{y})$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$. Thus the values of v_0 and its derivatives up to second order and those of v_1 and its derivatives of first order at the point $\tilde{y}$, and hence the values at the point $\tilde{x}$ of u_0 and u_1 and their corresponding derivatives are subject to some relation, that is, they cannot be, generally speaking, arbitrary.

A point x on the surface S of class C^1 and given by the equation $F = 0$ (F is a real-valued function, $\nabla F \neq 0$ on S) is called the *characteristic point* for Eq. (1) if at this point

$$(A(x)\ \nabla F(x),\ \nabla F(x)) = 0.$$

The surface S is called a *characteristic surface* for Eq. (1) or *characteristic* (for) of Eq. (1) if all its points are characteristic points.

In this section we shall study the Cauchy problem for Eq. (1), that is, the problem of finding solution of (1) satisfying conditions (4) and (5) with given functions u_0 and u_1 in the case when the surface S does not contain characteristic points.

The case when the surface S contains characteristic points is far more difficult. As it was shown, if the point $x^0 \in S$ is a characteristic point, then there are (smooth) functions u_0 and u_1 such that Eq. (1) has no smooth solution (in $C^2(U)$) in any neighbourhood of this point that satisfies conditions (4) and (5) on $S_0 = S \cap U$. It is easy to see that if U^+ is one of the parts into which S_0 divides U (it is assumed that U is a ball of sufficiently small radius with centre at x^0), then there is no solution in $C^2(U^+ \cup S_0)$ too which satisfies conditions (4) and (5) on S_0. If still a smooth solution exists, it may not be unique.

Suppose, for example, $n = 2$. In a disc U with the origin as its centre let us consider the equation

$$u_{x_1x_2} = f(x),$$

for which the line $x_2 = 0$ is a characteristic (to this form is transformed the wave equation $u_{x_1x_1} - u_{x_2x_2} = f_1$ by a change of variables). It is easy to see that for the existence in U of a smooth solution (belonging to $C^2(U)$) of this equation that satisfies the conditions, $u|_{x_2=0} = u_0(x_1)$, $u_{x_2}|_{x_2=0} = u_1(x_1)$ it is necessary and sufficient that $\frac{du_1(x_1)}{dx_1} \equiv f(x_1, 0)$. Furthermore, if this condition is satisfied, then the solution can be expressed in the form

$$u(x_1, x_2) = \int_0^{x_1} d\xi_1 \int_0^{x_2} f(\xi_1, \xi_2)\, d\xi_2 + u_0(x_1) + g(x_2),$$

where g is a twice continuously differentiable arbitrary function satisfying the conditions $g(0) = 0$, $\frac{dg(0)}{dx_2} = u_1(0)$.

If S is a characteristic surface, then there may be situations where the problem for Eq. (1) should be posed in analogy with the Cauchy problem for an ordinary differential equation not of the second order but of the first order. Thus, for example, for the equation (again $n = 2$)

$$u_{x_1x_1} - u_{x_2} = f(x)$$

(the heat equation) having the line $x_2 = 0$ as a characteristic in Chap. VI we shall study the problem (the Cauchy problem) of finding a solution of this equation in the half-plane $x_2 > 0$ satisfying only one condition (4): $u|_{x_2=0} = u_0(x_1)$.

We shall now consider the problem (1), (4), (5) in the case where the surface S does not contain characteristic points. Let Q be an n-dimensional region, and let S be an $(n - 1)$-dimensional surface given by Eq. (3) that lies in Q and divides Q into two disjoint regions Q^+ and Q^-, that is, $Q \setminus S = Q^+ \cup Q^-$, $Q^+ \cap Q^- = \varnothing$. Suppose that Eq. (1) is given in Q (that is, the coefficients and the free term of Eq. (1) are defined in Q), and a vector field $l(x) = (l_1(x), \ldots, l_n(x))$, $|l(x)| > 0$ on S, is defined on S that is nowhere tangent to S. Let there be given two functions $u_0(x)$ and $u_1(x)$. Suppose that S does not contain characteristic points of Eq. (1), that is, on S

$$(A(x)\nabla F, \nabla F) \neq 0. \tag{11}$$

It is required to find a function $u(x)$ belonging to $C^2(Q)$ and satisfying Eq. (1) in Q together with initial conditions (4) and (5) on S. This problem will be called the noncharacteristic Cauchy problem. The given functions, that is, the coefficients and the free term of Eq. (1), the function F of (3), the vector function l and the functions u_0 and u_1 will be referred to as the data of the problem.

We shall assume that the data of the problem (1), (4), (5) are infinitely differentiable: the coefficients and the free term of Eq. (1) and the function $F(x)$ of (3) belong to $C^\infty(Q)$, while the functions $l_1(x), \ldots, l_n(x), u_0(x), u_1(x)$ belong to $C^\infty(S)$ (that is, the functions $l_1(x(y)), \ldots, u_1(x(y))$, where $x = x(y)$ is a mapping, defined by (6), of a neighbourhood U of any point $x^0 \in S$ into a neighbourhood V of the origin, are infinitely differentiable in an $(n-1)$-dimensional region $\Sigma = V \cap \{y' \in R_{n-1}, y_n = 0\}$). We also assume that there is a solution $u(x)$ of the problem (1), (4), (5) that is infinitely differentiable in Q.

As shown above, all the derivatives of $u(x)$ up to second order are determined uniquely on S in terms of the data of the problem. We shall show that on the surface S all the derivatives of $u(x)$ of any order are uniquely determined in terms of the data of the problem. Since in this case the mapping (6) of the neighbourhood U of any point $x^0 \in S$ into the neighbourhood V of the origin is given by functions $F_i(x)$, $i = 1, \ldots, n$, that are infinitely differentiable in U, as a consequence of mapping (6) the problem (1), (4), (5) in U (by this we mean the problem of finding in U a solution of Eq. (1) satisfying the initial conditions $u|_{S_0} = u_0(x)$, $\frac{\partial u}{\partial l}|_{S_0} = u_1(x)$, where $S_0 = U \cap S$) is replaced by an equivalent problem (8)-(10) for the function $v(y)$ in V with infinitely differentiable data. And because there exists a solution $u(x)$ of the problem (1), (4), (5) that is infinitely differentiable in U, the problem (8)-(10) has also an infinitely differentiable solution $v(y) = u(x(y))$ in V. To establish this assertion, it suffices to show that all the derivatives $D_y^\alpha v(y)$ are uniquely determined on Σ in terms of the data of the problem (8)-(10).

For any $\alpha' = (\alpha_1, \ldots, \alpha_{n-1})$, $|\alpha'| \geqslant 0$, the values of the derivatives $D^{(\alpha', 0)}v(y)$ and $D^{(\alpha', 1)}v(y)$ on Σ are determined directly from (8) and (9):

$$D^{(\alpha', 0)}v|_\Sigma = D^{\alpha'} v_0, \quad D^{(\alpha', 1)}v|_\Sigma = D^{\alpha'} v_1.$$

If v_α denotes the value of the function $\frac{1}{\alpha!} D^\alpha v$ $(\alpha! = \alpha_1! \ldots \alpha_n!)$ at the origin:

$$v_\alpha = \frac{1}{\alpha!} D^\alpha v(0), \quad |\alpha| \geqslant 0, \tag{12}$$

then $v_{\alpha', 0}$ and $v_{\alpha', 1}$, $|\alpha'| \geqslant 0$, are uniquely determined in terms of v_0 and v_1:

$$v_{\alpha', 0} = \frac{1}{(\alpha)'!} D^{\alpha'} v_0 \,|_{y'=0}, \tag{13}$$

$$v_{\alpha', 1} = \frac{1}{(\alpha')!} D^{\alpha'} v_1 \,|_{y'=0} \tag{14}$$

$((\alpha')! = \alpha_1! \ldots \alpha_{n-1}!)$.

To find the value of $D^{(\alpha', 2)} v(y)$, $|\alpha'| \geqslant 0$, on Σ, we use (10). Differentiating (10) with respect to $y_1, \ldots, y_{n-1}$ and setting $y_n = 0$, we obtain

$$D^{(\alpha', 2)} v \,|_{\Sigma} = D^{(\alpha', 0)} H_1 \,|_{\Sigma}, \qquad |\alpha'| \geqslant 0,$$

where the function $H_1(y)$ is defined in V by the formula

$$H_1(y) = \sum_{i, j=1}^{n-1} \gamma_{ij}(y) v_{y_i y_j} + \sum_{i=1}^{n-1} \gamma_{in}(y) v_{y_i y_n} + \sum_{i=1}^{n} \gamma_i(y) v_{y_i} + \gamma(y) v + h(y)$$

(in which $v(y)$ is the solution of the problem (8)-(10)). Now $D^{(\alpha', 0)} H_1 |_{\Sigma}$ is a function (which is linear with known coefficients) of known quantities $D^{(\beta', 0)} v |_{\Sigma}$ and $D^{(\gamma', 1)} v |_{\Sigma}$ for $0 \leqslant |\beta'| \leqslant |\alpha'| + 2$, $0 \leqslant |\gamma'| \leqslant |\alpha'| + 1$. Therefore, on Σ, all the derivatives $D^{(\alpha', 2)} v(y)$, $|\alpha'| \geqslant 0$, are uniquely determined in terms of the data of the problem, and, in particular,

$$v_{\alpha', 2} = (2! (\alpha')!)^{-1} D^{(\alpha', 0)} H_1(y) \,|_{y=0}, \qquad |\alpha'| \geqslant 0.$$

We assume that for some $k \geqslant 2$ all the derivatives $D^{(\alpha', k-1)} v(y)$, $|\alpha'| \geqslant 0$, have been uniquely determined on Σ in terms of the data of the problem. We now find the derivative $D^{(\alpha', k)} v(y) |_{\Sigma}$, $|\alpha'| \geqslant 0$. For this, we differentiate in V Eq. (10) α_1 times with respect to $y_1, \ldots, \alpha_{n-1}$ times with respect to y_{n-1} and $k - 2$ times with respect to y_n, and then set $y_n = 0$. This yields

$$D^{(\alpha', k)} v(y) \,|_{\Sigma} = D^{(\alpha', k-2)} H_1(y) \,|_{\Sigma}$$

Now $D^{(\alpha', k-2)} H_1 |_{\Sigma}$ is a function (linear with known coefficients) of already-known quantities $D^{(\beta', i)} v |_{\Sigma}$, $0 \leqslant i \leqslant k-1$ ($0 \leqslant |\beta'| \leqslant |\alpha'| + 2$ for $0 \leqslant i \leqslant k-2$ and $0 \leqslant |\beta'| \leqslant |\alpha'| + 1$ for $i = k-1$). Hence all the derivatives $D^{(\alpha', k)} v$, $|\alpha'| \geqslant 0$, are uniquely determined on Σ in terms of the data of the problem, and, in particular,

$$v_{\alpha', k} = ((\alpha')! \, k!)^{-1} D^{(\alpha', k-2)} H_1(y) \,|_{y=0}. \tag{15}$$

This proves the assertion.

Remark. Let $v(y)$ be any infinitely differentiable function in V. Consider the following infinitely differentiable function in V:

$$H(y) \equiv v_{y_n y_n} - \sum_{i,j=1}^{n-1} \gamma_{ij} v_{y_i y_j} - \sum_{i=1}^{n-1} \gamma_{in} v_{y_i y_n} - \sum_{i=1}^{n} \gamma_i v_{y_i} - \gamma v - h. \quad (16)$$

It follows from the above discussion that if the values of the function $v(y)$ and its derivatives satisfy conditions (12), where the numbers v_α, $|\alpha| \geqslant 0$, are defined by (13)-(15), then

$$D^\alpha H(y)|_{y=0} = 0 \text{ for all } \alpha, \ |\alpha| \geqslant 0. \quad (17)$$

Thus we have shown that if the surface S does not contain characteristic points, then the data of the problem uniquely determine on S all the derivatives of the infinitely differentiable solution of the problem (1), (4), (5). Hence the solution of the problem (1), (4), (5) is unique in the class of functions that are uniquely determined by their values and those of all their derivatives on S. One of such classes is the class of analytic functions. Later in this section we shall show that in the class of analytic functions the problem (1), (4), (5) is solvable with analytic data.

It should be noted that in contrast to the case of an ordinary differential equation, the analyticity condition of the data in such a generality (if no additional conditions are imposed on the coefficients of Eq. (1)) is, in a definite sense, necessary for the solvability of the problem. The following example, due to H. Lewy, shows that a partial differential equation with infinitely differentiable coefficients and free term may not, in general, have a solution.

Example 1. The differential equation

$$u_{x_1x_3} + iu_{x_2x_3} + 2i(x_1 + ix_2) u_{x_3x_3} = f(x_3) \quad (18)$$

does not have twice continuously differentiable solutions in any neighbourhood of the origin (in R_3) if the real-valued function $f(x_3)$ is not analytic.

To prove this statement, it is obviously enough to check that the equation

$$u_{x_1} + iu_{x_2} + 2i(x_1 + ix_2) u_{x_3} = f(x_3) \quad (19)$$

does not have continuously differentiable solutions in any neighbourhood of the origin.

Suppose, on the contrary, that in the cylinder $Q = \{x_1^2 + x_2^2 < R^2, \ |x_3| < H\}$ for some $R > 0$ and $H > 0$, there is a solution $u(x)$ belonging to $C^1(\overline{Q})$ of Eq. (19) with a real-valued function $f(x_3)$ that is nonanalytic on the interval $|x_3| < H$. Then in the parallelepiped $\Pi = \{0 < \rho < R, \ 0 < \varphi < 2\pi, \ |x_3| < H\}$ the function $u_1(\rho, \varphi, x_3) = u(\rho\cos\varphi, \rho\sin\varphi, x_3)$ is a solution of the equation

$$u_{1\rho}e^{i\varphi}+i\frac{u_{1\varphi}}{\rho}e^{i\varphi}+2i\rho e^{i\varphi}u_{1x_3}=f(x_3),$$

where $u_1\in C^1(\overline{\Pi})$ and $u_1(\rho,\ 0,\ x_3)\equiv u_1(\rho,\ 2\pi,\ x_3)$. Integrating this equation (for fixed ρ and x_3) with respect to $\varphi\in(0,\ 2\pi)$, we find that in the rectangle $K_1=\{0<\rho<R,\ |x_3|<H\}$ the function

$$u_2(\rho,\ x_3)=\int_0^{2\pi}u_1(\rho,\ \varphi,\ x_3)\,e^{i\varphi}\,d\varphi,$$

$u_2(\rho,\ x_3)\in C^1(\overline{K}_1)$, satisfies the equation

$$u_{2\rho}+\frac{u_2}{\rho}+2i\rho u_{2x_3}=2\pi f(x_3).$$

Therefore the function

$$v(r,\ x_3)=\sqrt{r}\,u_2(\sqrt{r},\ x_3)$$

belonging to $C^1(K_2)\cap C(\overline{K}_2)$, where K_2 is the rectangle $\{0<r<R^2,\ |x_3|<H\}$, is a solution of the equation

$$v_r+iv_{x_3}=\pi f(x_3)$$

in K_2, or, which is the same, is a solution of the equation

$$\Big(v(r,\ x_3)+i\pi\int_0^{x_3}f(\xi)\,d\xi\Big)_r+i\Big(v(r,\ x_3)+i\pi\int_0^{x_3}f(\xi)\,d\xi\Big)_{x_3}=0.$$

But the last equation is the Cauchy-Riemann equation for the function

$$w(r,\ x_3)=v(r,\ x_3)+i\pi\int_0^{x_3}f(\xi)\,d\xi.$$

Hence the function $w(r,\ x_3)$ as a function of the complex variable $r+ix_3$, $w(r,\ x_3)=g(r+ix_3)$, is analytic in K_2 and continuous in $\overline{K}_2$. Since $\operatorname{Re} g|_{r=0}=0$, by the principle of symmetry the function $g(r+ix_3)$ can be continued analytically into the rectangle $K_3=\{|r|<R^2,\ |x_3|<H\}$, and, in particular, is analytic in x_3 on the segment $\{r=0,\ |x_3|<H\}$. But $g|_{r=0}=i\pi\int_0^{x_3}f(\xi)\,d\xi$; consequently, for $|x_3|<H$ the function $f(x_3)$ is also analytic, contradicting the hypothesis.

Let us note that the plane $x_1=0$, for example, does not contain characteristic points for Eq. (18). Thus for any initial functions the

Cauchy problem for Eq. (18) (with initial conditions given on the plane $x_1 = 0$) does not have solutions in any neighbourhood that contains the origin.

2. Analytic Functions of Several Variables. Let Q be a region in the n-dimensional space R_n and $g(x)$ a complex-valued function defined in Q.

The function $g(x)$ is said to be *analytic at the point* $x^0 \in Q$ if in a neighbourhood U of this point it can be represented as an absolutely convergent power series

$$g(x) = \sum_{\alpha_1=0}^{\infty} \sum_{\alpha_2=0}^{\infty} \cdots \sum_{\alpha_n=0}^{\infty} g_{\alpha_1 \ldots \alpha_n} (x_1 - x_1^0)^{\alpha_1} \ldots (x_n - x_n^0)^{\alpha_n}$$

$$= \sum_{\alpha} g_\alpha (x - x^0)^\alpha, \quad (20)$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $(x - x^0)^\alpha = (x_1 - x_1^0)^{\alpha_1} \ldots (x_n - x_n^0)^{\alpha_n}$.

The function $g(x)$ is said to be *analytic in a region* if it is analytic at every point of this region.

Let the function $g(x)$ be analytic at the point $x^0 \in Q$. Then in the cube $K_R(x^0) = \{|x_i - x_i^0| < R, \ i = 1, \ldots, n\}$, $R > 0$, this function is represented by an absolutely convergent (in $K_R(x^0)$) series (20). Since a power series that converges absolutely in $K_R(x^0)$ converges uniformly in any strictly interior subregion K of the cube $K_R(x^0)$, $K \Subset K_R(x^0)$, together with all its derivatives, the function $g(x) \in C^\infty(\bar{K})$, and, consequently, $g(x) \in C^\infty(K_R(x^0))$. Moreover, it is evident that $g_\alpha = \frac{1}{\alpha!} D^\alpha g(x^0)$, where $\alpha! = \alpha_1! \ldots \alpha_n!$, that is, the series (20) is the Taylor series of $g(x)$ at the point x^0. Hence it follows that a function which is analytic in the region Q is uniquely determined in all of Q by its own value and the values of all its derivatives at an arbitrary point of Q; in particular, if the function vanishes together with all its derivatives at a point of Q, then $g(x) \equiv 0$ in Q.

Let us show that *if the function $g(x)$ is analytic at the point $x^0 \in Q$, then it is also analytic in some neighbourhood of this point.* For this it is enough to prove that if $K_R(x^0)$ is a cube in which $g(x)$ is represented by the (absolutely convergent) series (20), then $g(x)$ is analytic in the cube $K_{R/4}(x^0)$.

Since the series (20) is absolutely convergent in $K_R(x^0)$, it follows that for any $\rho \in (0, R)$

$$\sum_{\alpha} |g_\alpha| \rho^{|\alpha|} < \infty. \quad (21)$$

We take an arbitrary point $x^1 \in K_{R/4}(x^0)$. Then for all $x \in K_{R/8}(x^1)$ we have

$$g(x) = \sum_{\alpha} g_\alpha \Big(\sum_{p_1=0}^{\alpha_1} C_{\alpha_1}^{p_1} (x_1 - x_1^1)^{p_1} (x_1^1 - x_1^0)^{\alpha_1 - p_1} \Big)$$

$$\ldots \times \Big(\sum_{p_n=0}^{\alpha_n} C_{\alpha_n}^{p_n} (x_n - x_n^1)^{p_n} (x_n^1 - x_n^0)^{\alpha_n - p_n} \Big)$$

$$= \sum_{\alpha} \sum_{p_1=0}^{\alpha_1} \ldots \sum_{p_n=0}^{\alpha_n} g_\alpha C_{\alpha_1}^{p_1} \ldots C_{\alpha_n}^{p_n}$$

$$\times (x_1 - x_1^1)^{p_1} \ldots (x_n - x_n^1)^{p_n} (x_1^1 - x_1^0)^{\alpha_1 - p_1} \ldots (x_n^1 - x_n^0)^{\alpha_n - p_n}.$$

Since for all $x \in K_{R/8}(x^1)$ and any $\alpha = (\alpha_1, \ldots, \alpha_n)$, $p = (p_1, \ldots, p_n)$, $p_i \leqslant \alpha_i$, $i = 1, \ldots, n$,

$$| g_\alpha C_{\alpha_1}^{p_1} \ldots C_{\alpha_n}^{p_n} (x_1 - x_1^1)^{p_1} \ldots (x_n^1 - x_n^0)^{\alpha_n - p_n} |$$

$$\leqslant | g_\alpha | 2^{|\alpha|} \left(\frac{R}{8}\right)^{|p|} \left(\frac{R}{4}\right)^{|\alpha| - |p|} = | g_\alpha | \left(\frac{R}{2}\right)^{|\alpha|} \frac{1}{2^{|p|}}$$

and since by (21) the series

$$\sum_{\alpha} \sum_{p} | g_\alpha | \left(\frac{R}{2}\right)^{|\alpha|} \frac{1}{2^{|p|}} = 2^n \sum_{\alpha} | g_\alpha | \left(\frac{R}{2}\right)^{|\alpha|} < \infty.$$

it follows that the function $g(x)$ is represented in $K_{R/8}(x^1)$ by an absolutely convergent series

$$g(x) = \sum_{p} g'_p (x - x^1)^p,$$

where $g'_p = \sum_{\alpha_1 = p_1}^{\infty} \ldots \sum_{\alpha_n = p_n}^{\infty} g_\alpha C_{\alpha_1}^{p_1} (x_1^1 - x_1^0)^{\alpha_1 - p_1} \ldots C_{\alpha_n}^{p_n} (x_n^1 - x_n^0)^{\alpha_n - p_n}$.
Therefore the function $g(x)$ is analytic at the point x^1, and hence in $K_{R/4}(x^0)$ because $x^1 \in K_{R/4}(x^0)$ is arbitrary. The assertion is proved.

A real-valued function $\tilde{g}(x) = \sum_{\alpha} \tilde{g}_\alpha (x - x^0)^\alpha$ which is analytic at x^0 is called *majorant* at x^0 of the function $g(x)$ (of (20)) if for all α, $| \alpha | \geqslant 0$, $| g_\alpha | \leqslant \tilde{g}_\alpha$.

Let $\{g_\alpha, | \alpha | \geqslant 0\}$ be a complex number sequence for which there is a real number sequence $\{\tilde{g}_\alpha, | \alpha | \geqslant 0\}$ such that for all α, $| \alpha | \geqslant 0$, $| g_\alpha | \leqslant \tilde{g}_\alpha$ and the series $\sum_{\alpha} \tilde{g}_\alpha (x - x^0)^\alpha$ converges absolutely in a neighbourhood of the point x^0. Then it is evident that the function $g(x) = \sum_{\alpha} g_{\alpha}(x - x^0)^{\alpha}$ is analytic at the point x^0, and the function $\tilde{g}(x) = \sum_{\alpha} \tilde{g}_{\alpha}(x - x^0)^{\alpha}$ is its majorant at x^0.

It is also evident that any function which is analytic at the point x^0 has a majorant (at x^0). As a majorant for the function $g(x)$ of (20) one can take, for example, the function $\sum_{\alpha} | g_{\alpha} | (x - x^0)^{\alpha}$. A majorant of $g(x)$ from (20) can also be constructed as follows. Suppose that the series (20) converges absolutely in the cube $K_R(x^0)$ for some $R > 0$. Take some ρ from the interval $(0, R)$. In view of (21), there is a positive M such that $| g_{\alpha} | \rho^{|\alpha|} \leqslant M$, that is, $| g_{\alpha} | \leqslant M/\rho^{|\alpha|}$ for all $\alpha = (\alpha_1, \ldots, \alpha_n)$. This means that a majorant of the function $g(x)$ at the point x^0 is the function

$$\tilde{g}(x) = \sum_{\alpha} \frac{M(x-x^0)^{\alpha}}{\rho^{|\alpha|}} = \frac{M}{\left(1-\frac{x_1-x_1^0}{\rho}\right) \cdots \left(1-\frac{x_n-x_n^0}{\rho}\right)}.$$

The function, with any $N \geqslant 1$,

$$\tilde{g}(x) = \frac{M}{1-\frac{(x_1-x_1^0)+\cdots+(x_{n-1}-x_{n-1}^0)+N(x_n-x_n^0)}{\rho}}$$

is also a majorant of $g(x)$ at the point x^0, since for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\frac{MN^{\alpha_n}(|\alpha|)!}{\rho^{|\alpha|}\alpha!} \geqslant |g_{\alpha}|$.

3. **Kovalevskaya's Theorem.** In this subsection we shall study the Cauchy problem in the class of analytic functions, that is, we shall consider solutions of the problem (1), (4), (5) that are analytic in Q or in its subregion Q' containing the surface S. We shall assume that the data of the problem (1), (4), (5) are analytic, that is, we shall assume that the coefficients and the free term of Eq. (1) and the function F in (3) (defining S) are analytic in Q, and the functions $l_1(x), \ldots, l_n(x), u_0(x), u_1(x)$ are analytic on S (that is, the functions $l_1(x(y)), \ldots, l_n(x(y)), u_0(x(y)), u_1(x(y))$, where $x(y)$, given by formula (6), is the mapping of some neighbourhood U of an arbitrary point $x^0 \in S$ into a neighbourhood V of the origin, are analytic in an $(n-1)$-dimensional region $\Sigma = V \cap \{y' \in R_{n-1}, y_n = 0\}$). We recall that the solution of the problem, the coefficients and free term of Eq. (1) and the initial functions are complex-valued, whereas the components $l_1(x), \ldots, l_n(x)$ of the vector $l(x)$ and the function $F(x)$ are real-valued. We shall assume that S does not contain characteristic points for Eq. (1).

First of all, we shall prove a local theorem regarding the existence and uniqueness of the solution of this problem.

Theorem 1 (Kovalevskaya). *Let the data of the problem* (1), (4), (5) *be analytic, and let the surface* S *not contain characteristic points for Eq.* (1). *Then for any point* $x^0 \in S$ *there is a neighbourhood* U_{x^0} *of this point where this problem has an analytic solution. Moreover, there cannot be more than one analytic solution of this problem in any neighbourhood of* x^0.

We recall (Subsec. 1) that by problem (1), (4), (5) in the region U_{x^0} we mean the problem of finding solution $u(x)$ of Eq. (1) in U_{x^0} satisfying the initial conditions $u|_{S_0} = u_0$, $\left.\frac{\partial u}{\partial l}\right|_{S_0} = u_1$, where $S_0 = S \cap U_{x^0}$; moreover, the neighbourhood U_{x^0} is supposed to be so small that the surface S_0 divides it into two disjoint regions.

Let x^0 be an arbitrary point on S. We consider one-to-one mapping (6) of a sufficiently small neighbourhood U of this point into a neighbourhood V of the origin—the image of x^0. Since the data of the problem (1), (4), (5) and the functions $F_i(x)$, $i = 1, \ldots, n-1$, are analytic, the Cauchy problem (1), (4), (5) in U transforms under this mapping to an equivalent problem (8)-(10) (in V) with analytic data. For the proof of Theorem 1, it is enough to show that we can find a neighbourhood V of the origin where the problem (8)-(10) has an analytic solution $v(y)$ and that this solution is unique.

We start by proving the uniqueness. Suppose that in a neighbourhood V_1 of the origin there is an analytic solution $v(y)$ of the problem (8)-(10). As $v(y) \in C^\infty(V_1)$, it follows from considerations of Subsec. 1 that the values on the surface Σ of all the derivatives $D^\alpha v$, $|\alpha| \geqslant 0$, are determined uniquely by the data of the problem. In particular, all the values $D^\alpha v(0)$, $|\alpha| \geqslant 0$, are uniquely determined. Hence (see Subsec. 2) the solution $v(y)$ is uniquely determined by the data of the problem in V_1. This establishes the uniqueness.

Now we prove the existence of a solution. First, we note that it is enough to prove the existence of a function $v(y)$ that is analytic at the origin (it is also analytic in a neighbourhood V of the origin, by the properties of analytic functions; see Subsec. 2) and is a solution of the problem (8)-(10) in V.

According to (12)-(15) (see Subsec. 1), the quantities v_α, $|\alpha| \geqslant 0$, are (uniquely) determined by the data of the problem (8)-(10). We shall show that the formal power series

$$\sum_\alpha v_\alpha y^\alpha \tag{22}$$

is an analytic function at the origin. Then the sum of this series, denoted by $v(y)$, which converges absolutely in a neighbourhood V of the origin will be the desired solution.

In fact, it follows from (13) that the value of the function $v(y', 0)$ analytic in y' and the values of all its derivatives with respect to $y_1, \ldots, y_{n-1}$ for $y' = 0$ coincide with corresponding values for the function $v_0(y')$ which is analytic in y'. Consequently, $v(y', 0) \equiv \equiv v_0(y')$ on $\Sigma = V \cap \{y' \in R_{n-1}, y_n = 0\}$. Similarly, from (14) we obtain $v_{y_n}(y', 0) \equiv v_1(y')$ on Σ. That the function $v(y)$ satisfies Eq. (10) in V can be verified in the following manner. Let us consider a function $H(y)$ which is analytic in V and is defined by (16), where for $v(y)$ we take the analytic function (22) which is under consideration. In view of the choice of quantities v_α, $|\alpha| \geqslant 0$, by Remark in Subsec. 1, equalities (17) hold, that is, the function $H(y)$ and all its derivatives at the origin vanish. Accordingly, $H(y)$ is analytic in V, and $H(y) \equiv 0$. This means that $v(y)$ satisfies Eq. (10) in V.

Thus we must prove that the series (22) converges absolutely in some neighbourhood of the origin. For this (see Subsec. 2) it is enough to show that this series has majorants at the origin.

The Cauchy problem (in V)

$$\tilde{v}_{y_n y_n} = \sum_{i,j=1}^{n-1} \tilde{\gamma}_{ij} \tilde{v}_{y_i y_j} + \sum_{i=1}^{n-1} \tilde{\gamma}_{in} \tilde{v}_{y_i y_n} + \sum_{i=1}^{n} \tilde{\gamma}_i \tilde{v}_{y_i} + \tilde{\gamma} \tilde{v} + \tilde{h}, \qquad (\tilde{10})$$

$$\tilde{v}\,|_{y_n=0} = \tilde{u}_0(y'), \qquad (\tilde{8})$$

$$\tilde{v}_{y_n}|_{y_n=0} = \tilde{u}_1(y') \qquad (\tilde{9})$$

with analytic data will be referred to as a majorant problem for the problem (8)-(10) if the data of the former are majorants at the origin for corresponding data of the problem (8)-(10).

If the problem $(\tilde{8})$-$(\tilde{10})$ has an analytic solution

$$\tilde{v}(y) = \sum_{\alpha} \tilde{v}_\alpha y^\alpha, \qquad (\tilde{22})$$

at the origin, then this solution is a majorant there for the series (22) and, consequently, (22) represents an analytic function at the origin.

In order to prove this statement, we must check the validity of the inequalities $|v_\alpha| \leqslant \tilde{v}_\alpha$ for all α, $|\alpha| \geqslant 0$. According to the definition of the majorant problem, the functions $\tilde{u}_0$ and $\tilde{u}_1$ are majorants at the origin of functions u_0 and u_1, respectively. Therefore (see (13) and (14)) $|v_{\alpha', 0}| \leqslant \tilde{v}_{\alpha', 0}$ and $|v_{\alpha', 1}| \leqslant \tilde{v}_{\alpha', 1}$ for all α', $|\alpha'| \geqslant 0$.

Now assume that for some $k \geqslant 1$ we have already established the inequalities $|v_{\alpha', s}| \leqslant \tilde{v}_{\alpha', s}$ for all s, $0 \leqslant s \leqslant k-1$, and all α', $|\alpha'| \geqslant 0$. We shall demonstrate that then $|v_{\alpha', k}| \leqslant \tilde{v}_{\alpha', k}$, $|\alpha'| \geqslant 0$. According to (15)

$$v_{\alpha', k} = \sum_{|\beta'| \leqslant |\alpha'|+1} c_{\beta', k-1} v_{\beta', k-1} + \sum_{s=0}^{k-2} \sum_{|\beta'| \leqslant |\alpha'|+2} c_{\beta', s} v_{\beta', s} + h_{\alpha', k},$$

and

$$\tilde{v}_{\alpha',\,k} = \sum_{|\beta'|\leqslant|\alpha'|+1} \tilde{c}_{\beta',\,k-1}\tilde{v}_{\beta',\,k-1} + \sum_{s=0}^{k-2}\ \sum_{|\beta'|\leqslant|\alpha'|+2} \tilde{c}_{\beta',\,s}\tilde{v}_{\beta',\,s} + \tilde{h}_{\alpha',\,k},$$

where

$$h_{\alpha',\,k} = \frac{1}{(\alpha')!\,k!}\,D^{(\alpha',\,k)}h(0),$$

$$\tilde{h}_{\alpha',\,k} = \frac{1}{(\alpha')!\,k!}\,D^{(\alpha',\,k)}\tilde{h}(0),$$

the constants $c_{\beta',\,s}$ are linear combinations with nonnegative coefficients of values at the origin of derivatives of the coefficients in Eq. (10), while $\tilde{c}_{\beta',\,s}$ are the same linear combinations of corresponding (nonnegative!) derivatives of the coefficients of Eq. ($\widetilde{10}$). Since ($\tilde{8}$)-($\widetilde{10}$) is a majorant problem for the problem (8)-(10), it follows that $|h_{\alpha',\,k}| \leqslant \tilde{h}_{\alpha',\,k}$ and $|c_{\beta',\,s}| \leqslant \tilde{c}_{\beta',\,s}$. Hence $|v_{\alpha',\,k}| \leqslant \tilde{v}_{\alpha',\,k}$.

Thus for the proof of absolute convergence of the series (22) in some neighbourhood of the origin it is enough to construct the majorant problem ($\tilde{8}$)-($\widetilde{10}$) which has an analytic solution at the origin. While constructing the majorant problem, it is more convenient to deal with homogeneous initial conditions (8) and (9):

$$v\,|_{y_n=0} = 0, \tag{8_0}$$

$$v_{y_n}\,|_{y_n=0} = 0. \tag{9_0}$$

We note that for the proof of the existence of the analytic solution of the problem (8)-(10) it is enough to show the existence of the analytic solution $w\,(y)$ of the following problem with homogeneous initial conditions:

$$w_{y_ny_n} - \sum_{i,\,j=1}^{n-1} \gamma_{ij}w_{y_iy_j} - \sum_{i=1}^{n-1} \gamma_{in}w_{y_iy_n} - \sum_{i=1}^{n} \gamma_i w_{y_i} - \gamma w - h' = 0,$$

$$w\,|_{y_n=0} = 0,$$

$$w_{y_n}\,|_{y_n=0} = 0,$$

where

$$h' = h - w'_{y_ny_n} + \sum_{i,\,j=1}^{n-1} \gamma_{ij}w'_{y_iy_j} + \sum_{i=1}^{n-1} \gamma_{in}w'_{y_iy_n} + \sum_{i=1}^{n} \gamma_i w'_{y_i} + \gamma w',$$

$$w'\,(y) = v_0\,(y') + y_n v_1\,(y').$$

Indeed, it is easy to see that if w is an analytic solution of this problem, the function $v = w + w'$, is the analytic solution of the problem (8)-(10).

Consequently, we can regard the initial conditions (8) and (9) as homogeneous, that is, it suffices to construct a majorant problem for the problem (8_0), (9_0), (10). Since the coefficients and the free term of Eq. (10) are analytic at the origin, as (see Subsec. 2) Eq. $(\tilde{10})$ of the majorant problem we can take the following equation

$$\tilde{v}_{y_n y_n} = \frac{M}{1-\frac{y_1+\dots+y_{n-1}+Ny_n}{\rho}} \times \left(\sum_{i,j=1}^{n-1} \tilde{v}_{y_i y_j} + \sum_{i=1}^{n-1} \tilde{v}_{y_i y_n} + \sum_{i=1}^{n} \tilde{v}_{y_i} + \tilde{v} + 1 \right) \qquad (\tilde{\tilde{10}})$$

for some $\rho > 0$, $M > 0$ and arbitrary $N \geqslant 1$. Let us consider solutions $\tilde{v} = Y(\eta)$ of Eq. $(\tilde{\tilde{10}})$ that depend only on $\frac{y_1+\dots+y_{n-1}+Ny_n}{\rho} = \eta$. All such solutions are solutions of the ordinary differential equation

$$Y'' = \frac{AY' + B(Y+1)}{a-\eta}, \qquad (23)$$

where $A = \frac{M\rho(n-1+N)}{N^2}$, $B = \frac{M\rho^2}{N^2}$, $a = 1 - \frac{M(n-1)^2}{N^2} - \frac{(n-1)M}{N}$. Choose N so large that the number a is positive, $0 < a < 1$.

Consider the solution $Y_0(\eta)$ of (23) that satisfies the homogeneous initial conditions $Y_0(0) = Y_0'(0) = 0$. Since the coefficients of Eq. (23) are analytic when $\eta = 0$ (even when $|\eta| < a$), it is easy to see that $Y_0(\eta)$ is also analytic at zero.* Since all the derivatives of the function $\frac{1}{a-\eta}$ are positive at the point $\eta = 0$, by (23)

$$\frac{d^k Y_0(0)}{d\eta^k} \geqslant 0 \quad \text{for all } k = 0, 1, \dots.$$

Thus the function $\tilde{v}(y) = Y_0\left(\frac{y_1+\dots+y_{n-1}+Ny_n}{\rho}\right)$ which is analytic at the origin is a solution of Eq. $(\tilde{\tilde{10}})$, and all its deriva-

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* It can be most easily exhibited in the following manner. Consider the equation

$$\tilde{Y}'' = \frac{1}{a-\eta}\left(A\tilde{Y}' + \frac{B(\tilde{Y}+1)}{a-\eta}\right), \qquad (\tilde{23})$$

whose coefficients are majorants (since $0 < a < 1$) of the corresponding coefficients of Eq. (23). Eq. (23) is the Euler's equation for the function $\tilde{Y} + 1$. The solution of Eq. $(\tilde{23})$ $\tilde{Y}_0(\eta) = \frac{1}{\sigma_1-\sigma_2}[\sigma_1(1-\eta/a)^{\sigma_2} - \sigma_2(1-\eta/a)^{\sigma_1}] - 1$ satisfying the initial conditions $\tilde{Y}(0) = \tilde{Y}'(0) = 0$, where $\sigma_1 = [1 - A + \sqrt{(1-A)^2+4B}]/2$, $\sigma_2 = [1 - A - \sqrt{(1-A)^2+4B}]/2$, is analytic at zero and is a majorant of the function $Y_0(\eta)$ at the point $\eta = 0$.

tives are nonnegative at the origin. Consequently, we have constructed an analytic solution of the Cauchy problem which is a majorant problem for the problem (8_0), (9_0), (10) at the origin. ■ *

The following proposition is a consequence of Theorem 1.

Theorem 2. *Let the data of the problem* (1), (4), (5) *be analytic, and let the surface* S *not have characteristic points. Then there is a region* Q' $(Q' \subset Q)$ *containing* S *in which this problem has an analytic solution, and this problem cannot have more than one analytic solution in any region containing* S.

First of all we note that the statement regarding uniqueness of the solution follows directly from Theorem 1 and the properties of analytic functions.

We shall now prove the existence of the solution. According to Theorem 1, for every point x^0 on the surface S there is a neighbourhood of this point where the problem (1), (4), (5) is uniquely solvable. It is easy to see that by contracting each of the neighbourhoods U_{x^0}, $x^0 \in S$, it is possible to obtain a cover $\{U'_{x^0}, x^0 \in S\}$ for the surface S which has the following property: if the intersection of any two neighbourhoods is not empty, then it is an open set each of connected components of which contains points of S (that is, this intersection can be expressed as a union of not more than a countable number of disjoint regions each of which contains points of S).

In fact, in U_{x^0} consider the ball $\{|x - x^0| < r_0\}$ of sufficiently small radius $r_0 = r_0(x^0) > 0$ such that the angle between normals to S at any two points of the intersection of this ball with S is less than $\pi/4$. Let us take for U'_{x^0} the region $\{x: x = x^1 + tn(x'), x^1 \in S \cap \{|x - x^0| < r_0/4\}, t \in (-\delta_0, \delta_0)\}$, where $n(x^1)$ is normal to S at the point x^1; moreover, we assume $\delta_0 = \delta_0(x^0) < r_0/4$ to be so small that through every point of this region there passes only one normal to the surface $S \cap \{|x - x^0| < r_0\}$ (that is, corresponding to each point $x \in U'_{x^0}$ there is only one point $x^1(x)$ belonging to $S \cap \{|x - x^0| < r_0\}$ such that x lies on the straight line $\{x: x = x^1 + n(x^1)t, t \in R_1\}$). It is evident that the cover $\{U'_{x^0}, x^0 \in S\}$ for the surface S is the desired one.

Since for every $x_0 \in S$ $U'_{x^0} \subset U_{x^0}$, the problem (1), (4), (5) has a unique analytic solution in U'_{x^0}; let it be denoted by $u_{x^0}(x)$. Note that if x^0 and x^1 are two arbitrary points on S and $U'_{x^0} \cap U'_{x^1} \neq \varnothing$, then $u_{x^0}(x) \equiv u_{x^1}(x)$ in $U'_{x^0} \cap U'_{x^1}$. Consequently, it is possible to define the analytic function $u(x)$ by the equality $u(x) = u_{x^0}(x)$ for $x \in U'_{x^0}$ in the region $Q' = \bigcup_{x^0 \in S} U'_{x^0}$, $Q' \subset Q$. The function $u(x)$ is the desired analytic solution in Q' of the problem (1), (4), (5). ■

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* We will use Halmos' ■ to indicate the conclusion of the proof.

When the surface S does not contain characteristic points, the Cauchy problem, as Kovalevskaya's theorem shows, for the second-order partial differential equation which was formulated in Subsec. 1 in analogy with the Cauchy problem for ordinary second-order differential equation is in fact analogous to it in a definite sense. The well-known Cauchy theorem in the theory of ordinary differential equations states that the ordinary differential equation (2) whose coefficients and the free term are analytic on the interval $a < x < b$ has in a neighbourhood of the point x^0, $a < x^0 < b$, where the initial conditions are prescribed, a unique analytic solution satisfying these initial conditions. Kovalevskaya's theorem generalizes the Cauchy theorem to the case of partial differential equations: if the surface S where the initial conditions are prescribed does not have characteristic points and the data of the problem (1), (4), (5) are analytic, then the problem (1), (4), (5) has a unique solution in some "neighbourhood" of S.

Nevertheless, the Cauchy problem for an ordinary differential equation and the Cauchy problem for a partial differential equation, and more so, the theory of ordinary differential equations and that of partial differential equations are not totally analogous — the case of partial differential equations is far more complicated.

In Subsec. 1 it was shown that when there are characteristic points on the surface S, the existence of an analytic (even twice continuously differentiable) solution of the Cauchy problem cannot be guaranteed: if $x^0 \in S$ is a characteristic point for Eq. (1), then there are smooth, even analytic, initial functions u_0 and u_1 such that the problem (1), (4),(5) has no solution (belonging to $C^2(U)$) in any neighbourhood U of this point. Moreover, it was noted that if S is a characteristic surface, then there may be cases where the Cauchy problem must be formulated in analogy with the first-order ordinary differential equation. (For example, for the equation $u_{x_1x_1} - u_{x_2} = f(x)$, for which the straight line $x_2 = 0$ is a characteristic, the Cauchy problem will be examined in Chap. VI. The problem is to find a solution of this equation in the half-plane $x_2 > 0$ satisfying one initial condition $u|_{x_2=0} = u_0(x_1)$.) In this case also, as the following example, due to Kovalevskaya, shows, the analyticity of the data of the problem does not guarantee the existence of an analytic solution.

Example 2. The equation

$$u_{x_1x_1} - u_{x_2} = 0$$

has no analytic solution at the origin satisfying the initial condition

$$u\,|_{x_2=0} = \frac{1}{1+x_1^2}.$$

It is directly verified that if the analytic solution of this problem at the origin exists:

$$u(x_1, x_2) = \sum_{\alpha_1, \alpha_2} u_{\alpha_1, \alpha_2} x_1^{\alpha_1} x_2^{\alpha_2},$$

then the coefficients u_{α_1, α_2} have the form $u_{2s, k} = \frac{(2s+2k)!}{(2s)!\,k!}(-1)^{k+s}$ and $u_{2s+1, k} = 0$, where $s \geqslant 0$, $k \geqslant 0$. But the above series does not converge in any neighbourhood of the origin since it diverges, for example, at any point $(0, x_2)$ when $x_2 \neq 0$.

As is well-known, the solution of the Cauchy problem for ordinary differential eqnation (2) depends continuously on the initial data. The following example, due to Hadamard, shows that, generally speaking, partial differential equations do not have this property.

Example 3. Consider the Cauchy problem in the disc $Q = \{x_1^2 + x_2^2 < 1\}$:

$$u_{x_2x_2} = -u_{x_1x_1},$$

$$u|_{x_2=0} = u_{n\ 0} \equiv e^{-\sqrt{n}} e^{inx_1},$$

$$u_{x_2}|_{x_2=0} = u_{n,1} \equiv 0,$$

where n is a natural number (the straight line $x_2 = 0$, obviously, does not contain characteristic points for the equation $u_{x_2x_2} = -u_{x_1x_1}$). As is easy to check, the solution of this problem (unique in the class of analytic functions) is of the form $u = u_n(x) = e^{-\sqrt{n}} \cosh nx_2 e^{inx_1}$. Consequently, for any point $x = (x_1, x_2)$ of the disc Q, not lying on the initial line $x_2 = 0$, $|u_n(x)| \to \infty$ as $n \to \infty$, in spite of the fact that $u_{n,0}(x_1) \to 0$ $(|u_{n,0}| = e^{-\sqrt{n}})$ and even for any $k \geqslant 1$ $\frac{d^k u_{n,0}}{dx_1^k} \to 0$, as $n \to \infty$, uniformly on $[-1, 1]$.

What is more, it is well known that any ordinary differential equation (2) with continuous coefficients and free term on some interval has always a solution (on all of this interval). However, for partial differential equations in such a general situation, which we considered up-to-now, the similar statement does not hold: as shown by Lewy's example (Subsec. 1), there are linear partial differential equations of the second order that have no solution in any neighbourhood of a given point; what is more, no conditions regarding smoothness of coefficients (and even analyticity of coefficients) can be imposed which would guarantee the existence of the solution with any smooth (even infinitely differentiable) free term. Consequently, in order to study nonanalytic solutions of linear second-order partial differential equations, it is necessary to impose additional conditions on the structure of the equation. In the next section we shall select some classes of equations which will be the subject of our study in the sequel.

§ 2. CLASSIFICATION OF LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

We consider a linear partial differential equation of the second order

$$\mathcal{L}u=\sum_{i,j=1}^{n} a_{ij}(x)\,u_{x_ix_j}+\sum_{i=1}^{n} a_i(x)\,u_{x_i}+a(x)\,u=f(x) \qquad (1)$$

in an n-dimensional region Q.

The coefficients $a_{ij}(x)$, i, $j=1,\dots,n$, are assumed to be real-valued, and the solutions of Eq. (1) are assumed to belong to $C^2(Q)$. The matrix $A(x)=\|a_{ij}(x)\|$ composed of the coefficients of highest derivatives of the operator $\mathcal{L}$ can be taken as symmetric. In fact, $\sum a_{ij}u_{x_ix_j}=\sum a'_{ij}u_{x_ix_j}+\sum a''_{ij}u_{x_ix_j}$, where $a'_{ij}=\frac{1}{2}(a_{ij}+a_{ji})$, $a''_{ij}=\frac{1}{2}(a_{ij}-a_{ji})$. Since $u_{x_ix_j}=u_{x_jx_i}$, it follows that $\sum a''_{ij}u_{x_ix_j}\equiv 0$, hence $\sum a_{ij}u_{x_ix_j}\equiv\sum a'_{ij}u_{x_ix_j}$, where $\|a'_{ij}(x)\|$ is symmetric.

Let x^0 be an arbitrary point of Q, and $\lambda_1(x^0), \dots, \lambda_n(x^0)$ the eigenvalues (evidently, real) of the matrix $A(x^0)$. The number of positive eigenvalues is denoted by $n_+=n_+(x^0)$, while that of negative eigenvalues by $n_-=n_-(x^0)$ and the number of zero eigenvalues by $n_0=n_0(x^0)$; $n=n_++n_-+n_0$.

Eq. (1) is called an *equation of elliptic type at the point* x^0 (or simply, *elliptic at the point* x^0) if $n_+=n$ or $n_-=n$. This equation is said to be *elliptic on the set* E, $E\subset Q$, if it is elliptic at every point of this set. An example of elliptic equation in R_n is Poisson's equation

$$\Delta u=f,$$

where $\Delta=\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_n^2}$ is the Laplace operator.

Eq. (1) is said to be *hyperbolic at the point* $x^0\in Q$ (or an *equation of the hyperbolic type at* x^0) if $n_+=n-1$ and $n_-=1$, or if $n_+=1$ and $n_-=n-1$. If the equation is hyperbolic at every point of the set E, $E\subset Q$, then it is said to be *hyperbolic on* E. An example of the equation that is hyperbolic in the whole of space R_n of variables $x_1, \dots, x_n$ is the wave equation

$$u_{x_1x_1}+\dots+u_{x_{n-1}x_{n-1}}-u_{x_nx_n}=f.$$

Eq. (1) is termed *ultra-hyperbolic at the point* x^0 if $n_0=0$ and $1<n_+<n-1$. Eq. (1) is *ultra-hyperbolic on* E, $E\subset Q$, if it is ultra-hyperbolic at each point of E. The equation

$$u_{x_1x_1}+u_{x_2x_2}-u_{x_3x_3}-u_{x_4x_4}=f(x)$$

is ultra-hyperbolic in all of R_4.

Eq. (1) is designated *parabolic* (or, an *equation of the parabolic type*) *at the point* $x^0 \in Q$ if $n_0 > 0$. Eq. (1) is said to be *parabolic on the set* $E \subset Q$ if it is parabolic at every point of E. The heat equation

$$u_{x_1x_1} + \ldots + u_{x_{n-1}x_{n-1}} - u_{x_n} = f(x)$$

provides an example of the equation that is parabolic in the whole space R_n of variables $x_1, \ldots, x_n$.

Of course, an equation need not be of the same type at all points of a region. For example, Chapligin's equation ($n = 2$)

$$u_{x_1x_1} + T(x_1)\, u_{x_2x_2} = f(x),$$

where the function $T(x_1) > 0$ for $x_1 > 0$, $T(x_1) < 0$ for $x_1 < 0$, and $T(x_1) = 0$ for $x_1 = 0$, is elliptic for $x_1 > 0$, hyperbolic for $x_1 < 0$ and parabolic for $x_1 = 0$.

We recall that (see Sec. 1.1*) the surface S lying in Q and given by the equation $F(x) = 0$ (the real-valued function $F \in C^1(Q)$ and $|\nabla F| \neq 0$ on S) is called the characteristic surface (characteristic) for Eq. (1) if for all points $x \in S$

$$(A(x)\,\nabla F,\ \nabla F) = 0. \tag{2}$$

If Eq. (1) is elliptic in Q, then the matrix $A(x)$ is positive- or negative-definite at each point $x \in Q$. This means that (2) holds only when $|\nabla F| = 0$. Hence elliptic equations do not have characteristic surfaces (what is more, no surface S contains a characteristic point of the elliptic equation).

If Eq. (1) is hyperbolic in Q, then it can be shown that a characteristic surface can be made to pass through every point of Q. For example, for the wave equation $u_{x_1x_1} + \ldots + u_{x_{n-1}x_{n-1}} = u_{x_nx_n}$ Eq. (2) has the form

$$F_{x_1}^2 + \ldots + F_{x_{n-1}}^2 - F_{x_n}^2 = 0. \tag{2'}$$

This equation is, in particular, satisfied by the function $(x - x^0, m) = (x_1 - x_1^0)\, m_1 + \ldots + (x_n - x_n^0)\, m_n$, where x^0 is an arbitrary point of R_n and the vector $m = (m_1, \ldots, m_n)$, $|m| = 1$, is subject to the condition $m_1^2 + \ldots + m_{n-1}^2 = m_n^2$. Eq. (2′) is also satisfied by the function $(x_1 - x_1^0)^2 + \ldots + (x_{n-1} - x_{n-1}^0)^2 - (x_n - x_n^0)^2$, where x^0 is an arbitrary point of R_n. Hence the plane $(x - x^0, m) = 0$ and the canonical surface $(x_1 - x_1^0)^2 + \ldots + (x_{n-1} - x_{n-1}^0)^2 = (x_n - x_n^0)^2$ are characteristics for the wave equation.

For the heat equation $u_{x_1x_1} + \ldots + u_{x_{n-1}x_{n-1}} = u_{x_n}$ Eq. (2) is of the form

$$F_{x_1}^2 + \ldots + F_{x_{n-1}}^2 = 0.$$

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* Here and in what follows the first number will denote the section and the second its subsection.

It is evident that any solution of this equation has the form $F = \Phi(x_n)$, where Φ is an arbitrary continuously differentiable function ($\Phi' \neq 0$). Thus the characteristics for the heat equation are the planes $x_n = \text{const}$.

Let x^0 be a point of the region Q. We denote by $y = y(x)$ ($y_i = y_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n$) a transformation which maps one-to-one a neighbourhood U of the point x^0 into the neighbourhood V of the corresponding point y^0, $y^0 = y(x^0)$, and by $x = x(y)$ the inverse transformation. We shall assume that the functions $y_i(x) \in C^2(\bar{U})$, $i = 1, \ldots, n$, and that the Jacobian matrix $J(x) = \left\| \frac{\partial y_i}{\partial x_j} \right\|$ of the transformation $y = y(x)$ is nonsingular, that is, the Jacobian of transformation $\det J(x) \neq 0$ in $\bar{U}$. Let $v(y)$ denote the function $u(x(y))$. Since

$$u_{x_i} = \sum_{k=1}^{n} v_{y_k} y_{kx_i}, \quad u_{x_i x_j} = \sum_{k,\, s=1}^{n} v_{y_k y_s} y_{kx_i} y_{sx_j} + \sum_{k=1}^{n} v_{y_k} v_{kx_i x_j},$$

Eq. (1), as a result of the change of variables, assumes the form

$$\sum_{k,\, s=1}^{n} \tilde{a}_{ks}(x(y))\, v_{y_k v_s} = F(y, v, \nabla v), \tag{3}$$

where $\tilde{a}_{ks}(x) = \sum_{i,\, j=1}^{n} a_{ij}(x)\, y_{kx_i} y_{sx_j}$ and F is a function that does not depend on the second derivatives of v. Since the matrices $\tilde{A}(x) = \| \tilde{a}_{ks}(x) \|$ and $A(x) = \| a_{ij}(x) \|$ are related by $\tilde{A}(x) = JAJ^*$, by a well-known theorem of algebra the numbers of positive, negative and zero eigenvalues of $\tilde{A}(x)$ coincide with the corresponding numbers for the matrix $A(x)$. This implies that at any point $y \in V$ Eq. (3) is of the same type as Eq. (1) at the corresponding point $x \in U$. Hence the above classification of second-order equations is invariant under smooth one-to-one nondegenerate transformations of independent variables. This fact can be utilized for simplifying Eq. (1).

Let us take an arbitrary point $x^0 \in Q$. It is known that for the matrix $A(x^0)$ there exists a nonsingular matrix $T = T(x^0) = \| t_{ij} \|$ such that

$$TA(x^0)T^* = \Lambda(x^0) = \left\| \begin{matrix} \overbrace{+1 \cdots +1}^{n_+} & & 0 \\ & \overbrace{-1 \cdots -1}^{n_-} & \\ 0 & & \overbrace{0 \cdots 0}^{n_0} \end{matrix} \right\|$$

We effect a linear change of variables $y = T(x^0)\,x$. Since the Jacobian matrix of this transformation is T, Eq. (1), under this transformation, reduces to (2) in which the matrix of coefficients of highest derivatives is $TA(x)\,T^*$. This implies that for $x = x^0$ Eq. (3) has the form

$$v_{y_1y_1} + \dots + v_{y_{n_+}y_{n_+}} - v_{y_{n_++1}y_{n_++1}} - \dots - v_{y_{n_++n_-}y_{n_++n_-}} = F_1,$$

where the function F_1 does not depend on second derivatives of v. This form is called the *canonical form* of Eq. (1) at the point x^0.

Hence for any point $x = x^0 \in Q$ we can find a nonsingular linear transformation of independent variables which reduces Eq. (1) to the canonical form at the point $x = x^0$. Since the transformation depends only on the values of the coefficients of highest derivatives in (1) at $x = x^0$, in the case when the coefficients are constants in Q the resulting linear transformation reduces Eq. (1) to the canonical form at each point of Q (in the region Q).

§ 3. FORMULATION OF SOME PROBLEMS

In this section we shall examine some physical problems which lead to problems in partial differential equations.

1. Problems of the Equilibrium and Motion of a Membrane. Let us consider the problem of determining the equilibrium position of a membrane (a thin elastic plate) which is subject to the action of a system of forces.

We shall assume that in any admissible position the membrane is a surface lying in the space $(x, u) = (x_1, x_2, u)$ that projects uniquely onto a region Q of the plane x_1Ox_2, and is given by the equation $u = u(x)$, $x \in Q$, where the function $u(x) \in C^1(\bar{Q})$. If $u = \varphi(x)$, $x \in Q$, is any admissible position of the membrane, then we assume that any other admissible position $u = u(x)$ is obtained from the position $u = \varphi(x)$ in such a manner that every point of the membrane is displaced parallel to the Ou-axis.

Suppose that the external force acting on the membrane is directed parallel to the Ou-axis and has continuous density $f_1(x, u)$ equal to $f(x) - a(x)\,u$ (the membrane is subject to the force with density $f(x)$, $x \in Q$, and to the resistance force of the elastic medium whose density is $a(x)\,u$ is proportional to the displacement and opposite in sign, $a(x) \geqslant 0$ is the elasticity coefficient of the medium). The work done by this force in displacing the membrane from the position $\varphi(x)$ to the position $u(x)$ is given by

$$\int_Q \int_{\varphi(x)}^{u(x)} f_1(x, u)\,du\,dx = \int_Q [f(x)\,(u(x) - \varphi(x)) - \frac{a(x)}{2}\,(u^2(x) - \varphi^2(x))]\,dx.$$

Besides this, the membrane is subject to the internal elastic force. We assume that the work done by this force in displacing the membrane from the position $\varphi(x)$ to the position $u(x)$ is equal to

$$-\int_Q k(x)\left[\sqrt{1+|\nabla u|^2}-\sqrt{1+|\nabla\varphi|^2}\right]dx$$

(the work done by this force with regard to the element $(x_1, x_1+\Delta x_1)\times \times (x_2, x_2+\Delta x_2)$ of Q is proportional to the change in the surface area of that part of the membrane which is projected onto this element; the coefficient $k(x) > 0$ is called the tension of the membrane).

If at the points of the boundary of the membrane a force is applied with linear density $g_1(x, u) = g_1(x) - \sigma_1(x)u$ ($\sigma_1(x) \geqslant 0$ is the coefficient of elastic fastening of the boundary), then the work done by this force in displacing the membrane from the position $\varphi(x)$ to the position $u(x)$ is given by

$$\int_{\partial Q}\left[g_1(x)(u(x)-\varphi(x))-\frac{\sigma_1(x)}{2}(u^2(x)-\varphi^2(x))\right]dS.$$

Thus, in the position $u(x)$, the potential energy of the membrane is

$$U(u)=U(\varphi)+\int_Q\left[k(x)\left(\sqrt{1+|\nabla u|^2}-\sqrt{1+|\nabla\varphi|^2}\right.\right.$$
$$\left.+\frac{a}{2}(u^2-\varphi^2)-f(u-\varphi)\right]dx+\int_{\partial Q}\left[\frac{\sigma_1}{2}(u^2-\varphi^2)-g_1(u-\varphi)\right]dS,$$

where $U(\varphi)$ is the potential energy of the membrane in the position φ.

To simplify the problem, we assume that the gradient of the function $u(x)$ is small for all the admissible positions $u(x)$ which the membrane can have, and we neglect the terms of the order $|\nabla u|^4$. Then the potential energy of the membrane in position u becomes

$$U(u)=U(\varphi)+\int_Q\left[\frac{k}{2}(|\nabla u|^2-|\nabla\varphi|^2)+\frac{a}{2}(u^2-\varphi^2)-f(u-\varphi)\right]dx$$
$$+\int_{\partial Q}\left[\frac{\sigma_1}{2}(u^2-\varphi^2)-g_1(u-\varphi)\right]dS.$$

If u is the equilibrium position of the membrane, then, by the principle of virtual displacements, the polynomial (in t)

$$P(t)=U(u+tv)$$
$$=U(u)+t\left[\int_Q(k\nabla u\,\nabla v+auv-fv)\,dx+\int_{\partial Q}(\sigma_1uv-g_1v)\,dS\right]$$
$$+\frac{t^2}{2}\left[\int_Q(k|\nabla v|^2+av^2)\,dx+\int_{\partial Q}\sigma_1v^2\,dS\right]$$

has, for any admissible v, a stationary point for $t = 0$. Hence $\frac{dP(0)}{dt} = 0$, that is, for all $v \in C^1(\overline{Q})$ the function $u(x)$ describing the equilibrium position of the membrane satisfies the integral identity

$$\int_Q (k\nabla u\, \nabla v + auv)\, dx + \int_{\partial Q} \sigma_1 uv\, dS = \int_Q fv\, dx + \int_{\partial Q} g_1 v\, dS. \tag{1}$$

If the boundary of the membrane is fixed (tightly stretched), then all the admissible positions u of the membrane satisfy the condition

$$u|_{\partial Q} = \varphi|_{\partial Q}. \tag{2}$$

In this case the potential energy of the membrane in any position u is (neglecting the terms of the order $|\nabla u|^4$)

$$U(u) = U(\varphi) + \int_Q \left[\frac{k}{2}(|\nabla u|^2 - |\nabla\varphi|^2 + \frac{a}{2}(u^2 - \varphi^2) - f(u - \varphi)\right] dx.$$

Let u be the equilibrium position of the tightly stretched membrane. Then for any function $v \in C^1(\overline{Q})$ satisfying the condition

$$v|_{\partial Q} = 0 \tag{3}$$

the function $u + tv$ satisfies condition (2) for all t. Therefore for all such v the polynomial

$$P(t) = U(u + tv) = U(u) + t\int_Q (k\nabla u\, \nabla v + auv - fv)\, dx$$
$$+ \frac{t^2}{2}\int_Q (k|\nabla v|^2 + av^2)\, dx$$

has a minimum when $t = 0$. This means that for all $v \in C^1(\overline{Q})$ satisfying (3) the function $u(x)$ which describes the position of a tightly stretched membrane satisfies the integral identity

$$\int_Q (k\nabla u\, \nabla v + auv)\, dx = \int_Q fv\, dx. \tag{4}$$

It will be shown in Chap. V that under suitable conditions on the given functions k, a, σ_1, f, g_1, and in the case of a tightly stretched membrane also on $\varphi|_{\partial Q}$, the integral identities (1) and (4) determine unique functions $u(x)$ subject to the condition (2). Furthermore, it will be shown that if the boundary ∂Q is sufficiently smooth, then the functions $u(x)$ belong to $C^2(\overline{Q})$.

However, assuming here that $u(x) \in C^2(\overline{Q})$, $k(x) \in C^1(\overline{Q})$, $k(x) \geqslant k_0 > 0$, $a(x) \in C(\overline{Q})$, $\sigma_1(x) \in C(\partial Q)$, $g_1 \in C(\partial Q)$, $\varphi \in C(\partial Q)$, we shall obtain local conditions, in place of integral identities (1) and (4), that must be satisfied by the desired function $u(x)$.

Since, by Ostrogradskii's formula,

$$\int_Q k\nabla u\,\nabla v\,dx = -\int_Q v\,\mathrm{div}\,(k\nabla u)\,dx + \int_{\partial Q} k\,\frac{\partial u}{\partial n}\,v\,dS$$

for any $v \in C^1(\overline{Q})$, we can rewrite (1) and (4), respectively, in the form

$$\int_Q (\mathrm{div}\,(k\nabla u) - au + f)\,v\,dx - \int_{\partial Q}\left(k\,\frac{\partial u}{\partial n} + \sigma_1 u - g_1\right) v\,dS = 0 \qquad (1')$$

and

$$\int_Q (\mathrm{div}\,(k\nabla u) - au + f)\,v\,dx = 0. \qquad (4')$$

Because the function $\mathrm{div}\,(k\nabla u) - au + f$ is continuous, from (4′) we obtain

$$\mathrm{div}\,(k\nabla u) - au + f = 0, \quad x \in Q, \qquad (5)$$

which together with the boundary condition (2) gives the required local conditions that must be satisfied by the function $u(x)$ in the case of a tightly stretched membrane. The problem of finding the solution of Eq. (5) satisfying the boundary condition (2) is called the *first boundary-value problem* (*the Dirichlet problem*) for Eq. (5).

Since the function $v(x)$ in (1′) is an arbitrary function belonging to $C^1(\overline{Q})$, we find, in particular, that for v satisfying condition (3) $u(x)$ in this case also satisfies Eq. (5). Therefore the identity (1′) may be written as

$$\int_{\partial Q}\left(k\,\frac{\partial u}{\partial n} + \sigma_1 u - g_1\right) v\,dS = 0.$$

Because any function in $C^1(\partial Q)$ admits of an extension into Q which belongs to $C^1(\overline{Q})$ (see Sec. 4.2, Chap. III), the last identity yields the boundary condition

$$\left.\frac{\partial u}{\partial n} + \sigma u\right|_{\partial Q} = g, \qquad (6)$$

where $\sigma = \sigma_1/k \geqslant 0$, $g = g_1/k$.

The problem of finding a solution of Eq. (5) satisfying the boundary condition (6) is called the *third boundary-value problem* for Eq. (5). When $\sigma \equiv 0$, the third boundary-value problem is known as the *second boundary-value problem* (*the Neumann problem*). In this case the boundary condition becomes

$$\left.\frac{\partial u}{\partial n}\right|_{\partial Q} = g. \qquad (7)$$

Hence the equilibrium position of a membrane is described by the solution of Eq. (5) satisfying some boundary condition. Eq. (5) is elliptic and is called the *equation of equilibrium for the membrane.*

We shall now investigate the problem of the motion of a membrane.

Let $u(x, t)$ define the position of the membrane at a given time t. Then the function $u_t(x, t)$ and $u_{tt}(x, t)$ (these derivatives are assumed to exist) define the velocity and acceleration of the membrane at a point $x \in Q$. Suppose that at a certain (initial) time moment $t = t_0$ the position of the membrane and its velocity are given, that is,

$$u|_{t=t_0} = \psi_0(x), \qquad x \in \bar{Q}, \tag{8}$$

$$u_t|_{t=t_0} = \psi_1(x), \qquad x \in \bar{Q}. \tag{9}$$

The conditions (8) and (9) are referred to as initial conditions.

By the D'Alembert's principle, the equation of motion of the membrane is the equilibrium equation (5) in which $f(x)$ has been replaced by the function $-\rho(x)\, u_{tt} + f(x, t)$ ($-\rho(x)\, u_{tt}$ is the density of the force of inertia at the point x, $\rho(x) > 0$ is the density of the membrane at the point x, and $f(x, t)$ is the density of the external force which, generally speaking, depends on t):

$$\operatorname{div}(k\nabla_x u) - au + f(x, t) - \rho(x)\, u_{tt} = 0, \; x \in Q, \; t > t_0. \tag{10}$$

As in the static case, the boundary conditions have the form (2), (6) or (7), depending on the conditions defined on the boundary ∂Q and are fulfilled for all values of time $t \geqslant t_0$ in question. The problems of finding a solution of Eq. (10) subject to conditions (2), (8), (9); (7), (8), (9); (6), (8), (9) are called, respectively, the *first*, *second* and *third mixed problems* for Eq. (10).

Hence the motion of the membrane is described by the solution of Eq. (10) satisfying the initial and some boundary conditions. Eq. (10) is hyperbolic (in the three-dimensional space) and is called the *equation of motion of a membrane.*

When the membrane is expanded infinitely ($Q = R_2$), the function $u(x, t)$ describing the motion of the membrane for all $x \in R_2$ and $t > t_0$ is the solution of Eq. (10) and satisfies initial conditions (8) and (9). In this case we say that $u(x, t)$ is a solution of the *initial-value problem* (*the Cauchy problem*) for Eq. (10).

If the coefficients in Eqs. (10) and (5) are constants: $k(x) \equiv k$, $\rho(x) \equiv \rho$, and $a(x) \equiv 0$, then these equations are called, respectively, the *wave equation*

$$\frac{1}{a^2} u_{tt} - \Delta u = \frac{f(x, t)}{k}, \quad x \in Q, \quad t > t_0, \tag{10'}$$

$a = \sqrt{\frac{k}{\rho}}$, and *Poisson's equation*

$$\Delta u = -\frac{f(x)}{k}, \quad x \in Q. \tag{5'}$$

In the case of one space variable, Eq. (10′) has the form

$$\frac{1}{a^2}u_{tt}-u_{xx}=\frac{f(x,\ t)}{k},\quad x\in(\alpha,\ \beta),\quad t>t_0. \tag{10″}$$

It describes the motion of a string situated over the interval $(\alpha,\ \beta)$. When $x=(x_1,\ x_2,\ x_3)$, the equation

$$\frac{1}{a^2}u_{tt}-\Delta u=\frac{f(x,\ t)}{k},\qquad x\in Q,\quad t>t_0, \tag{10‴}$$

describes the motion of a gas in a region Q (the function $u(x,\ t)$ characterizes, for example, small deviations at the point $x\in Q$ at time t in the pressure of gas from the constant pressure). The quantity a in this case is the velocity of sound propagation in the gas.

2. The Problem of Heat Conduction. Suppose that a substance in the three-dimensional region Q has the density $\rho(x)>0$, heat capacity $c(x)>0$, and the coefficient of heat conduction $k(x)>0$. Let $u(x,\ t)$ denote the temperature at $x\in Q$ at a given time t. Assuming that the temperature at the initial time $t=t_0$ is known:

$$u(x,\ t)\,|_{t=t_0}=\psi_0(x),\qquad x\in Q; \tag{11}$$

it is required to determine it for $t>t_0$.

Let Q' be a subregion of Q. By Newton's law, the amount of heat passing through the boundary $\partial Q'$ into Q' in an interval of time $(t_1,\ t_2)$, $t_0\leqslant t_1<t_2$, is equal to

$$-\int_{t_1}^{t_2}dx\int_{\partial Q'}k(x)\frac{\partial u}{\partial n}\,dS,$$

where n is the normal to $\partial Q'$ which is outward with respect to Q'.

If the source of heat is present in Q with a given density $f(x,\ t)$, then the increase in the amount of heat in Q' in an interval of time $(t_1,\ t_2)$ is equal to

$$\int_{t_1}^{t_2}dt\int_{Q'}f(x,t)\,dx-\int_{t_1}^{t_2}dt\int_{\partial Q'}k(x)\frac{\partial u}{\partial n}\,dS$$

and therefore the equation of heat balance in Q' has the form

$$-\int_{t_1}^{t_2}dt\int_{\partial Q'}k(x)\frac{\partial u}{\partial n}\,dS+\int_{t_1}^{t_2}dt\int_{Q'}f(x,\ t)\,dx$$

$$=\int_{Q'}c(x)\rho(x)\left(u(x,\ t_2)-u(x,\ t_1)\right)dx.$$

Noting that $u(x, t_2) - u(x, t_1) = \int\limits_{t_1}^{t_2} \frac{\partial u}{\partial t} dt$ and using the Ostrogradskii's formula, we obtain

$$\int\limits_{t_1}^{t_2} dt \int\limits_{Q'} \left[c(x)\,\rho(x)\frac{\partial u}{\partial t} - \operatorname{div}(k(x)\nabla u) - f(x, t)\right] dx = 0.$$

If the integrand is continuous in Q, then, because Q' and the interval of time (t_1, t_2) are arbitrary, the last equation is equivalent to the differential equation

$$c(x)\,\rho(x)\frac{\partial u}{\partial t} - \operatorname{div}(k(x)\nabla u) = f(x, t), \qquad x \in Q, \quad t > t_0. \quad (12)$$

This equation is of parabolic type (in a four-dimensional space of the variables x_1, x_2, x_3, t). When the functions $c(x)$, $\rho(x)$ and $k(x)$ are constants, Eq. (12) is called the *heat equation*:

$$\frac{1}{a^2}u_t - \Delta u = \frac{f(x, t)}{c\rho}, \quad (12')$$

where $a^2 = \frac{k}{c\rho}$.

It should be emphasized that Eq. (12) holds only for $t > t_0$ and only for the interior points of Q. The behaviour of the function $u(x, t)$ for $t = t_0$ is given by the initial condition (11), and for $x \in \partial Q$ $u(x, t)$ should be subject to some additional conditions. This is motivated by the concrete physical problem establishing heat relation of Q with the external medium.

In the simplest case, the value of the temperature $u(x, t)$ is given on the boundary ∂Q:

$$u|_{\partial Q} = f_0(x, t) \quad (13)$$

for all values of t under consideration. In this case the temperature is described by the solution $u(x, t)$ of Eq. (12) satisfying the conditions (11) and (13).

If the density $q_0(x, t)$ of the heat flow through the boundary is known, then, by Newton's law, the boundary condition is of the form

$$k(x)\frac{\partial u}{\partial n}\Big|_{\partial Q} = q_0(x, t). \quad (14)$$

If the temperature $u_0(x, t)$ of the medium external to Q is given, and the density $q_0(x, t)$ of the heat flow through the boundary ∂Q is proportional to the difference in temperatures $u|_{\partial Q}$ and $u_0|_{\partial Q}$, then the boundary condition is of the form

$$k\frac{\partial u}{\partial n} + k_1 u|_{\partial Q} = k_1 u_0|_{\partial Q}, \quad (15)$$

where $k_1(x) > 0$ is the coefficient of heat exchange of the body with the surrounding medium.

The problem of finding solutions of Eq. (12) satisfying the conditions (11), (13); (11), (14); (11), (15) are called, respectively, the *first*, *second*, and *third mixed problems* for Eq. (12).

When the substance occupies the whole of R_3, $Q = R_3$, the temperature $u(x, t)$ satisfies Eq. (12) for $t > t_0$ and the initial condition (11) for $t = t_0$. In this case we say that $u(x, t)$ is the solution of the *initial value problem* (*the Cauchy problem*) for Eq. (12).

PROBLEMS ON CHAPTER I

1. Let a surface S of class C^2 divide the region Q into two disjoint regions Q^+ and Q^-, and let the function $u(x)$ belong to $C^1(Q) \cap C^2(Q^+ \cup S) \cap C^2(Q^- \cup S)$ and satisfy in Q^+ and Q^- the second-order linear differential equation

$$\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} a_i(x)\, u_{x_i} + a(x)\, u = f(x) \qquad (1)$$

when coefficients and free term are continuous in Q. Show that if for a neighbourhood U_{x^0} of a point $x^0 \in S$ the function $u(x)$ does not belong to $C^2(U_{x^0})$, then x^0 is a characteristic point for Eq. (1).

2. Suppose that in a two-dimensional region Q there is given the second-order linear differential equation (1) whose coefficient and free term are analytic, and suppose that the two lines L_1 and L_2 which intersect at a point $x^0 \in Q$ are characteristics for this equation. Show that the problem (the Goursat problem) has a unique solution $u(x)$ of Eq. (1) satisfying the conditions $u|_{L_1} = u_1$ and $u|_{L_2} = u_2$, where the functions u_1 and u_2 are analytic, in a neighbourhood of the point x^0 in the class of analytic functions ($u_1(x^0) = u_2(x^0)$).

3. Suppose that a second-order linear differential equation with continuous coefficients is given in a region Q. Show the following.

If the equation is elliptic (hyperbolic) at a point of Q, then it is also elliptic (hyperbolic) in a neighbourhood of this point.

If there are two points in Q at one of which the equation is elliptic and at the other is hyperbolic, then there is a point in Q where the equation is parabolic.

SUGGESTED READING ON CHAPTER I

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CHAPTER II

THE LEBESGUE INTEGRAL AND SOME QUESTIONS OF FUNCTIONAL ANALYSIS

§ 1. THE LEBESGUE INTEGRAL

The notion of an integral and that of an integrable function associated with it are fundamental for mathematical analysis. Due to the need of applied sciences, and of mathematics itself, these ideas have undergone radical changes during the course of their development. To solve some problems, it was enough to know how to integrate continuous or even analytic functions, while for other problems it became necessary to enlarge these sets and sometimes even consider the set of all functions which are integrable in the sense of Riemann. What is more, in order to express mathematically a number of phenomena even the set of Riemann-integrable functions is not rich enough. Naturally, this set was not sufficient even for the need of mathematics itself.

In particular, some processes are described approximately by means of a sequence of "well-behaved" functions $f_k(x)$, $k = 1, 2, \ldots$, whose convergence can be asserted only in some integral sense. Thus, for example, the sequence $f_k(x)$, $k = 1, 2, \ldots$, may have one of the following properties: $\int | f_k - f_m | \, dx \to 0$ as m, $k \to \infty$ (the sequence is fundamental in the mean), $\int (f_k - f_m)^2 \, dx \to 0$ (the sequence is fundamental in the mean square) or, in more complicated cases, the integrals containing derivatives of the functions converge to zero (the sequence is fundamental in the energy). These properties, in particular when the sequence is fundamental in the mean square, may not guarantee convergence in the ordinary sense: the sequence may not converge at any point. However, we can still show (it will be done below) that there exists a unique function to which this sequence converges in a definite sense (in the mean square). This function, generally speaking, is nonintegrable in the sense of Riemann, therefore the integral in the definition of convergence must be understood in some wider sense—in the sense of Lebesgue.

1. Set of Measure Zero. A set $E \subset R_n$ is said to be a *set of* (n-dimensional) *measure zero* if it can be covered by a countable system of (n-dimensional) open cubes the sum of whose volumes (total volume) is arbitrarily small, that is, for a given $\varepsilon > 0$ we can find a countable system of cubes $K_1, K_2, \ldots$ such that $E \subset \bigcup\limits_{i=1}^{\infty} K_i$ and the total volume of these cubes $\sum\limits_{i=1}^{\infty} |K_i| < \varepsilon$, where $|K_i|$ is the volume of the cube K_i, $i = 1, 2, \ldots$.

It readily follows from the definition that the set composed of a countable number of points has measure zero. The intersection and union of a countable number of sets of measure zero are sets of measure zero. The smooth surfaces of dimension $k < n$ are also sets of measure zero.

The following criterion will prove useful in what follows.

Lemma 1. *A set E is a set of measure zero if and only if it can be covered by a countable system of cubes having finite total volume so that every point is covered by infinitely-many sets of these cubes.*

First, suppose that the covering mentioned in Lemma 1 exists. Deleting from it a finite number of cubes with maximum volumes, we can make the total volume of the remaining covering arbitrarily small. This means that E is a set of measure zero. Conversely, if E is a set of measure zero, then it can be covered by a countable system of cubes with total volume less than 2^{-k} for any integer $k \geqslant 1$. The required covering is obtained by taking the union over $k = 1, 2, \ldots$ of these covers. ■

If a property holds for all the points x of a set G except, possibly, for the set of measure zero, then we say that this property holds *for almost all points $x \in G$, almost everywhere in G, a.e. (in G)*. Thus, the Dirichlet's function $\chi(x)$ which equals 1 at points whose all coordinates are rational and vanishes elsewhere, vanishes a.e. in R_n.

Let Q be a region of R_n. Together with the functions defined everywhere in Q (that is, having finite value at every point of Q), we shall also consider functions that are defined a.e. in Q, that is, functions which are undefined on a set of measure zero. The functions $f + g$, $f \cdot g$ (f and g are defined a.e.) are defined at those points where both the functions f and g are defined.

2. Measurable Functions. Let Q be a region of the space R_n. A sequence of functions (defined a.e. in Q) $f_k(x)$, $k = 1, 2, \ldots$, is said to *converge a.e. in Q* if for almost all $x^0 \in Q$ the number sequence of values of these functions at the point x^0 has a (finite) limit. A function $f(x)$ is called the *limit of an a.e. convergent sequence* $f_k(x)$, $k = 1, 2, \ldots$, $f_k(x) \to f(x)$ a.e. in Q, as $k \to \infty$, if for almost all $x^0 \in Q$ $\lim\limits_{k \to \infty} f_k(x^0) = f(x^0)$. It is evident that if two functions

$f(x)$ and $g(x)$ are limits of the same sequence of functions, that converges a.e., they coincide a.e.

A function $f(x)$ is said to be *measurable* in Q if it is the limit of an a.e. convergent sequence of functions in $C(\bar{Q})$.

Let us note some of the obvious properties of measurable functions.

Any linear combination of measurable functions is measurable; the function $f_1 \cdot f_2$ is measurable if so are f_1 and f_2. Together with f the function $|f|$ is also measurable. If the functions $f_1, f_2, \ldots$ are measurable, then so are $\max\limits_{i \leqslant k} (f_i(x))$, $\min\limits_{i \leqslant k} (f_i(x))$, and $\lim\limits_{k\to\infty} f_k(x)$ (the limit is understood in the sense of a.e.). Since $\sup\limits_{k} (f_k(x)) = \lim\limits_{k\to\infty} \max\limits_{i \leqslant k} (f_i(x))$ and $\inf\limits_{k} (f_k(x)) = \lim\limits_{k\to\infty} \min\limits_{i \leqslant k}(f_i(x))$, it follows that these functions are also measurable provided so are f_k, $k = 1, 2, \ldots$. If the derivative of a measurable function exists a.e., it is also measurable.

It follows from the definition that a function $f(x)$ belonging to $C(\bar{Q})$ is measurable. An arbitrary function $f(x)$ belonging to $C(Q)$ is also measurable, because it can be expressed as the limit of a sequence of functions belonging to $C(\bar{Q})$ that converges in Q: $f(x) = \lim\limits_{\delta\to 0} f(x)\,\zeta_\delta(x)$, where $\zeta_\delta(x)$ is the slicing function for the region Q (see Chap. I).

3. Lebesgue Integral of Nonnegative Functions. We shall often consider sequences $f_k(x)$, $k = 1, 2, \ldots$, of measurable functions that are monotone nondecreasing (nonincreasing) a.e. in Q, that is, the sequences which, for all $k \geqslant 1$, satisfy the inequalities $f_{k+1}(x) \geqslant f_k(x)$ ($f_{k+1}(x) \leqslant f_k(x)$) a.e. in Q. If such a sequence of functions is bounded a.e. (that is, for almost all $x^0 \in Q$ the number sequence $f_k(x^0)$, $k = 1, 2, \ldots$, is bounded), then it converges a.e. to some function. We shall denote this as follows: $f_k \uparrow f$ a.e. as $k \to \infty$ if the sequence is monotone nondecreasing, and $f_k \downarrow f$ a.e. as $k \to \infty$ if the sequence is monotone nonincreasing.

A function $f(x)$ which is nonnegative a.e. in Q is said to be *Lebesgue-integrable in Q* (*over Q*) if there is a monotone nondecreasing sequence of functions $f_k(x)$, $k = 1, 2, \ldots$, in $C(\bar{Q})$ which converges to $f(x)$ a.e. in Q and is such that the sequence of (Riemann) integrals $\int\limits_Q f_k(x)\,dx \leqslant C$, $k = 1, 2, \ldots$, is bounded above. The exact upper bound of the set $\left\{\int\limits_Q f_k(x)\,dx,\ k = 1, 2, \ldots\right\}$ is called the *Lebesgue integral* of the function $f(x)$:

$$(L)\int\limits_Q f\,dx = \sup_k \int\limits_Q f_k(x)\,dx = \lim_{k\to\infty} \int\limits_Q f_k(x)\,dx. \qquad (1)$$

Let us show that if the function $f(x)$ nonnegative a.e. in Q is Lebesgue-integrable, then for any monotone nondecreasing sequence of functions $f'_k(x)$, $k = 1, 2, \dots$, in $C(\overline{Q})$ that converges a.e. to f the sequence of integrals $\int_Q f'_k(x)\,dx$, $k = 1, 2, \dots$, is bounded above and $\sup_k \int_Q f'_k(x)\,dx = (L) \int_Q f\,dx$, that is, *the Lebesgue integral is independent of the choice of the approximating sequence.*

Before proving the above statement, we shall demonstrate that if $f_k(x)$, $k = 1, 2, \dots$, is an arbitrary sequence of functions in $C(\overline{Q})$ such that $f_k \uparrow f$ a.e. as $k \to \infty$ ($f(x) \geqslant 0$ a.e.), then $\sup_k \int_Q f_k(x)\,dx \geqslant 0$, and hence for any a.e. nonnegative function $f(x)$ that is Lebesgue-integrable

$$(L) \int_Q f\,dx \geqslant 0. \tag{2}$$

Let $f_k(x) \uparrow f(x)$ a.e. as $k \to \infty$. Take an arbitrary $\varepsilon > 0$. The set E consisting of points where the sequence f_k, $k = 1, 2, \dots$, does not converge to f or where $f < 0$ is a set of measure zero, hence it can be covered by a countable set of open cubes $\{K_i,\ i = 1, 2, \dots\}$ with total volume less than ε. Let K denote the union of all the cubes of this cover. For any point $x^0 \in \overline{Q}\setminus K$ $f_k(x^0) \uparrow f(x^0) \geqslant 0$ as $k \to \infty$, therefore, there is an $N = N(x^0)$ such that $f_N(x^0) > -\varepsilon$. Since the function $f_N(x) \in C(\overline{Q})$, the last inequality holds also in the intersection $U_{x^0} \cap \overline{Q}$ of the set $\overline{Q}$ with an open cube U_{x^0} centred at the point x^0. Because the sequence is monotone, the inequalities $f_k(x) > -\varepsilon$ also hold in $U_{x^0} \cap \overline{Q}$ for all $k \geqslant N$. The aggregate $\{U_x,\ x \in \overline{Q}\setminus K\} \cup \{K_i,\ i = 1, 2, \dots\}$ of open sets covers the set $\overline{Q}$, and, since $\overline{Q}$ is closed, from this cover we can select a finite subcover $U_{x^1}, \dots, U_{x^l}$, $K_{i_1}, \dots, K_{i_s}$. Put $K' = \bigcup_{j=1}^{s} K_{i_j}$. Since $|K'| < \varepsilon$ and there exists an N_0 such that for all $x \in \overline{Q}\setminus K' \subset \left(\bigcup_{j=1}^{l} U_{x^j}\right) \cap \overline{Q}$ we have $f_k(x) > -\varepsilon$ for all $k \geqslant N_0$, it follows that for these k

$$\int_Q f_k(x)\,dx = \int_{Q\setminus K'} f_k(x)\,dx + \int_{K'} f_k(x)\,dx \geqslant -\varepsilon|Q| + A_1\varepsilon$$
$$= \varepsilon(A_1 - |Q|),$$

where $|Q|$ is the volume of Q, $A_1 = \min\limits_{x \in \overline{Q}} f_1(x)$. Since $\varepsilon > 0$ is arbitrary, the last inequality yields (2).

Now suppose that $f(x)$ is an arbitrary a.e. nonnegative Lebesgue-integrable function, and let $f_k(x)$, $k = 1, 2, \ldots$, $f_k \uparrow f$ a.e. as $k \to \infty$, be a sequence of functions in $C(\overline{Q})$ for which the sequence of integrals is bounded. Taking an arbitrary sequence of functions $f'_k(x)$, $k = 1, 2, \ldots$, in $C(\overline{Q})$ such that $f'_k(x) \uparrow f(x)$ a.e. as $k \to \infty$, we shall show that

$$\lim_{k\to\infty} \int\limits_Q f'_k \, dx = \lim_{k\to\infty} \int\limits_Q f_k \, dx.$$

Consider the sequence $f_k - f'_m$, $k = 1, 2, \ldots$, with arbitrary m. Since, as $k \to \infty$, $f_k - f'_m \uparrow f - f'_m \geqslant 0$ a.e. in Q, it follows that $\lim\limits_{k\to\infty} \int\limits_Q (f_k - f'_m)\, dx = \lim\limits_{k\to\infty} \int\limits_Q f_k \, dx - \int\limits_Q f'_m \, dx \geqslant 0$. Hence the sequence $\int\limits_Q f'_m \, dx$, $m = 1, 2, \ldots$, is bounded and $\lim\limits_{m\to\infty} \int\limits_Q f'_m \, dx \leqslant \lim\limits_{k\to\infty} \int\limits_Q f_k \, dx$, and, because the reverse inequality obviously holds, the assertion is established.

Let us consider a cube containing the region Q and sides parallel to the coordinate planes, and decompose it by planes parallel to the sides into a finite number of parallelepipeds. The nonempty intersection of an open parallelepiped of the resulting decomposition with the region Q will be called a cell (of decomposition of Q), and the aggregate of all the cells the decomposition Π of Q. A measurable function $f(x)$ is called *step-function in Q* if it assumes only constant values inside each cell of a decomposition Π of Q.

By the integral of a step-function we shall obviously mean the sum of volumes of all the cells multiplied by the value of the function in the corresponding cell.

Lemma 2. *For every monotone nondecreasing sequence of functions $f_k(x)$, $k = 1, 2, \ldots$, in $C(\overline{Q})$ there exists an a.e. monotone nondecreasing sequence $f'_k(x)$, $k = 1, 2, \ldots$, of step-functions such that $f'_k(x) \leqslant f_k(x)$, $k = 1, 2, \ldots$, a.e., and $f_k(x) - f'_k(x) \to 0$ a.e. as $k \to \infty$.*

Proof. Since the function $f_k(x)$ is uniformly continuous, we can find a number $\delta_k > 0$ such that $|f_k(x') - f_k(x'')| < 2^{-k}$ for any points $x', x'' \in \overline{Q}$ satisfying $|x' - x''| < \delta_k$, $k = 1, 2, \ldots$. We denote by Π_1 a decomposition of Q with maximum diameter of the cell $\leqslant \delta_1$. The step-function $f'_1(x)$, which is equal to $\min\limits_{x \in \bar{k}} f_1(x)$ in every cell K of the decomposition Π_1, has the property that $0 \leqslant f_1(x) - f'_1(x) \leqslant 2^{-1}$ for almost all $x \in Q$. By taking finer decomposition of Π_1 we can construct a new decomposition Π_2 with maximum diameter of the cell $\leqslant \delta_2$. The step-function $f_2'(x)$, which is equal to $\min\limits_{x\in\overline{K}} f_2(x)$ in every cell K of Π_2, satisfies the inequalities $0 \leqslant f_2(x) - f_2'(x) \leqslant 2^{-2}$ for almost all $x \in Q$. Furthermore, $f_2'(x) \geqslant \geqslant f_1'(x)$ a.e. in Q. Continuing this process, we obtain for any $k \geqslant 1$ a decomposition Π_k of Q and together with it a step-function $f_k'(x)$ having the properties that $0 \leqslant f_k(x) - f_k'(x) < 2^{-k}$, $f_k'(x) \leqslant f_{k-1}'(x)$ for almost all $x \in Q$. Hence $f_k'(x) \leqslant f_k(x)$ a.e. in Q, and a.e. in Q the sequence $f_k(x) - f_k'(x)$, $k = 1, 2, \ldots$, has a limit equal to zero. ∎

Lemma 2'. *For any sequence of step-functions* $f_k'(x)$, $k = 1, 2, \ldots$, *which is monotone nondecreasing a.e. in* Q, *there exists a monotone nondecreasing sequence of functions* $f_k(x)$, $k = 1, 2, \ldots$, *in* $C(\overline{Q})$ *such that* $f_k(x) \leqslant f_k'(x)$ *a.e. and* $f_k'(x) - f_k(x) \to 0$, *as* $k \to \infty$, *a. e. in* Q.

Proof. Evidently, it suffices to establish the lemma for $f_1'(x) \geqslant 0$ a.e.

Consider the function $f_k'(x)$ ($f_1'(x) \geqslant 0$ a.e. implies that $f_k'(x) \geqslant 0$ a.e.), and let the corresponding decomposition Π_k' of a cube containing Q (a_0 denotes the length of the side of the cube) consist of m_k cells (when the decomposition Π_k of Q corresponding to f_k' does not contain more than m_k cells). Take $\delta_k = \min\left\{\frac{a_k}{2}, \frac{1}{2na_0^{n-1}m_k2^k}\right\}$, where a_k is the length of the smallest of all sides of all the parallelepipeds, the cells of decomposition Π_k', and let $\zeta_{\delta_k}^p(x)$, $0 \leqslant \zeta_{\delta_k}^p(x) \leqslant 1$, where $\zeta_{\delta_k}^p$ is the slicing function (see Introduction, Chap. I) for the pth cell of decomposition Π_k', (δ_k is chosen so that the total volume of the intersection of parallelepipeds, where $\sum\limits_{p=1}^{m_k} \zeta_{\delta_k}^p(x) < 1$, with Q does not exceed 2^{-k}).

Let $\psi_k(x)$ denote the function $f_k'(x)\cdot \sum\limits_{p=1}^{m_k} \zeta_{\delta_k}^p(x)$. It is easy to see that $\psi_k(x) \in C(\overline{Q})$, $\psi_k(x) \leqslant f_k'(x)$ a.e., and $f_k'(x) - \psi_k(x) \to 0$, as $k \to \infty$, a.e. Then the functions $f_k(x) = \max\limits_{m\leqslant k} \psi_m(x)$, which are continuous in $\overline{Q}$, satisfy a.e. the inequalities $f_k(x) \leqslant f_k'(x)$, $k = 1, 2, \ldots$, and $f_k'(x) - f_k(x) \to 0$, as $k \to \infty$, a. e. ∎

Lemmas 2 and 2' immediately imply the following proposition.

Theorem 1. *In order that a function* $f(x)$ *nonnegative a.e. in* Q *may be Lebesgue-integrable over* Q, *it is necessary and sufficient that there exist an a.e. monotone nondecreasing sequence* $f_k(x)$, $k = 1, 2, \ldots$, *which converges to $f(x)$ a.e., of step-functions with bounded sequence of integrals. Furthermore,* $(L)\int_Q f\,dx = \sup_k \int_Q f_k\,dx$.

Lemma 3. *A monotone nondecreasing sequence of functions in $C(\overline{Q})$ having bounded sequence of integrals converges a.e. in Q.*

It follows from Lemma 2 that in order to prove Lemma 3 it suffices to establish the following proposition: If the sequence of step-functions f_k, $k = 1, 2, \ldots$, is monotone nondecreasing a.e. and the sequence of their integrals is bounded, then the sequence f_k, $k = 1, 2, \ldots$, converges a.e. in Q.

Let us cover the boundary ∂Q ($\partial Q \in C^1$, see Introduction, Chap. I) by a finite number of closed cubes $K_1, \ldots, K_l$ with sufficiently small total volume so that the set $Q' = Q \setminus \bigcup_{i=1}^{l} K_i$ is a region. Evidently, it is enough to show that an a.e. monotone nondecreasing sequence of step-functions f_k, $k = 1, 2, \ldots$, converges a.e. in the polyhedron Q'.

Consider an arbitrary function $f_k(x)$ from this sequence, and let Π_k be the corresponding decomposition of the polyhedron Q'.

Let S denote the union of sides of all such polyhedra which belong to at least one of the decompositions Π_k, $k = 1, 2, \ldots$, and let $\mathscr{E}$ denote the aggregate of all the points x of the set $Q' \setminus S$ at which the number sequence $f_k(x)$, $k = 1, 2, \ldots$, is unbounded. Because S is a set of measure zero, it is enough to show that $\mathscr{E}$ is a set of measure zero.

Taking an arbitrary $\varepsilon > 0$, we denote by $\mathscr{E}_{k,\varepsilon}$ the set which consists of (a finite number) of cells of the decomposition Π_k on which $f_k(x) \geqslant 1/\varepsilon$. Since $C \geqslant \int_{Q'} f_k(x)\,dx \geqslant -|A_1||Q'| + \frac{1}{\varepsilon}|\mathscr{E}_{k,\varepsilon}|$, where A_1 is the least value that the function $f_1(x)$ assumes on the cells of the decomposition Π_1 ($f_k(x) \geqslant A_1$ a.e. in Q'), it follows that $|\mathscr{E}_{k,\varepsilon}| \leqslant \varepsilon(C + |A_1||Q|)$. Since

$$\mathscr{E} \subset \bigcup_{k=1}^{\infty} \mathscr{E}_{k,\varepsilon} = \mathscr{E}_{1,\varepsilon} \cup \bigcup_{k=1}^{\infty} (\mathscr{E}_{k+1,\varepsilon} \setminus \mathscr{E}_{k,\varepsilon}),$$

the set $\mathscr{E}$ is covered by a countable system of polyhedra; furthermore, the total volume of these polyhedra does not exceed $\varepsilon(C + |A_1||Q|)$, because, in view of the fact that the sequence of step-functions $f_k(x)$, $k = 1, 2, \ldots$, is monotone a.e., for any $N \geqslant 1$ $\mathscr{E}_{1,\varepsilon} \cup \bigcup_{k=1}^{N-1} (\mathscr{E}_{k+1,\varepsilon} \setminus \mathscr{E}_{k,\varepsilon}) \subset \overline{\mathscr{E}}_{N,\varepsilon}$ and hence

$$|\mathscr{E}_{1,\varepsilon}| + \sum_{k=1}^{N-1} |\mathscr{E}_{k+1,\varepsilon} \setminus \mathscr{E}_{k,\varepsilon}| \leqslant |\mathscr{E}_{N,\varepsilon}| \leqslant \varepsilon(C + |A_1||Q|).$$

But then the set $\mathscr{E}$ can be covered by a countable system of open cubes with total volume less than $2\varepsilon(C + |A_1||Q|)$. ▮

4. **Lebesgue-Integrable Functions.** Any real-valued function $f(x)$ can be expressed as

$$f(x) = f^+(x) - f^-(x), \tag{3}$$

where the functions $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$ are nonnegative.

The function $f(x)$ is said to be *Lebesgue-integrable over* Q if the functions $f^+(x)$ and $f^-(x)$ in (3) are Lebesgue-integrable over Q. The integral of $f(x)$ is defined by

$$(L)\int_Q f\,dx = (L)\int_Q f^+\,dx - (L)\int_Q f^-\,dx. \tag{4}$$

Let $\Lambda(Q)$ denote the set of functions that are Lebesgue-integrable over Q. From the definition of $\Lambda(Q)$ it follows that the function $C_1f_1(x) + C_2f_2(x) \in \Lambda(Q)$ if the functions $f_i(x) \in \Lambda(Q)$ and C_i are any constants, $i = 1, 2$. Furthermore,

$$(L)\int_Q (C_1f_1 + C_2f_2)\,dx = C_1(L)\int_Q f_1\,dx + C_2(L)\int_Q f_2\,dx.$$

Therefore, in particular, $|f(x)| = f^+(x) + f^-(x) \in \Lambda(Q)$ if $f(x) \in \Lambda(Q)$, that is, a Lebesgue-integrable function is absolutely integrable. Since $|f| + f = 2f^+ \geqslant 0$ and $|f| - f = 2f^- \geqslant 0$, inequality (2) implies

$$\left|(L)\int_Q f\,dx\right| \leqslant (L)\int_Q |f|\,dx \tag{5}$$

for $f \in \Lambda(Q)$.

For functions f_1, f_2 in $\Lambda(Q)$ satisfying the inequality $f_1 \leqslant f_2$ for almost all $x \in Q$, by the same inequality (2) we have the inequality

$$(L)\int_Q f_1\,dx \leqslant (L)\int_Q f_2\,dx. \tag{6}$$

The functions $\max(f_1(x), f_2(x)) = \frac{1}{2}(f_1 + f_2 + |f_1 - f_2|)$ and $\min(f_1(x), f_2(x)) = \frac{1}{2}(f_1 + f_2 - |f_1 - f_2|)$ belong to $\Lambda(Q)$ if so do the functions f_1 and f_2, and therefore also $\max(f_1(x), \ldots, f_m(x)) \in \Lambda(Q)$, $\min(f_1(x), \ldots, f_m(x)) \in \Lambda(Q)$, if $f_i(x) \in \Lambda(Q)$, $i = 1, \ldots, m$.

Theorem 6. *In order that an a.e. nonegative function* $f(x)$ *which is Lebesgue-integrable over* Q *may vanish a.e., it is necessary and sufficient that* $(L)\int_Q f\,dx = 0$.

Proof. If $f(x) = 0$ a.e. in Q, then the sequence $f_k(x)$, $k = 1, 2, \ldots$, of functions that are identically zero in Q has the property that $f_k(x) \uparrow f(x)$, as $k \to \infty$, a.e. in Q. Therefore, by definition, $(L) \int\limits_Q f\,dx = 0$.

Conversely, suppose that $(L) \int\limits_Q f\,dx = 0$. Then there exists a sequence of functions $f_k(x)$, $k = 1, 2, \ldots$, in $C(\bar{Q})$ such that $f_k(x) \uparrow f(x)$, as $k \to \infty$, a. e. Now consider the sequence $f_k^+(x)$, $k = 1, 2, \ldots$. Evidently, $f_k^+(x) \in C(\bar{Q})$, $k = 1, 2, \ldots$, and $f_k^+(x) \uparrow f(x)$, as $k \to \infty$, a.e., thereby implying that $0 \leqslant \int\limits_Q f_k^+(x)\,dx \leqslant (L) \int\limits_Q f\,dx = 0$, that is $f_k^+(x) \equiv 0$ for all $k \geqslant 1$, which, in turn, yields that $f = 0$ a.e. ∎

5. Comparison of Riemann and Lebesgue Integrals. If a function $f(x)$ is Riemann-integrable (it should be recalled that Riemann integral is defined only for bounded functions), then, as is known, there exist two sequences f'_k, f''_k, $k = 1, 2, \ldots$, of step-functions, with f'_k, $k = 1, 2, \ldots$, a.e. monotone nondecreasing and f''_k, $k = 1, 2, \ldots$, a.e. monotone nonincreasing, that converge to $f(x)$ a.e. and are such that the sequences of their integrals have a common limit equal to the Riemann integral of $f(x)$. In the more "economical" process of constructing a Lebesgue integral, it is enough to have (in view of Theorem 1) only the first of these sequences (the bounded function $f(x)$ can, by adding an appropriate contant, be assumed nonnegative).

Hence *if a function $f(x)$ is Riemann-integrable, then it is also Lebesgue-integrable and the two integrals coincide.* Henceforth the letter L before the integral sign will be dropped, and an integral will always mean a Lebesgue integral, while an integrable function will mean a function belonging to $\Lambda(Q)$.

The set of bounded functions belonging to $\Lambda(Q)$ is larger than the set of Riemann-integrable functions, because, for instance, the Dirichlet function $\chi(x) \in \Lambda(Q)$ is bounded but not Riemann-integrable.

Furthermore, in the construction of the Lebesgue integral of $f(x)$ it was not assumed that the function is bounded; for example, the unbounded function $|x|^{-\alpha}$ for $0 < \alpha < n$ belongs to $\Lambda(|x| < 1)$. In Courses of Analysis the idea of Riemann integral is extended to unbounded functions (improper integrals). It can be easily shown that an absolutely Riemann-integrable (in the improper sense) function $f(x)$ belongs to $\Lambda(Q)$ and that its Lebesgue integral coincides with the improper Riemann integral.

Let us note that in regions of dimension not less than 2 all the improper Riemann-integrable functions are absolutely integrable in the improper sense. Therefore only in the one-dimensional case the existence of improper Riemann integral of a function may not imply its integrability in the sense of Lebesgue. An example of this is the function $\frac{1}{x}\sin\frac{1}{x}$ defined on $(0, 1)$.

6. Sufficient Conditions for Lebesgue-Integrability. Levy's Theorem. We shall now establish a relationship between measurability and integrability of a function. By definition, an integrable function is measurable; however, as illustrated by the function $|x|^{-\alpha}$, $\alpha > n$, defined in the ball $\{|x| < 1\}$, not every measurable function is integrable. Let us establish some sufficient conditions for integrability. For this, we shall require theorems on the passage to limit under the integral sign, which are also important in their own right.

First, we examine monotone sequences of functions and prove a theorem which states that the set of integrable functions is "closed" with respect to monotone limit processes.

Theorem 3 (B. Levi). *An a.e. monotone sequence of functions* $f_k(x)$, $k = 1, 2, \ldots$, *integrable over* Q *with bounded sequence of integrals converges a.e. in* Q *to an integrable function* $f(x)$, *and*

$$\lim_{k\to\infty}\int_Q f_k\,dx = \int_Q f\,dx. \tag{7}$$

Proof. It is enough to prove the theorem for a monotone nondecreasing sequence. By changing the sign of all the functions, the case of a monotone nonincreasing sequence can be reduced to the previous case.

So, let $f_k(x)$, $k = 1, 2, \ldots$, be an a.e. monotone nondecreasing sequence of integrable functions. Without loss of generality, we may assume that $f_k(x) \geqslant 0$ a.e., $k = 1, 2, \ldots$ (otherwise, instead of the sequence $f_k(x)$, $k = 1, 2, \ldots$, we would consider the sequence $F_k(x) = f_k(x) - f_1(x)$, $k = 1, 2, \ldots$, consisting of a.e. nonnegative functions).

For every $k \geqslant 1$, we examine the sequence of functions $f_{km}(x)$, $m = 1, 2, \ldots$, in $C(\overline{Q})$, $f_{km}(x) \uparrow f_k(x)$, as $m \to \infty$, a.e. in Q. The functions $\varphi_m(x) = \max\limits_{i\leqslant m}(f_{im}(x))$, $m = 1, 2, \ldots$, belong to $C(\overline{Q})$ and have the following properties:

(a) $\varphi_m(x) \leqslant \varphi_{m+1}(x)$,

(b) $f_{km}(x) \leqslant \varphi_m(x) \leqslant f_m(x)$ for $k \leqslant m$ (the second inequality holds a.e.),

(c) $\int_Q \varphi_m(x)\,dx \leqslant \int_Q f_m(x)\,dx \leqslant C$,

(d) $\int_Q f_{km}(x)\,dx \leqslant \int_Q \varphi_m(x)\,dx$ for $k \leqslant m$.

From (a), (c) and Lemma 3, we find that $\varphi_m \uparrow f$, as $m \to \infty$, a.e. in Q, where f is a measurable function. Passing to limit as $m \to \infty$ in the left inequality of (b) and using the right inequality of (b), we obtain, for all k, $\varphi_k(x) \leqslant f_k(x) \leqslant f(x)$ a.e. in Q. Then, as $k \to \infty$, $f_k(x) \to f(x)$ a.e. in Q and $f(x) \geqslant 0$ a.e. in Q. Hence $f(x) \in \Lambda(Q)$ and $\int\limits_Q f\,dx = \lim\limits_{k\to\infty} \int\limits_Q \varphi_k(x)\,dx$. Taking the limits, as $m \to \infty$, in (c) and (d), we have $\int\limits_Q f_k\,dx \leqslant \int\limits_Q f\,dx \leqslant \lim\limits_{m\to\infty} \int\limits_Q f_m\,dx$ for every k, that is, $\int\limits_Q f\,dx = \lim\limits_{m\to\infty} \int\limits_Q f_m\,dx$. $\blacksquare$

By means of Levi's theorem the following sufficient condition for integrability of a function is established.

Theorem 4 (Fatou's Lemma). *If the sequence* $f_k(x)$, $k = 1, 2, \ldots,$ *of a.e. nonnegative integrable functions converges a.e. to a function* $f(x)$ *and* $\int\limits_Q f_k dx \leqslant A$, $k = 1, 2, \ldots,$ *then* $f(x)$ *is indegrable and* $\int\limits_Q f\,dx \leqslant A$.

Proof. Consider the integrable functions $\psi_{mk}(x) = \min\limits_{m\leqslant i\leqslant k}(f_i(x))$ with $m \leqslant k$. In view of the fact that as $k \to \infty$ $\psi_{mk}(x) \downarrow \psi_m(x) = \inf\limits_{i\geqslant m}(f_i(x))$ a.e. and $0 \leqslant \psi_{mk}(x) \leqslant f_m(x)$ a.e., the inequality (6) and Levi's theorem imply that $\psi_m(x) \in \Lambda(Q)$ and $0 \leqslant \int\limits_Q \psi_m(x)\,dx \leqslant \int\limits_Q f_m(x)\,dx \leqslant A$. The assertion of the theorem now follows from Levi's theorem, since a.e. $\psi_m(x) \uparrow f(x)$ as $m \to \infty$. $\blacksquare$

Another sufficient condition for a function to belong to the set $\Lambda(Q)$ is contained in the following theorem.

Theorem 5. *If a function* $f(x)$ *is measurable and* $|f(x)| \leqslant g(x)$ *a.e., where* $g(x)$ *is an integrable function, then* $f(x)$ *is also integrable.*

Thus a measurable function with integrable modulus is integrable, and, in particular (the region Q is bounded!), every bounded (that is, $|f(x)| \leqslant$ const a.e. in Q) measurable function is integrable.

Proof of Theorem 5. Since the function $f(x)$ is measurable, there is a sequence of integrable (in fact, of even continuous in Q) functions $f_k(x)$, $k = 1, 2, \ldots,$ which converges to $f(x)$ a.e. in Q. The sequence of integrable functions $f'_k(x) = \max(-g(x), \min(f_k(x), g(x)))$, $k = 1, 2, \ldots,$ also converges to $f(x)$ a.e. and has an additional property that $|f'_k(x)| \leqslant g(x)$ a.e., $k = 1, 2, \ldots$. Then the sequence $f'_k(x) + g(x)$, $k = 1, 2, \ldots,$ is composed of a.e. nonnegative functions, converges a.e. to $f + g$, and for all k $\int\limits_Q (f_k' + g)\,dx \leqslant 2\int\limits_Q g\,dx$. By Fatou's lemma, $f + g \in \Lambda(Q)$, thereby implying $f \in \Lambda(Q)$. ∎

From Levi's theorem and Theorem 5 we obtain the following result.

Corollary. *If* $f_k(x) \in \Lambda(Q)$, $k = 1, 2, \ldots$, *and the series* $\sum\limits_{k=1}^{\infty} \int\limits_Q |f_k|\,dx < \infty$, *then the series* $\sum\limits_{k=1}^{\infty} f_k(x)$ *converges a.e. in* Q *absolutely* $\Big($*that is, the sequence* $\sum\limits_{k=1}^{m} |f_k(x)|$, $m = 1, 2, \ldots$, *converges a.e.*$\Big)$, *and the function* $f(x) = \sum\limits_{k=1}^{\infty} f_k(x) \in \Lambda(Q)$.

7. Lebesgue's Theorem on Passage to Limit under the Integral Sign. One of the fundamental results of the theory of Lebesgue integration is the following theorem, due to Lebesgue, regarding the passage to limit under the integral sign.

Theorem 6 (Lebesgue). *Let the sequence of measurable functions* $f_k(x)$, $k = 1, 2, \ldots$, *converge a.e. in* Q *to a function* $f(x)$, *and let* $|f_k(x)| \leqslant g(x)$ *a.e.*, $k = 1, 2, \ldots$, *where* $g(x)$ *is integrable. Then* $f(x)$ *is also integrable and*

$$\lim_{k\to\infty} \int\limits_Q f_k\,dx = \int\limits_Q f\,dx. \tag{7}$$

Proof. By Theorem 5, the functions $f_k(x)$, $k = 1, 2, \ldots$, are integrable.

Consider the measurable functions $\varphi_s(x) = \sup\limits_{k\geqslant s}(f_k(x))$ and $\psi_s(x) = \inf\limits_{k\geqslant s}(f_k(x))$, $s = 1, 2, \ldots$. Since a.e. $|\varphi_s(x)| \leqslant g(x)$ and $|\psi_s(x)| \leqslant g(x)$, $s = 1, 2, \ldots$, the functions $\varphi_s(x)$ and $\psi_s(x)$, $s = 1, 2, \ldots$, are also integrable. But, as $s \to \infty$, $\varphi_s(x) \downarrow f(x)$ and $\psi_s(x) \uparrow f(x)$ a.e. which, according to Levi's theorem, implies that $f(x) \in \Lambda(Q)$ and $\int\limits_Q f\,dx = \lim\limits_{s\to\infty} \int\limits_Q \varphi_s\,dx = \lim\limits_{s\to\infty}\int\limits_Q \psi_s\,dx$. Now (7) is a consequence of the obvious inequalities $\psi_s(x) \leqslant f_s(x) \leqslant \varphi_s(x)$ a.e., $s = 1, 2, \ldots$. ∎

The relation (7) may not hold if the majorant of the sequence is not an integrable function. For instance, the sequence $f_k(x) = \frac{k^2|x|^k}{\sigma_n}(1 - |x|)$, $k = 1, 2, \ldots$, where σ_n is the surface area of unit sphere in the n-dimensional space, defined in the ball $Q =$

$= \{|x| < 1\}$ tends to zero everywhere in $\bar{Q}$ but $\int\limits_Q f_k\, dx =$ $= \frac{k^2}{(k+n)(k+n+1)} \to 1$ as $k \to \infty$.

Lebesgue's theorem yields the following result.

Theorem 7. *Suppose that for some* $s \geqslant 0$ *the function* $f(x, y)$, $x = (x_1, \ldots, x_n) \in Q \subset R_n$, $y = (y_1, \ldots, y_m) \in \bar{\Omega} \subset R_m$, *belongs to the space* $C^s(\bar{\Omega})$ *for almost all* $x \in Q$, *and for all* $y \in \bar{\Omega}$ *and* $|\alpha| \leqslant s$ $|D_y^\alpha f(x, y)| \leqslant g(x)$ *for almost all* $x \in Q$, *where the function* $g(x)$ *is integrable over* Q. *Then* $\int\limits_Q f(x, y)\, dx \in C^s(\bar{\Omega})$.

8. Change of Variables under the Integral Sign. As regards the change of independent variables, the Lebesgue integral behaves exactly like a Riemann integral.

Suppose that the transformation

$$y = y(x) \quad (y_i = y_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n) \tag{8}$$

that is continuously differentiable in the region Q maps Q one-to-one into the region Q'. First of all, we shall show that *this transformation maps a set of measure zero into a set of measure zero.*

Indeed, let E, $E \subset Q$, be a set of measure zero. Since the union of a countable number of sets of measure zero is a set of measure zero, it is enough to show that under transformation (8) the image of the set $E_\delta = E \cap Q_\delta$, with sufficiently small $\delta > 0$, is a set of measure zero.

Let $\varepsilon > 0$ be arbitrary. The set E_δ can be covered by a countable set of cubes with total volume less than ε. We may assume that all the cubes of this cover have diameters less than $\delta/2$, thereby implying that all of them belong to $Q_{\delta/2}$. Since every cube, with diameter d, of this system is mapped by (8) into a region with diameter $d' \leqslant d\sqrt{n} \max\limits_{\substack{1 \leqslant i \leqslant n \\ x \in Q_{\delta/2}}} |\nabla y_i| = Cd$, the image of the set E_δ can be covered by a countable system of cubes whose total volume is less than $C^n (\sqrt{n})^n \varepsilon$. This proves the assertion.

Theorem 8. *Let the transformation* (8), *which is continuously differentiable in* Q *and has a nonvanishing Jacobian* $J(x)$ *in* Q, *map* Q *one-to-one into* Q'. *In order that a function* $f(y)$ *may belong to* $\Lambda(Q')$ *it is necessary and sufficient that the function* $f(y(x))\,|J(x)|$ *belong to* $\Lambda(Q)$. *Furthermore,*

$$\int\limits_{Q'} f(y)\, dy = \int\limits_Q f(y(x))\,|J(x)|\, dx. \tag{9}$$

Proof. The inverse transformation corresponding to (8) maps Q' one-to-one into Q, is continuously differentiable in Q', and has a nonvanishing Jacobian in Q'. Therefore it suffices to establish Theorem 8 only in one direction, and we may confine to the case where $f(y)$ is a.e. nonnegative in Q'.

Let $f(y)$ be an a.e. nonnegative function integrable over Q', and let $f_k(y)$, $k = 1, 2, \ldots$, be a sequence of functions in $C(\bar{Q}')$, $f_k(y) \uparrow f(y)$, as $k \to \infty$, a.e. in Q'. Consider the following sequence of continuous functions in $\bar{Q}'$: $f'_k(y) = f_k(y)\, \zeta(k\rho(y)), k = 1, 2, \ldots$, where the function $\zeta(t)$ defined over $[0, \infty)$ equals zero for $0 \leqslant t \leqslant 1/2$, $2t - 1$ for $1/2 < t < 1$, 1 for $t \geqslant 1$, and $\rho(y)$ is the distance of $y \in Q'$ from the boundary $\partial Q'$ ($\rho(y) \in C(\bar{Q}')$).

Obviously, for any k $f'_k(y) \leqslant f_k(y) \leqslant f(y)$ a.e. in Q' (this implies that the sequence $\int\limits_Q f'_k\, dy$, $k = 1, 2, \ldots$, is bounded) and, as $k \to \infty$, $f'_k(y) \uparrow f(y)$ a.e. in Q'. Hence

$$\lim_{k\to\infty} \int\limits_{Q'} f'_k(y)\, dy = \int\limits_Q f(y)\, dy.$$

Since the functions $f'_k(y)$ are continuous in $\bar{Q}'$, we have $\int\limits_{Q'} f'_k(y)\, dy = \int\limits_Q f'_k(y(x))\, |J(x)|\, dx$, $k = 1, 2, \ldots$. Hence the function $f(y(x))\, |J(x)|$, being the limit of a monotone nondecreasing sequence of functions $f_k(y(x))\, |J(x)|$, $k = 1, 2, \ldots$, in $C(\bar{Q})$ that converges a.e. in Q, is integrable over Q and relation (9) holds. ■

Remark. Theorem 8 readily implies that if in Q the inequalities $C_0 \leqslant |J(x)| \leqslant C_1$, where C_0 and C_1 are some positive constants, hold, then a necessary and sufficient condition for the function $f(y)$ to be integrable over Q' is that the function $f(y(x))$ be integrable over Q. Moreover, we have the inequalities

$$C_0 \int\limits_Q |f(y(x))|\, dx \leqslant \int\limits_Q |f(y)|\, dy \leqslant C_1 \int\limits_Q |f(y(x))|\, dx. \tag{10}$$

9. Measurable Sets. Integrals over Measurable Sets. Consider a subset E of Q. The function $\chi_E(x)$ equal to 1 for $x \in E$ and zero for $x \in Q \setminus E$ is called the *characteristic function* of the set E.

A set E is called *measurable* if its characteristic function is measurable. The *measure* of a measurable set E (mes E) is defined by the relation

$$\text{mes } E = \int\limits_Q \chi_E(x)\, dx. \tag{11}$$

(The integral on the right side exists in view of Theorem 5.)

If Q' is a subregion of Q, then it is measurable, since $\chi_{Q'}(x) = \lim_{\delta\to 0} \zeta_\delta(x)$, where $\zeta_\delta(x)$ is the slicing function for Q'. Further, $\operatorname{mes} Q' = |Q'|$.

The sets of measure zero defined in Subsec. 1 are measurable, and they are the only sets with zero measure (according to the definition just given). To prove this statement, it suffices to use Theorem 2 of Subsec. 4.

If E is a measurable subset of Q and $f(x)$ is integrable over Q, then by definition we consider this function to be also integrable over E, and the integral is defined by

$$\int_E f\,dx = \int_Q f\chi_E\,dx. \tag{12}$$

(Again, in view of Theorem 5, the integral on the right-hand side exists.)

If E is a subregion Q' of Q, then, as is easy to see, these new definitions of integrability and of integral over Q', of course, do not contradict the respective definitions (Subsec. 4) given directly for Q'.

10. Absolute Continuity of an Integral. The following property is known as *absolute continuity of the Lebesgue integral.*

Theorem 9. *Let a function $f(x)$ be integrable over Q. Then for any $\varepsilon > 0$ there is a $\delta > 0$ such that for an arbitrary measurable set $E \subset Q$, $\operatorname{mes} E < \delta$, the inequality*

$$\left|\int_E f\,dx\right| < \varepsilon \tag{13}$$

holds.

Proof. In view of the inequality (5), it is enough to prove the theorem for $|f(x)|$, that is, we may assume that $f(x) \geqslant 0$ a.e. in Q.

Taking an arbitrary $\varepsilon > 0$, we choose a function $f_\varepsilon(x) \in C(\overline{Q})$ such that $f(x) \geqslant f_\varepsilon(x) \geqslant 0$ a.e. in Q and $0 \leqslant \int_Q f\,dx - \int_Q f_\varepsilon\,dx \leqslant \varepsilon/2$.

Then $\int_E f\,dx = \int_Q f\chi_E\,dx = \int_Q (f - f_\varepsilon)\chi_E\,dx + \int_Q f_\varepsilon\chi_E\,dx \leqslant \frac{\varepsilon}{2} + M_\varepsilon \operatorname{mes} E$, where $M_\varepsilon = \max_{x\in\overline{Q}} f_\varepsilon(x)$. Hence (13) follows if we set $\delta = \varepsilon/(2M_\varepsilon)$. ∎

11. Relationship between Multiple and Iterated Integrals. We shall now discuss the question of reducing a Lebesgue multiple integral to an iterated integral, and simultaneously the question of interchanging the order of integration.

Let Q_n be an n-dimensional bounded region in variables $x = (x_1, \ldots, x_n)$ and Q_m an m-dimensional bounded region in variables $y = (y_1, \ldots, y_m)$. We consider a function $f(x, y)$ in the bounded region $Q_{m+n} = Q_m \times Q_n$ of $(m+n)$-dimensional space in variables (x, y).

Theorem 10 (Fubini). *Let the function* $f(x, y)$ *be integrable over* Q_{m+n}. *Then* $f(x, y)$ *is integrable with respect to* $y \in Q_m$ *for almost all* $x \in Q_n$, *is integrable with respect to* $x \in Q_n$ *for almost all* $y \in Q_m$, *the functions* $\int\limits_{Q_m} f(x, y)\, dy$ *and* $\int\limits_{Q_n} f(x, y)\, dx$ *are integrable with respect to* $x \in Q_n$ *and* $y \in Q_m$, *respectively, and*

$$\int\limits_{Q_{m+n}} f\, dx\, dy = \int\limits_{Q_n} dx \int\limits_{Q_m} f\, dy = \int\limits_{Q_m} dy \int\limits_{Q_n} f\, dx. \qquad (14)$$

It is, of course, enough to prove Fubini's theorem (see Subsec. 9) for the case where Q_n is the cube $K_n = \{|x_i| < a,\ i = 1, \ldots, n\}$, Q_m is the cube $K_m = \{|y_i| < a,\ i = 1, \ldots, m\}$, and Q_{m+n} is the cube $K_{m+n} = \{|x_i| < a,\ |y_j| < a,\ i = 1, \ldots, n,\ j = 1, \ldots, m\}$ for some $a > 0$. Before proving the above theorem, we shall establish the following assertion.

Lemma 4. *Let* E *be a set of* $(m+n)$-*dimensional measure zero situated in* K_{m+n}, *and let* $E_2(\bar{x})$ *and* $E_1(\bar{y})$ *be its m- and n-dimensional sections by the planes* $x = \bar{x}$ *and* $y = \bar{y}$, *respectively. Then for almost all* $x \in K_n$ *the set* $E_2(x)$ *has m-dimensional measure zero and for almost all* $y \in K_m$ *the set* $E_1(y)$ *has n-dimensional measure zero.*

By Lemma 1 (Subsec. 1), the set E can be covered by a countable system of cubes with finite total volume in such a manner that every point of the set belongs to infinitely-many cubes. We may assume that the sides of these cubes are parallel to coordinate planes. The series formed by integrals of characteristic functions $\chi_k(x, y)$ of these cubes converges. Since $\int\limits_{K_{m+n}} \chi_k(x, y)\, dx\, dy = \int\limits_{K_n} dx \int\limits_{K_m} \chi_k\, dy$, according to Corollary to Theorems 3 and 5 (Subsec. 6) the series composed of integrals $\int\limits_{K_m} \chi_k(x, y)\, dy$ converges for almost all x. This, in turn, means that for almost all x the set $E_2(x)$ is covered by a countable system of m-dimensional cubes with finite total volume in a manner such that each of the points lies in infinitely-many such cubes. ∎

Proceeding now to the proof of Fubini's theorem, we first note that we can confine to the case $f(x, y) \geqslant 0$ a.e. in K_{m+n}.

Take a sequence of functions $f_k(x, y)$, $k = 1, 2, \ldots$, in $C(\bar{K}_{m+n})$ such that $f_k(x, y) \uparrow f(x, y)$ a.e. in K_{m+n}. Denote by E a set having $(m+n)$-dimensional measure zero such that for all $(x, y) \in K_{m+n} \setminus E$ the sequence $f_k(x, y)$ converges monotonically to $f(x, y)$.

By the definition of an integral,

$$\int_{K_n} dx \int_{K_m} f_k(x,\ y)\,dy = \int_{K_{m+n}} f_k(x,\ y)\,dx\,dy \to \int_{K_{m+n}} f\,dx\,dy$$

as $k \to \infty$.

By Levi's theorem, monotone sequence $F_k(x) = \int\limits_{K_m} f_k(x,\ y)\,dy$, $k = 1, 2, \ldots$, of functions belonging to $C(\overline{K}_n)$ converges a.e. in K_n to a function $F(x)$ integrable over K_n, and

$$\int_{K_n} F\,dx = \lim_{k\to\infty} \int_{K_n} F_k(x)\,dx = \int_{K_{m+n}} f\,dx\,dy. \qquad (15)$$

Let us take an arbitrary point $\overline{x} \in K_n$ at which the number sequence $F_k(\overline{x})$, $k = 1, 2, \ldots$, converges to $F(\overline{x})$ and the set $E_2(\overline{x})$ (the intersection of E with the plane $x = \overline{x}$) has m-dimensional measure zero. According to Lemma 4, the set of points of K_n not satisfying these properties has n-dimensional measure zero. The sequence $f_k(\overline{x},\ y)$, $k = 1, \ldots$, converges monotonically for all $y \in K_m \setminus E_2(\overline{x})$ (consequently, a.e. in K_m) to $f(\overline{x},\ y)$. Levy's theorem asserts that $f(\overline{x},\ y) \in \Lambda(K_m)$, and, as $k \to \infty$,

$$\int_{K_m} f_k(\overline{x},\ y)\,dy \uparrow \int_{K_m} f(\overline{x},\ y)\,dy. \qquad (16)$$

Hence the functions $\int\limits_{K_m} f(x,\ y)\,dy$ and $F(x)$ coincide a.e. in K_n. ■

In what follows, we shall often make use of the following proposition which is a consequence of Fubini's theorem.

Corollary. *If $f(x,\ y)$ is measurable in Q_{m+n}, is a.e. nonnegative, and if one of the iterated integrals in* (14) *exists (that is, for instance, for almost all x $f(x,\ y)$ is integrable with respect to y and the function $\int\limits_{Q_m} f\,dy$ is integrable with respect to x), then the function $f(x,\ y)$ is integrable over Q_{m+n}, and hence the second iterated integral also exists and* (14) *holds.*

To establish this, it suffices to verify that $f(x,\ y) \in \Lambda(Q_{m+n})$. The sequence $f_k(x,\ y) = \min(f(x,\ y),\ k)$, $k = 1, 2, \ldots$, has the property that $f_k(x,\ y) \uparrow f(x,\ y)$ a.e. in Q_{m+n},

$$\int_{Q_{m+n}} f_k(x,\ y)\,dx\,dy = \int_{Q_n} dx \int_{Q_m} f_k(x,\ y)\,dy \leqslant \int_{Q_n} dx \int_{Q_m} f\,dy$$

(the equality here has been written on the basis of Fubini's theorem as applied to a measurable and bounded, and hence integrable over Q_{m+n}, function $f_k(x, y)$). The fact that $f(x, y) \in \Lambda(Q_{m+n})$ now follows from Levi's theorem.

12. Integrals of the Potential Type. Let a function $\rho(x)$ be measurable and bounded a.e. in Q, $|\rho(x)| \leqslant M$ a.e. Then for every $x \in R_n$ the function $u(x) = \int\limits_Q \frac{\rho(y)\,dy}{|x-y|^\alpha}$, $\alpha < n$, is defined and is known as *integral of the potential type.*

Let us show that $u(x) \in C(R_n)$. For $\alpha \leqslant 0$ it is obvious, so let $\alpha > 0$. We first note that for any points x^0 and x and any $\delta > 0$ we have the inequality

$$\begin{aligned} |u(x^0)-u(x)| &\leqslant \int\limits_Q |\rho(y)| \left|\frac{1}{|x^0-y|^\alpha}-\frac{1}{|x-y|^\alpha}\right| dy \\ &\leqslant M \int\limits_{|x^0-y|<\delta} \left(\frac{1}{|x^0-y|^\alpha}+\frac{1}{|x-y|^\alpha}\right) dy \\ &+ M \int\limits_{Q\cap\{|x^0-y|\geqslant\delta\}} \left|\frac{1}{|x^0-y|^\alpha}-\frac{1}{|x-y|^\alpha}\right| dy. \qquad (17) \end{aligned}$$

Fix x^0 and take an arbitrary $\varepsilon > 0$. Since for $\alpha \geqslant 0$ and $x \neq x^0$

$$\inf_{y\in\{|x-y|<\delta\}\cap\{|x^0-y|>\delta\}} \frac{1}{|x-y|^\alpha} = \frac{1}{\delta^\alpha} \geqslant \sup_{y\in\{|x-y|>\delta\}\cap\{|x^0-y|<\delta\}} \frac{1}{|x-y|^\alpha},$$

and $\operatorname{mes}\{(|x-y|<\delta) \cap (|x^0-y|>\delta)\} = \operatorname{mes}\{(|x-y|>\delta) \cap \cap (|x^0-y|<\delta)\}$, it follows that

$$\int\limits_{|x^0-y|<\delta} \frac{dy}{|x-y|^\alpha} \leqslant \int\limits_{|x^0-y|<\delta} \frac{dy}{|x^0-y|^\alpha}.$$

Hence

$$\int\limits_{|x^0-y|<\delta} \left(\frac{1}{|x^0-y|^\alpha}+\frac{1}{|x-y|^\alpha}\right) dy \leqslant 2 \int\limits_{|x^0-y|<\delta} \frac{dy}{|x^0-y|^\alpha} = \text{const}\,\delta^{n-\alpha}.$$

Accordingly, we can find (and fix) a $\delta > 0$ so that the first term on the right-hand side of (17) is less than $\varepsilon/2$.

The function $F(x, y) = \left|\frac{1}{|x^0-y|^\alpha}-\frac{1}{|x-y|^\alpha}\right|$ is continuous on the closed set $\Omega = \left\{|x-x^0| \leqslant \frac{\delta}{2},\ y \in \overline{Q} \cap (|y-x^0| \geqslant \delta)\right\}$ and $F(x, y)|_{x=x^0} = 0$. Therefore an η, $0 < \eta < \delta/2$, can be found such that $F(x, y) < \frac{\varepsilon}{2M|Q|}$ whenever $|x-x^0| < \eta$ for all $y \in \overline{Q} \cap \cap (|y-x^0| \geqslant \delta)$. Accordingly, if $|x-x^0| < \eta$, the second term in (17) also does not exceed $\varepsilon/2$. Then, for $|x-x^0| < \eta$, $|u(x^0)-u(x)| < \varepsilon$, that is, $u(x)$ is continuous.

Now assume $n - \alpha > 1$. We shall demonstrate that in this case $u(x)$ is continuously differentiable in R_n and that

$$u_{x_i}(x) = \int\limits_Q \rho(y) \frac{\partial}{\partial x_i} \left(\frac{1}{|x-y|^\alpha} \right) dy = \alpha \int\limits_Q \rho(y) \frac{y_i - x_i}{|x-y|^{\alpha+2}} \, dy.$$

Since $\left| \frac{y_i - x_i}{|x-y|^{\alpha+2}} \right| \leqslant \frac{1}{|x-y|^{\alpha+1}}$, $i = 1, \ldots, n$, arguing in the same way as above for $u(x)$, we conclude that the functions

$$u_i(x) = \alpha \int\limits_Q \rho(y) \frac{y_i - x_i}{|x-y|^{\alpha+2}} \, dy, \qquad i = 1, \ldots, n,$$

are continuous in R_n. Further, by Fubini's theorem, for any i, $i = 1, \ldots, n$,

$$\int\limits_{x_i^0}^{x_i} u_i(x)\, dx_i = \alpha \int\limits_{x_i^0}^{x_i} dx_i \int\limits_Q \rho(y) \frac{y_i - x_i}{|x-y|^{\alpha+2}} \, dy$$

$$= \alpha \int\limits_Q \rho(y)\, dy \int\limits_{x_i^0}^{x_i} \frac{y_i - x_i}{|x-y|^{\alpha+2}} \, dx_i = \int\limits_Q \rho(y)\, dy \int\limits_{x_i^0}^{x_i} \frac{\partial}{\partial x_i} \left(\frac{1}{|x-y|^\alpha} \right) dx_i$$

$$= u(x) - u(x_1, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots, x_n).$$

Therefore

$$u_i(x) = u_{x_i}(x), \qquad i = 1, \ldots, n,$$

establishing the assertion.

In exactly the same way, it can be shown that if $n - \alpha > s$, s is an integer, then $u(x)$ has continuous derivatives up to order s, and that

$$D^\alpha u(x) = \int\limits_Q \rho(y) D_x^\alpha \frac{1}{|x-y|^\alpha} \, dy$$

for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| \leqslant s$.

We remark that the function

$$u_1(x) = \int\limits_Q \rho(y) \ln |x-y| \, dy,$$

called *logarithmic potential*, is $(n - 1)$ times continuously differentiable in R_n and for all α, $|\alpha| \leqslant n - 1$,

$$D^\alpha u_1(x) = \int\limits_Q \rho(y) D_x^\alpha \ln |x-y| \, dy.$$

13. Lebesgue Integral of Complex-Valued Functions. Suppose that the function $f(x)$ defined in the region Q is complex-valued:

$$f(x) = \operatorname{Re} f(x) + i \operatorname{Im} f(x).$$

The function $f(x)$ is called measurable in Q if so are the functions $\operatorname{Re} f$ and $\operatorname{Im} f$, while $f(x)$ is integrable in Q if both $\operatorname{Re} f$ and $\operatorname{Im} f$ are integrable in Q. In this case the integral of $f(x)$ is defined by the relation

$$\int_Q f\,dx = \int_Q \operatorname{Re} f\,dx + i \int_Q \operatorname{Im} f\,dx.$$

Since $\frac{1}{2}(|\operatorname{Re} f| + |\operatorname{Im} f|) \leqslant |f| \leqslant |\operatorname{Re} f| + |\operatorname{Im} f|$, in order that the measurable function $f(x)$ be integrable it is necessary and sufficient that the function $|f(x)|$ be integrable.

14. Lebesgue Integral on an $(n-1)$-Dimensional Surface. Let S be an $(n-1)$-dimensional surface (of class C^1), and let S_m, $m = 1, 2, \ldots, N$, be a cover of S by simple pieces, $S = \bigcup_{m=1}^{N} S_m$ (see Chap. I, Introduction). Every simple piece S_m is described by the equation

$$x_p = \varphi_m(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n),$$
$$(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n) \in D_m, \quad \varphi_m \in C^1(\bar{D}_m) \qquad (18)$$

(D_m, the projection of S_m onto the coordinate plane $x_p = 0$, $1 \leqslant p = p(m) \leqslant n$, is an $(n-1)$-dimensional region having boundary of class C^1).

Formula (18) provides a one-to-one correspondence between the points $(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n)$ of the set $\bar{D}_m$ and the points $(x_1, \ldots, x_{p-1}, x_p, \ldots, x_n)$ of $\bar{S}_m$: with every point $(x_1, \ldots, x_{p-1}, \varphi_m(x_1, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n), x_{p+1}, \ldots, x_n) \in \bar{S}_m$ is associated a point $(x_1, x_2, \ldots, x_{p-1}, x_{p+1}, \ldots, x_n) \in \bar{D}_m$ (its projection onto the plane $x_p = 0$).

Suppose that the set E is contained in $\bar{S}_m$ for some m, $m = 1, \ldots, N$. Let $\mathscr{E}$ denote its original in $\bar{D}_m$ under the above mapping. We shall say that E is a set of surface measure zero if $\mathscr{E}$ is a set of $(n-1)$-dimensional measure zero.

A set E belonging to S is called a set of *surfaces of measure zero* if each of the sets $E \cap S_m$, $m = 1, \ldots, N$, is a set of surface measure zero.

It can be easily shown that the property of the set $E \subset S$ being a set of surface measure zero is independent of the choice of the cover $S_1, \ldots, S_N$ of the surface S.

The notion of a set of measure zero permits us to introduce, in complete analogy with the case of n-dimensional region (see Subsecs. 1-4), the concept of a.e. convergence on S and the related concepts of measurable and Lebesgue-integrable functions on S.

A function defined on S is called *measurable* (on S) if it is the limit of a sequence of functions in $C(\bar{S})$ that converges a.e. on S.

The nonnegative function $f(x)$ defined on S is said to be (Lebesgue) *integrable* on S if it is the limit of a monotone nondecreasing sequence, converging a.e. on S, of functions $f_k(x)$, $k = 1, 2, \ldots$, continuous on $\bar{S}$ for which the sequence of surface (Riemann) integrals is bounded above: $\int_S f_k(x)\,dS \leqslant C, k = 1, 2, \ldots$. The (Lebesgue) surface integral of $f(x)$ is defined by

$$\int_S f\,dS = \sup_k \int_S f_k(x)\,dS = \lim_{k\to\infty} \int_S f_k(x)\,dS.$$

A real-valued function $f(x)$ defined on S is said to be *Lebesgue-integrable* over S if the nonnegative functions $f^+(x) = \max(f(x), 0)$ and $f^-(x) = \max(-f(x), 0)$ are Lebesgue-integrable; in this case

$$\int_S f\,dS = \int_S f^+\,dS - \int_S f^-\,dS.$$

Let the function f be defined on a surface S, and let $S_1, \ldots, S_N$ be a cover of S by simple pieces. Suppose that $f^m(x_1, \ldots, x_{p(m)-1}, x_{p(m)+1}, \ldots, x_n)$ defined on D_m denotes the function $f(x_1, \ldots \ldots, x_{p(m)-1}, \varphi_m(x_1, \ldots, x_{p(m)-1}, x_{p(m)+1}, \ldots, x_n), x_{p(m)+1}, \ldots \ldots, x_n)$.

We shall show that the *function f is measurable if and only if all the functions f^m, $m = 1, \ldots, N$, are measurable; f is integrable over S if and only if each of the functions f^m is integrable over D_m, $m = 1, \ldots, N$; furthermore*

$$\int_S f\,dS = \sum_{m=1}^{N} \int_{D'_m} f^m \sqrt{1+|\nabla\varphi_m|^2}\,dx_1 \ldots dx_{p(m)-1}\,dx_{p(m)+1}\,dx_n, \quad (19)$$

where $D'_1 = D_1$, and D'_m, $m > 1$, is the projection of $S_m \setminus \bigcup_{i=1}^{m-1} \bar{S}_i$ onto the plane $x_{p(m)} = 0$.

If f is measurable (integrable), then, clearly, so is each of the functions f^m, $m = 1, \ldots, N$.

Let us show that if all the functions f^m, $m = 1, \ldots, N$, are integrable, then so is f (assertion regarding measurability is established similarly); this we shall do assuming (this is no loss of generality) that f, and hence, all f^m, $m = 1, \ldots, N$, are nonnegative a.e. (on S and correspondingly on D_m).

For each m, $m = 1, \ldots, N$, take a monotone nondecreasing sequence f_k^m, $k = 1, 2, \ldots$, of nonnegative functions in $C(\bar{D}_m)$ converging a.e. (in D_m) to the function f^m. We examine the sequence Π_k, $k = 1, 2, \ldots$, of decompositions of an $(n-1)$-dimensional cube K containing D_m: Π_1 is decomposition of K into 2^{n-1} equal subcubes with sides equal to half that of K, the decomposition Π_2 is obtained by taking finer divisions of Π_1 in which every cell (cube) of decomposition Π_1 is divided into 2^{n-1} equal subcubes, and so on. For all $m = 1, \ldots, N$, $k = 1, 2, \ldots$, we denote by $D'_{m,\,k}$ a closed set composed of (a finite number of) closures of all the cells of decomposition Π_k contained in D'_m, and by $\tilde{f}_k^m$ a function continuous in $\bar{D}_m$ which vanishes in $D_m \setminus D'_{m,\,k}$ and in $D'_{m,\,k}$ equals the function $f_k^m \cdot \zeta(k \cdot r_{m,\,k})$, where $\zeta(t) = 0$ for $0 < t < 1/2$, $\zeta(t) = 2t - 1$ for $1/2 \leqslant t \leqslant 1$, $\zeta(t) = 1$ for $t > 1$, and $r_{m,\,k}$ is the distance between the point $(x_1, \ldots, x_{p(m)-1}, x_{p(m)+1}, \ldots, x_n) \in D'_{m,\,k}$ and the boundary of $D'_{m,\,k}$. Clearly, for all m, $m = 1, \ldots, N$, the sequence $\tilde{f}_k^m$, $k = 1, 2, \ldots$, is monotone nondecreasing a.e. in D'_m and converges to f^m.

We define a function $\tilde{\tilde{f}}{}_k^m$, $m = 1, \ldots, N$, $k = 1, 2, \ldots$, continuous on $\bar{S}$ as follows:

$$\text{for } x \in S_m \qquad \tilde{\tilde{f}}{}_k^m = \tilde{f}_k^m(x_1, \ldots, x_{p(m)-1}, x_{p(m)+1}, \ldots, x_n),$$

$$\text{for } x \in S \setminus S_m \quad \tilde{\tilde{f}}{}_k^m = 0;$$

and put $f_k = \sum_{m=1}^{N} \tilde{\tilde{f}}{}_k^m$, $k = 1, 2, \ldots$. Clearly, each of the functions f_k, $k = 1, 2, \ldots$, is continuous on $\bar{S}$ and

$$\int_S f_k\, dS = \sum_{m=1}^{N} \int_S \tilde{\tilde{f}}{}_k^m\, dS = \sum_{m=1}^{N} \int_{S_m} \tilde{\tilde{f}}{}_k^m\, dS$$

$$= \sum_{m=1}^{N} \int_{D_m} \tilde{f}_k^m \sqrt{1 + |\nabla \varphi_m|^2}\, dx_1 \ldots dx_{p(m)-1}\, dx_{p(m)+1} \ldots dx_n$$

$$= \sum_{m=1}^{N} \int_{D'_m} \tilde{f}_k^m \sqrt{1 + |\nabla \varphi_m|^2}\, dx_1 \ldots dx_{p(m)-1}\, dx_{p(m)+1} \ldots dx_n. \qquad (20)$$

Furthermore, as $k \to \infty$, $f_k \uparrow f$ a.e. on S. Since the function $f^m \sqrt{1 + |\nabla \varphi_m|^2}$, $m = 1, \ldots, N$, is integrable on D'_m, (20) implies that the sequence $\int_S f_k\, dS$, $k = 1, 2, \ldots$, is bounded. Consequently,

f is integrable over S. Passing to the limit, as $k \to \infty$, in (20), we obtain (19).

From what has been just proved, it readily follows that all the properties that were established earlier for an n-dimensional region also hold for functions measurable and continuous over S.

§ 2. NORMED LINEAR SPACES. HILBERT SPACE

1. Linear Spaces. A set $\mathcal{F}$ is called a *linear space* if on its elements the operations of addition and multiplication by real (complex) numbers are defined and the resulting elements also belong to $\mathcal{F}$ and have the following properties:

(a) $f_1 + f_2 = f_2 + f_1$,
(b) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$,
(c) there is an element o in $\mathcal{F}$ such that $0 \cdot f = o$ for any $f \in \mathcal{F}$,
(d) $(c_1 + c_2) f = c_1 f + c_2 f$,
(e) $c (f_1 + f_2) = cf_1 + cf_2$,
(f) $(c_1 c_2) f = c_1 (c_2 f)$,
(g) $1 \cdot f = f$

for any f, f_1, . . . $\in \mathcal{F}$ and any real (complex) numbers c, c_1, ...

Depending on whether the numbers by which the elements of $\mathcal{F}$ are multiplied are real or complex, the space $\mathcal{F}$ is designated as a *real* or a *complex linear space*. For definiteness, in this chapter we shall consider only complex linear spaces. All the definitions and results are easily carried over to the case of a real linear space.

A subset of the linear space $\mathcal{F}$ which itself constitutes a linear space is called a *linear manifold* in $\mathcal{F}$.

Let f_m, $m = 1, 2, \ldots$, be a countable (or finite) system of elements of the linear space $\mathcal{F}$. A set composed of elements of the form $c_1 f_1 + \ldots + c_k f_k$ for all possible k and arbitrary complex $c_1, \ldots, c_k$ is a linear manifold in $\mathcal{F}$ and is called *linear manifold spanned by the elements* f_k, $k = 1, 2, \ldots$. The elements $f_1, \ldots, f_m$ of $\mathcal{F}$ are *linearly independent* if $c_1 f_1 + \ldots + c_m f_m = o$ holds only for $c_1 = \ldots = c_m = 0$; otherwise $f_1, \ldots, f_m$ are *linearly dependent.* When the set of elements of $\mathcal{F}$ is infinite, it is linearly independent if every finite subset of it is linearly independent.

A linear manifold is *finite-dimensional* (*n-dimensional*) if it has n linearly independent elements and the aggregate of any of its $(n + 1)$ elements is linearly dependent.

The linear manifold spanned by linearly independent elements f_k, $k = 1, \ldots, n$, of $\mathcal{F}$ is n-dimensional.

A linear manifold is called *infinite-dimensional* if one can find in it a linearly independent subset composed of infinite number of elements.

2. Normed Linear Spaces. The space $\mathcal{F}$ is a *normed linear space* if to each element f of this space we can assign the quantity $\| f \| = \| f \|_{\mathcal{F}}$ (*norm of* f) having the following properties:

(a) $\| cf \| = | c | \, \| f \|$ for any complex c and any $f \in \mathcal{F}$,

(b) $\| f_1 + f_2 \| \leqslant \| f_1 \| + \| f_2 \|$ for any $f_i \in \mathcal{F}$, $i = 1, 2$ (the triangle inequality),

(c) $\| f \| \geqslant 0$, where $\| f \| = 0$ if and only if $f = o$.

In a normed linear space one can introduce the notion of *distance* $\| f_1 - f_2 \|$ between two elements f_1 and f_2 as well as that of convergence.

The sequence f_m, $m = 1, 2, \ldots$, of elements of $\mathcal{F}$ is called *fundamental* if $\| f_k - f_m \| \to 0$ as $k, m \to \infty$.

The sequence f_m, $m = 1, 2, \ldots$, of elements of $\mathcal{F}$ is said to *converge to* $f \in \mathcal{F}$ ($f_m \to f$ as $m \to \infty$, or $\lim\limits_{m \to \infty} f_m = f$) if $\| f_m - f \| \to 0$ as $m \to \infty$.

A sequence cannot converge to two different elements, for if $\| f_m - f \| \to 0$ and $\| f_m - g \| \to 0$ as $m \to \infty$, then $\| f - g \| = \| f - f_m + f_m - g \| \leqslant \| f_m - f \| + \| f_m - g \| \to 0$ as $m \to \infty$, that is, $\| f - g \| = 0$, implying $f = g$.

If $f_m \to f$, then $\| f_m \| \to \| f \|$ (*continuity of the norm*). Indeed, by the triangle inequality, $\| f_m \| \leqslant \| f_m - f \| + \| f \|$ and $\| f \| \leqslant \| f_m - f \| + \| f_m \|$. Consequently, $| \, \| f_m \| - \| f \| \, | \leqslant \| f_m - f \| \to 0$ as $m \to \infty$.

If the sequence converges ($f_m \to f$), then it is fundamental, since

$$\| f_k - f_m \| = \| f_k - f + f - f_m \| \leqslant \| f_k - f \| + \\ + \| f - f_m \| \to 0 \text{ as } k, m \to \infty.$$

The converse, generally speaking, does not hold.

A normed linear space is called *complete* if corresponding to every fundamental sequence of its elements there is an element of this space to which this sequence converges.

A complete normed linear space B is known as *Banach space*.

A linear manifold in a Banach space B that is complete in the norm of B (and hence is itself a Banach space with the same norm) is called a *subspace* of B. The linear manifold spanned by a finite number of elements of B is a subspace of B.

If $\mathcal{M}$ is a linear manifold in B, then the set $\overline{\mathcal{M}}$ obtained by adding to $\mathcal{M}$ all those elements which are limits of all the fundamental sequences of elements of $\mathcal{M}$ (in B every fundamental sequence has a limit) is known as the *closure* (in B) of manifold $\mathcal{M}$.

The closure $\overline{\mathcal{M}}$ of a linear manifold $\mathcal{M}$ is clearly a linear manifold. Let us show that it is closed. To this end, let f_k, $k = 1, 2, \ldots$, be a fundamental sequence of elements of $\overline{\mathcal{M}}$, and put $f = \lim\limits_{k \to \infty} f_k$.

We show that $f \in \overline{\mathcal{M}}$. By the definition of $\overline{\mathcal{M}}$, for any $k = 1, 2, \ldots$, there is an element $f'_k \in \mathcal{M}$ such that $\| f'_k - f_k \| < 1/k$, therefore

$$\| f - f'_k \| = \| f - f_k + f_k - f'_k \| \leqslant \| f - f_k \| + 1/k \to 0$$

as $k \to \infty$, that is $f = \lim\limits_{k\to\infty} f'_k$, implying $f \in \overline{\mathcal{M}}$.

Thus the *closure of a linear manifold in B is a subspace.*

The closure of a linear manifold spanned by the elements f_k, $k = 1, 2, \ldots$, is called a *subspace spanned by these elements.*

A set $\mathcal{M}' \subset B$ is *bounded* if there is a constant C such that $\|f\| \leqslant C$ for all $f \in \mathcal{M}'$.

The set $\mathcal{M}' \subset B$ is said to be *everywhere dense in B* if for any $f \in B$ there is a sequence f'_k, $k = 1, 2, \ldots$, of elements of $\mathcal{M}'$ that converges to f.

A Banach space B is *separable* if it contains an everywhere dense countable set.

3. The Scalar Product. Hilbert Space. We say that a *scalar product* is introduced in a linear space H if with every pair of elements h_1, $h_2 \in E$ there is associated a complex quantity (h_1, h_2) (the scalar product of these elements) with the following properties:

(a) $(h_1, h_2) = \overline{(h_2, h_1)}$ (in particular, (h, h) is a real number),

(b) $(h_1 + h_2, h) = (h_1, h) + (h_2, h)$,

(c) for any complex c $(ch_1, h_2) = c\,(h_1, h_2)$,

(d) $(h, h) \geqslant 0$, where $(h, h) = 0$ if and only if $h = o$.

Let us establish the important inequality, known as *Bunyakovskii's inequality*,

$$| (h_1, h_2) |^2 \leqslant (h_1, h_1)\cdot(h_2, h_2), \qquad (1)$$

which holds for arbitrary h_1, h_2 in H. When $h_2 = o$, the inequality (1) is obvious, so let $h_2 \neq o$. For any complex t $0 \leqslant (h_1 + th_2, h_1 + th_2) = (h_1, h_1) + t\,\overline{(h_1, h_2)} + \bar{t}\,(h_1, h_2) + |t|^2\,(h_2, h_2)$. If we put $t = -\frac{(h_1, h_2)}{(h_2, h_2)}$, this inequality becomes $(h_1, h_1) - \frac{|(h_1, h_2)|^2}{(h_2, h_2)} \geqslant 0$, equivalent to (1).

The scalar product generates a norm $\| h \| = \sqrt{(h, h)}$ in H. Properties (a) and (c) of a norm are apparently satisfied. To show that (b) (the triangle inequality) is also satisfied, we make use of the Bunyakovskii's inequality

$$\begin{aligned} \| h_1 + h_2 \|^2 &= \| h_1 \|^2 + (h_1, h_2) + (h_2, h_1) + \| h_2 \|^2 \\ &\leqslant \| h_1 \|^2 + 2\,\| h_1 \|\,\| h_2 \| + \| h_2 \|^2 = (\| h_1 \| + \| h_2 \|)^2. \end{aligned}$$

A linear space with a scalar product that is complete in the norm generated by this scalar product (that is, is a Banach space in this norm) is called a *Hilbert space.*

Apart from the convergence (in norm), it proves convenient to introduce one more type of convergence in a Hilbert space. A sequence h_m, $m = 1, 2, \ldots$, in H is said to *converge weakly* to an element $h \in H$ if $\lim\limits_{m\to\infty} (h_m, f) = (h, f)$ for any $f \in H$.

Let us show that a sequence cannot converge weakly to different elements of H. Assume that there are two elements $h, h' \in H$ such that $\lim\limits_{m\to\infty} (h_m, f) = (h, f)$ and $\lim\limits_{m\to\infty} (h_m, f) = (h', f)$ for any $f \in H$. Then for all $f \in H$ $(h - h', f) = 0$ and, in particular, with $f = h - h'$ we have $(h - h', h - h') = 0$, implying $h = h'$.

If a sequence $h_m \in H$, $m = 1, 2, \ldots$, converges to $h \in H$, then it converges to it weakly as well. Indeed,

$$|(h_m, f) - (h, f)| = |(h_m - h, f)| \leqslant \|h_m - h\| \, \|f\| \to 0 \text{ as } m \to \infty.$$

4. Hermitian Bilinear Forms and Equivalent Scalar Products. A *Hermitian bilinear form* W is said to be defined on a Hilbert space H if with every pair of elements $h_1, h_2 \in H$ there is associated a complex number $W(h_1, h_2)$ with the following properties:

(a) $W(h_1 + h_2, h) = W(h_1, h) + W(h_2, h)$,

(b) $W(ch_1, h_2) = cW(h_1, h_2)$,

(c) $W(h_1, h_2) = \overline{W(h_2, h_1)}$

for arbitrary $h, h_1, h_2 \in H$ and arbitrary complex c.

The function $W(h, h)$ defined on H is called *quadratic form* corresponding to the Hermitian bilinear form $W(h_1, h_2)$. By Property (c), the quadratic form corresponding to a Hermitian bilinear form is real-valued.

An example of Hermitian bilinear form defined on H is the scalar product and the corresponding quadratic form is the square of the norm generated by the scalar product.

If a quadratic form corresponding to a Hermitian bilinear form has the property that $W(h, h) \geqslant 0$ for all $h \in H$ and $W(h, h) = 0$ for $h = o$ only, then the bilinear form $W(h_1, h_2)$ can be taken as the (new) scalar product in H: $W(h_1, h_2) = (h_1, h_2)'$, and the resulting (new) norm is defined by $\|h'\| = \sqrt{W(h, h)}$.

The norm $\| \ \|'$ is said to be *equivalent* to the norm $\| \ \|$ if there exist constants $C_1 > 0$, $C_2 > 0$ such that $\|h\|' \leqslant C_1 \|h\|$, $\|h\| \leqslant C_2 \|h\|'$ for any element $h \in H$. Two scalar products $(\ ,\ )$ and $(\ ,\ )'$ are said to be equivalent if so are the norms generated by them.

If the norm $\| \ \|'$ is equivalent to the norm $\| \ \|$, then the set H is a Hilbert (that is, a complete) space also with respect to the scalar product $(\ ,\ )'$.

In fact, let the sequence h_k, $k = 1, 2, \ldots$, of elements of H be fundamental in the norm $\| \ \|'$: $\|h_k - h_s\|' \to 0$ as $k, s \to \infty$; this sequence is also fundamental in the norm $\| \ \|$, since $\|h_k - h_s\| \leqslant C_2 \|h_k - h_s\|'$. Because H is complete in the norm $\| \ \|$, there exists an element $h \in H$ to which the sequence in question converges: $\| h_k - h \| \to 0$ as $k \to \infty$. But the sequence converges to h in the norm $\| \quad \|'$ also, since $\| h_k - h \|' \leqslant C_1 \| h_k - h \|$, and the conclusion follows.

5. Orthogonality. Orthonormal Systems. Two elements $h_1, h_2 \in H$ are *orthogonal* $(h_1 \perp h_2)$ if $(h_1, h_2) = 0$. An element h is said to be *orthogonal to a set* $H' \subset H$ if $(h, h') = 0$ for all $h' \in H'$. Two sets H' and H'' in H are orthogonal $(H' \perp H'')$ if $(h', h'') = 0$ for all $h' \in H'$, $h'' \in H''$.

If $h \in H$ is orthogonal to a set H' that is everywhere dense in H, then $h = o$. Indeed, let h'_k, $k = 1, 2, \ldots$, be a sequence of elements of H' and $h'_k \to h$ as $k \to \infty$. Since $(h'_k, h) = 0$ for all $k \geqslant 1$ and by weak convergence $(h'_k, h) \to \| h \|^2$, it follows that $\| h \| = 0$, implying $h = o$.

An element $h \in H$ is *normalized* if $\| h \| = 1$, and a set $H' \subset H$ is called *orthonormal* (*orthonormal system*) if its elements are normalized and are mutually orthogonal. An orthonormal set is, obviously, linearly independent.

A countably infinite (or finite) linearly independent set of elements h_k, $k = 1, 2, \ldots$, can be transformed into a countably infinite (or finite) orthonormal set in the following manner (Gramm-Schmidt's method):

$$e_1 = \frac{h_1}{\| h_1 \|}, \quad e_2 = \frac{h_2 - (h_2, e_1) e_1}{\| h_2 - (h_2, e_1) e_1 \|}, \ldots,$$

$$e_n = \frac{h_n - (h_n, e_1) e_1 - \ldots - (h_n, e_{n-1}) e_{n-1}}{\| h_n - (h_n, e_1) e_1 - \ldots - (h_n, e_{n-1}) e_{n-1} \|}, \ldots,$$

(according to the supposition that the set h_k, $k = 1, 2, \ldots$, is linearly independent, $h_n - (h_n, e_1) e_1 - \ldots - (h_n, e_{n-1}) e_{n-1} \neq o$ for any $n \geqslant 2$).

6. Fourier Series with Respect to an Arbitrary Orthonormal System. Suppose that f is an arbitrary element of H and $e_1, \ldots, e_n, \ldots$ a countable orthonormal system in H (if H is finite-dimensional, one must take an orthonormal system consisting of a finite number of elements). Denoting by H_p, for some $p \geqslant 1$, the subspace spanned by the elements $e_1, \ldots, e_p$, we try to find in H_p an element which is closest (in the norm of H) to the element f. Since any element of H_p is of the form $\sum\limits_{r=1}^{p} c_r e_r$ with certain constants $c_1, \ldots, c_p$, the problem reduces to determining constants $c_1, \ldots, c_p$ such that the quantity $\delta^2_{H_p}(f; c_1, \ldots, c_p) = \| f - \sum\limits_{r=1}^{p} c_r e_r \|^2$ attains its minimum.

The quantities $f_k = (f, e_k)$, $k = 1, 2, \ldots$, are called *Fourier coefficients* of f with respect to the system $e_1, e_2, \ldots$. Since

$$\delta^2_{H_p}(f;\ c_1, \ldots, c_p)\Big(f-\sum_{r=1}^{p} c_r e_r,\ f-\sum_{r=1}^{p} c_r e_r\Big)$$

$$=\|f\|^2-\sum_{r=1}^{p} c_r\bar{f}_r-\sum_{r=1}^{p} \bar{c}_r f_r+\sum_{r=1}^{p} |c_r|^2$$

$$=\sum_{r=1}^{p} |c_r-f_r|^2-\sum_{r=1}^{p} |f_r|^2+\|f\|^2,$$

the quantity $\delta^2_{H_p}(f;\ c_1, \ldots, c_p)$ attains its minimum only when $c_r=f_r$, $r=1, \ldots, p$, and this minimum, denoted by $\delta^2_{H_p}(f)$, is equal to $\|f\|^2-\sum_{r=1}^{p} |f_r|^2$:

$$\delta^2_{H_p}(f)=\|f\|^2-\sum_{r=1}^{p} |f_r|^2. \tag{2}$$

Thus, for a given f, we have the inequality

$$\Big\|f-\sum_{r=1}^{p} c_r e_r\Big\| \geqslant \Big\|f-\sum_{r=1}^{p} f_r e_r\Big\|$$

for all $c_1, \ldots, c_p$, describing the *minimal property* of Fourier coefficients; the equality is attained only for $c_r = f_r$, $r = 1, \ldots, p$.

Denoting by f^p the unique element closest to f in the subspace H_p:

$$f^p=\sum_{r=1}^{p} f_r e_r,$$

we have

$$\|f-f^p\|^2=\delta^2_{H_p}(f). \tag{3}$$

The element f^p is called the projection of f onto the subspace H_p.

Equality (2) implies that for any $f \in H$ and any $p \geqslant 1$, $\sum_{r=1}^{p} |f_r|^2 \leqslant \|f\|^2$. Accordingly, the number sequence $\sum_{r=1}^{\infty} |f_r|^2$ converges and *Bessel's inequality*

$$\sum_{r=1}^{\infty} |f_r|^2 \leqslant \|f\|^2$$

holds.

Lemma 1. *Let f_k, $k=1, 2, \ldots$, be a sequence of complex numbers and e_k, $k=1, 2, \ldots$, an orthonormal system in H. In order that the series $\sum_{k=1}^{\infty} f_k e_k$ may converge in the norm of H it is necessary and sufficient that the number series $\sum_{r=1}^{\infty} |f_r|^2$ converge.*

Proof. Let $S_p = \sum_{r=1}^{p} f_r e_r$ be the partial sum of the series $\sum_{r=1}^{\infty} f_r e_r$. For $p > q$ we have the equality $\| S_p - S_q \|^2 = \| \sum_{q+1}^{p} f_r e_r \|^2 = \sum_{q+1}^{p} | f_r |^2$ which implies that convergence of $\sum | f_r |^2$ is necessary as well as sufficient for the sequence of its partial sums to be fundamental, and hence for convergence of series in question, since H is complete. ▮

Let f be any element of H and f_k, $k = 1, 2, \ldots$, its Fourier coefficients with respect to the orthonormal system e_k, $k = 1, 2, \ldots$. The series

$$\sum_{k=1}^{\infty} f_k e_k$$

is called Fourier series of f with respect to the system e_k, $k = 1, 2, \ldots$.

Lemma 1 and Bessel's inequality yield the following result.

Lemma 2. *The Fourier series of any element $f \in H$ with respect to an arbitrary orthonormal system converges in the norm of H.*

Lemma 2 establishes the existence of an element $\tilde{f} \in H$ to which the Fourier series of f converges. A natural question can be asked: Is $\tilde{f} = f$ for all $f \in H$?

In the general case, unless additional conditions are imposed on the system $e_1, e_2, \ldots$ besides its orthonormality, the answer to above question is in negative.

7. Orthonormal Basis. It follows from (2) that for any $f \in H$ the quantity $\delta^2_{H_p}(f)$ decreases when p increases. Therefore a priori there are two cases to be examined:

(a) for all $f \in H$ $\delta^2_{H_p}(f) \to 0$ as $p \to \infty$,

(b) there is an element $f \in H$ for which $\delta^2_{H_p}(f) \to c > 0$ as $p \to \infty$.

When (a) holds, for any $f \in H$ we have, by (3),

$$f = \lim_{p \to \infty} \sum_{k=1}^{p} f_k e_k$$

or, which is the same

$$f = \sum_{k=1}^{\infty} f_k e_k, \tag{4}$$

that is, in case (a) the Fourier series of an element f converges (in the metric of H) to f. Furthermore, for any $f \in H$ we have

$$\| f \|^2 = \sum_{k=1}^{\infty} | f_k |^2, \tag{5}$$

known as the *Parseval-Steklov equality*, and its generalization

$$(f, g) = \sum_{k=1}^{\infty} f_k \bar{g}_k, \tag{5'}$$

true for all $f, g \in H$.

Equality (5) is a consequence of (2). To establish (5′) we first note that the series on the right side converges absolutely, because its general term has a majorant which is the general term of a convergent series: $|f_k \bar{g}_k| \leqslant \frac{1}{2}(|f_k|^2 + |g_k|^2)$. Further, by (4)

$$(f, g) = \lim_{p\to\infty} (f^p, g) = \lim_{p\to\infty} \Big(\sum_{k=1}^{p} f_k e_k, g\Big)$$
$$= \lim_{p\to\infty} \sum_{k=1}^{p} f_k (e_k, g) = \lim_{p\to\infty} \sum_{k=1}^{p} f_k \bar{g}_k = \sum_{k=1}^{\infty} f_k \bar{g}_k,$$

as desired.

In case (b), there is an element $f \in H$ whose Fourier series converges (by Lemma 2 of the preceding Subsection) to $\tilde{f} \neq f$, that is, the element $h = f - \tilde{f} \neq o$. Hence

$$f = h + \sum_{k=1}^{\infty} f_k e_k,$$

where $h \neq o$ and h is orthogonal to the subspace spanned by the system $e_1, e_2, \ldots$.

We again return to case (a) in which we are basically interested.

A countable orthonormal system $e_1, e_2, \ldots$ is called *complete* or an *orthonormal basis* for the space H if any element $f \in H$ can be expanded in a Fourier series (4) with respect to this system.

The aforementioned discussion leads to the following assertion.

Lemma 3. *For an orthonormal system $e_1, e_2, \ldots$ to be an orthonormal basis for H it is necessary and sufficient that the Parseval-Steklov equality* (5) *hold for any element $f \in H$ or* (5′) *hold for any two elements f and g of H.*

Lemma 4. *For an orthonormal system $e_1, e_2, \ldots$ to be an orthonormal basis for H it is necessary and sufficient that the linear manifold spanned by this system constitute an everywhere dense set in H.*

If the system $e_1, e_2, \ldots$ is an orthonormal basis for H, then every element $f \in H$ is approximated in the norm of H as closely as one pleases by the partial sums of its Fourier series which are linear combinations of this system. This shows that the condition is necessary.

To show that the condition is sufficient, take an arbitrary element $f \in H$. For an $\varepsilon > 0$ we can find a number $p = p(\varepsilon)$ and numbers $c_1(\varepsilon), \ldots, c_p(\varepsilon)$ such that $\| f - \sum_{k=1}^{p} c_k(\varepsilon) e_k \| < \varepsilon$. Since the

Fourier coefficients have the property of being minimal, it follows that

$$\|f-\sum_{k=1}^{p} f_k e_k\|^2 \leqslant \|f-\sum_{k=1}^{p} c_k(\varepsilon) e_k\|^2 \leqslant \varepsilon^2,$$

which shows that f has a Fourier expansion (4). ▮

Theorem 1. *In a separable Hilbert space there is an orthonormal basis.*

Proof. Let h_1', h_2', . . . be a set everywhere dense in H. By h_1 denote the first nonvanishing element h'_{k_1} ($h'_1 = \ldots = h'_{k_1-1} = o$), by h_2 the first nonvanishing element of the set h'_{k_1+1}, h'_{k_1+2}, . . . that forms with h_1 a pair of linearly independent elements, and so on. The countable (or finite) system h_1, h_2, . . . is linearly independent and the linear combinations of the elements of this system are everywhere dense in H. We can transform the system h_1, h_2, . . . (Subsec. 5) into a countable orthonormal system of elements e_1, e_2, . . . whose linear combinations are also dense in H. By Lemma 4, this system is an orthonormal basis for H. ▮

§ 3. LINEAR OPERATORS. COMPACT SETS. COMPLETELY CONTINUOUS OPERATORS

1. Operators and Functionals. Let B_1 and B_2 be Banach spaces and B_1' a set lying in B_1. An *operator* A (*operator* A *from* B_1 *into* B_2) is said to be defined on B_1' if to every element $f \in B_1'$ there corresponds an element $g \in B_2 : g = Af$. The set B_1' is called the *domain of definition of* A and is denoted by D_A, $D_A = B_1'$, while the set of elements of the form Af, $f \in D_A$, is known as its *range* $R_A \subset B_2$.

The operator A is a functional if the space B_2 is a set of complex numbers (the modulus of a complex number is taken as a norm of this set). The functionals will commonly be denoted by l.

Simplest examples of operators are the operator O, the null operator, and (for $B_1 = B_2$) the identity operator I defined as follows: $Of = o$ for all $f \in D_O$, $If = f$ for all $f \in D_I$.

An operator A is said to be *continuous on an element* $f \in D_A$ if it maps a sequence f_k, $k = 1, 2, \ldots$, of elements of D_A converging to f in the norm B_1 into a sequence Af_k, $k = 1, 2, \ldots$, that converges to Af in the norm B_2. Operator A is *continuous on the set* $E \subset D_A$ (in particular, on D_A) if it is continuous on every element $f \in E$. An operator A that is continuous on D_A will be referred to as *continuous*.

The operator A is *linear* if D_A is a linear manifold and $A(c_1f_1 + c_2f_2) = c_1Af_1 + c_2Af_2$ for any elements $f_i \in D_A$ and numbers c_i, $i = 1, 2$.

The null element of B_1 is mapped by the linear operator A into the null element of B_2, because

$$Ao = A(0 \cdot f) = 0 \cdot Af = o$$

(f is any element of D_A).

For a linear operator A to be continuous it is necessary and sufficient that it be continuous on the null element (or, generally, on some element of D_A).

The necessity of this condition is apparent. To show that it is sufficient as well, let f_k, $k = 1, 2, \ldots$, be a sequence of elements of D_A that converges to $f \in D_A$. Since $g_k = f_k - f$, $k = 1, 2, \ldots$, is a sequence of elements of D_A converging to zero, it follows that $Ag_k \to o$ as $k \to \infty$, thereby implying $Af_k \to Af$ as $k \to \infty$. ▮

Let A_i, $i = 1, 2$, be linear operators from B_1 into B_2, $D_{A_1} = D_{A_2}$, and let c_i, $i = 1, 2$, be certain numbers. We define a new operator $A = c_1A_1 + c_2A_2$ as follows: for any $f \in D_A = D_{A_1} = D_{A_2}$, $Af = c_1A_1f + c_2A_2f$. Operator A is also linear.

Thus the operations of addition and multiplication by complex numbers are defined on the set of linear operators with common domains of definition. It is easily seen that this set is a linear space.

A linear operator A is said to be *bounded* if there is a constant $C > 0$ such that $\| Af \|_{B_2} \leqslant C \| f \|_{B_1}$ for all $f \in D_A$ or, what is the same, $\| Af \|_{B_2} \leqslant C$ for all $f \in D_A$ satisfying $\| f \|_{B_1} = 1$.

The exact upper bound of the values of C is known as the *norm* of A and is denoted by $\| A \|$.

We can show that

$$\| A \| = \sup_{f \in D_A} \frac{\| Af \|_{B_2}}{\| f \|_{B_1}} = \sup_{\substack{f \in D_A \\ \| f \|_{B_1} = 1}} \| Af \|_{B_2}. \tag{1}$$

Put $\alpha = \sup\limits_{f \in D_A} \| Af \|_{B_2} / \| f \|_{B_1}$. For all $f \in D_A$, $\| Af \|_{B_2} / \| f \|_{B_1} \leqslant \alpha$, therefore $\| A \| \leqslant \alpha$. To establish the reverse inequality, we note that, by the definition of exact upper bound, for every $\varepsilon > 0$ there is an element $f_\varepsilon \in D_A$ such that $\| Af_\varepsilon \|_{B_2} / \| f_\varepsilon \|_{B_1} \geqslant \alpha - \varepsilon$. This implies that $\| A \| \geqslant \alpha - \varepsilon$ for any $\varepsilon > 0$, that is, $\| A \| \geqslant \alpha$. Hence $\| A \| = \alpha$.

In particular, when A is a bounded linear functional, $A = l$, its norm is given by

$$\| l \| = \sup_{f \in D_l} \frac{| lf |}{\| f \|_{B_1}} = \sup_{\substack{f \in D_l \\ \| f \|_{B_1} = 1}} | lf |.$$

Note that the set of bounded linear operators with common domains of definition is a linear manifold in the space of all the linear operators with the same domains of definition. The norm defined above for a bounded linear operator satisfies all the axioms of being a norm. It can be easily shown that this normed space is complete (that is, it is a Banach space).

The following proposition establishes a connection between the notions of boundedness and continuity for linear operators.

For a linear operator A to be continuous it is necessary and sufficient that it be bounded.

Sufficiency. Let the sequence f_k, $k=1, 2, \ldots$, in D_A converge (in B_1) to $f \in D_A$. Since $\|Af_k - Af\|_{B_2} = \|A(f_k - f)\|_{B_2} \leqslant \|A\| \times \times \|f_k - f\|_{B_1} \to 0$ as $k \to \infty$, it follows that $Af_k \to Af$, as $k \to \infty$, in B_2.

Necessity. Assume the contrary, that is, A is unbounded. Then there is a sequence f'_k, $k = 1, 2, \ldots$, of elements of D_A such that $\|Af'_k\|_{B_2} > k\|f'_k\|_{B_1}$, but this contradicts the continuity of A, because the sequence $f_k = f'_k/(k\|f'_k\|_{B_1})$, $k = 1, 2, \ldots$, belonging to D_A converges in B_1 to zero, while the sequence Af_k, $k = 1, 2, \ldots$, cannot converge to $Ao = o$ since $\|Af_k\|_{B_2} \geqslant 1$. ∎

A bounded linear operator A whose domain of definition D_A is everywhere dense in B_1 can always be assumed to be defined on the whole of B_1, by redefining it on $B_1 \setminus D_A$ as follows. Let f be an element of $B_1 \setminus D_A$ and f_k, $k = 1, 2, \ldots$, a sequence of elements of D_A that converges to f in the norm of B_1 (D_A is everywhere dense in B_1). Since A is bounded, the sequence Af_k, $k = 1, 2, \ldots$, of elements of B_2 is fundamental in B_2, and because B_2 is complete, the sequence Af_k, $k = 1, 2, \ldots$, has a limit in B_2. We show that this limit is independent of the choice of the sequence f_k, $k = = 1, 2, \ldots$. In fact, let f'_k, $k = 1, 2, \ldots$, be another sequence of D_A converging to f. Then $\|Af'_k - Af_k\|_{B_2} = \|A(f'_k - f_k)\|_{B_2} = = \|A\| \, \|f'_k - f_k\|_{B_1} \to 0$ as $k \to \infty$; accordingly, the limit depends only on the element f. We take this as the value Af of A on f. The *extension* of A obtained in this manner, and referred to as the *extension with respect to continuity*, is a bounded linear operator defined on the whole of B_1.

If A_1, A_2 are linear operators for which $R_{A_2} \subset D_{A_1}$, then the linear operator A_1A_2 on D_{A_2} with range in R_{A_1} is defined as follows: $A_1A_2f = A_1(A_2f)$. If A_1 and A_2 are bounded, then so is A_1A_2 and $\|A_1A_2\| \leqslant \|A_1\| \, \|A_2\|$.

Suppose that the equation $Af = g$ has a unique solution $f \in D_A$ for every $g \in R_A$. This means that on R_A an operator, denoted by A^{-1}, is defined which with every $g \in R_A$ associates the unique $f \in D_A$ such that $Af = g$. The operator A^{-1} is called *inverse* of A. Clearly, $D_{A^{-1}} = R_A$, $R_{A^{-1}} = D_A$, $A^{-1}A = I$, $AA^{-1} = I$; and A^{-1} is linear provided so is A.

2. Riesz's Theorem. An example of a bounded linear functional defined on a Hilbert space H is provided by the scalar product: if we fix an arbitrary element $h \in H$, then (f, h) is (with respect to f)

a bounded linear functional (boundedness is a consequence of Bunyakovskii's inequality). The most striking feature is that any bounded linear functional defined on H (or, by the results of Subsec. 1, on a set everywhere dense in H) may be expressed as a scalar product by a proper choice of $h \in H$. Namely, the following important assertion holds.

Theorem 1 (F. Riesz). *For every bounded linear functional defined on a Hilbert space H there is a unique element $h \in H$ such that for all $f \in H$*

$$l(f) = (f, h). \tag{2}$$

We shall prove this theorem only for a separable Hilbert space H (only for such spaces will this theorem be used in this book).

Proof. Let $e_1, e_2, \ldots, e_n, \ldots$ be an orthonormal basis for H (such a basis exists, by Theorem 1, Sec. 2), and let $\sum\limits_{k=1}^{\infty} f_k e_k$ be the Fourier expansion of some $f \in H$. Since, as $p \to \infty$, $\sum\limits_{k=1}^{p} f_k e_k \to f$, by the continuity of l

$$l(f) = \lim_{p\to\infty} l\left(\sum_{k=1}^{p} f_k e_k\right) = \lim_{p\to\infty} \sum_{k=1}^{p} f_k l(e_k) = \sum_{k=1}^{\infty} f_k \bar{h}_k, \tag{3}$$

where $h_k = \overline{l(e_k)}$, $k = 1, 2, \ldots$.

Consider the element $h^p = \sum\limits_{k=1}^{p} h_k e_k$. Since $|l(h^p)| \leqslant \|l\| \, \|h^p\|$ (l is bounded) and $l(h^p) = \sum\limits_{k=1}^{p} h_k l(e_k) = \sum\limits_{k=1}^{p} |h_k|^2 = \|h^p\|^2$, we have for all $p \geqslant 1$ $\sum\limits_{k=1}^{p} |h_k|^2 \leqslant \|l\|^2$, which implies that the series $\sum\limits_{k=1}^{\infty} |h_k|^2$ converges and $\sum\limits_{k=1}^{\infty} |h_k|^2 \leqslant \|l\|^2$. By Lemma 1 (Sec. 2.6), the series $\sum\limits_{k=1}^{\infty} h_k e_k$ converges in the norm of H to an element $h \in H$ (h_k are Fourier coefficients of h).

Substituting $f_k = (f, e_k)$ into (3) and again making use of the continuity of l, we obtain (2): $l(f) = \sum\limits_{k=1}^{\infty} (f, h_k e_k) = \left(f, \sum\limits_{k=1}^{\infty} h_k e_k\right) = (f, h)$.

Apart from representation (2) if there is another representation for l: $l(f) = (f, h')$, then for all $f \in H$ $(f, h - h') = 0$, implying $h = h'$. ∎

Let us note that, when proving Theorem 1, we have established that $\| h \| \leqslant \| l \|$. The reverse inequality $\| l \| \leqslant \| h \|$ follows from (2) and the Bunyakovskii's inequality. Thus $\| l \| = \| h \|$.

3. Adjoint Operator. Suppose that H is a Hilbert space and A a linear operator from H into H which is defined on a set D_A everywhere dense in H (operator A is, generally speaking, not bounded).

Let D_{A^*} denote a set of elements of H satisfying the following condition: for any $g \in D_{A^*}$ there is an element $h \in H$ such that for all $f \in D_A$

$$(Af, g) = (f, h).$$

The set D_{A^*} is nonempty, because the null element of H belongs to it: when $g = o$, $h = o$.

We show that to every element $g \in D_{A^*}$ there corresponds only one $h \in H$. Assume the contrary, that is, to a $g \in D_{A^*}$ let there correspond two elements h, h' of H. Then for all $f \in D_A$ $(f, h - h') = 0$, implying $h = h'$ (recall that D_A is everywhere dense in H).

Thus an operator, denoted by A^*, is defined on D_{A^*}: to every element $g \in D_{A^*}$ there corresponds a unique element $h = A^*g \in H$ such that

$$(Af, g) = (f, A^*g) \tag{4}$$

for any $f \in D_A$. Operator A^* is called *adjoint* of A. Its domain of definition is the set D_{A^*} consisting of those elements of H for which (4) holds for all $f \in D_A$.

If g_1, g_2 are arbitrary elements of D_{A^*} and c_1, c_2 arbitrary complex numbers, then for any $f \in D_A$ (4) yields

$$(f, c_1A^*g_1 + c_2A^*g_2) = \bar{c}_1 (f, A^*g_1) + \bar{c}_2 (f, A^*g_2)$$
$$= \bar{c}_1 (Af, g_1) + \bar{c}_2 (Af, g_2) = (Af, c_1g_1 + c_2g_2),$$

which implies that $c_1g_1 + c_2g_2 \in D_{A^*}$ (that is, D_{A^*} is a linear manifold) and $A^* (c_1g_1 + c_2g_2) = c_1A^*g_1 + c_2A^*g_2$. Thus the operator A^* is linear.

Now suppose that A is bounded. By Subsec. 1, it can be assumed to be defined on all of H. Take an *arbitrary* element $g \in H$. The linear functional $l (f) = (Af, g)$ is bounded, because $| l (f) | \leqslant \|Af \| \times \times \| g \| \leqslant (\| A \| \| g \|) \| f \|$. By Riesz's theorem (Subsec. 2), there is a (unique) element $h \in H$ such that $l (f) = (Af, g) = (f, h) = = (f, A^*g)$. Hence (4) holds for *all* $g \in H$, that is, $D_{A^*} = H$.

Let us show that A^* is bounded and that $\| A^* \| = \| A \|$. Setting in (4) $f = A^*g$ for any $g \in H$, we obtain

$$\| A^*g \|^2 = (AA^*g, g) \leqslant \| A (A^*g) \| \| g \|$$
$$\leqslant (\| A \| \| g \|) \| A^*g \|.$$

Therefore $\| A^*g \| \leqslant \| A \| \, \| g \|$, that is, A^* is bounded and $\| A^* \| \leqslant \| A \|$. Setting in (4) $g = Af$ for any $f \in H$, we similarly obtain $\| A^* \| \geqslant \| A \|$. Hence $\| A^* \| = \| A \|$.

Summarizing, the adjoint operator A^* of a bounded linear operator A is defined on the whole space, is linear, is bounded and its norm equals that of A.

It can be easily shown that $(A^*)^* = A$, $(cA)^* = \bar{c}A^*$ (c is a complex number), $(A + B)^* = A^* + B^*$, $(AB)^* = B^*A^*$.

4. Matrix Representation of a Bounded Linear Operator. While proving Riesz's theorem, it was established that a bounded linear functional defined on a separable Hilbert space is completely determined by its values on an orthonormal basis for this space. Same is the case with bounded linear operators.

Let A be a bounded linear operator acting from a separable Hilbert space H into H. Let $D_A = H$ and $e_1, \ldots, e_n, \ldots$ an orthonormal basis for H.

The infinite matrix $a_{ij} = (Ae_i, e_j) = (e_i, A^*e_j)$, $i \geqslant 1$, $j \geqslant 1$ will be called *matrix representation* of A in the basis $e_1, \ldots, e_n, \ldots$. Since $(A^*e_j, e_i) = \bar{a}_{ij}$, $i = 1, 2, \ldots$, are the Fourier coefficients of A^*e_j, by the Parseval-Steklov equality (equality (5), Sec. 2.7) the series $\sum\limits_{i=1}^{\infty} |a_{ij}|^2$ converges and for all $j = 1, 2, \ldots$ we have the inequality

$$\sum_{i=1}^{\infty} | a_{ij} |^2 = \| A^*e_j \|^2 \leqslant \| A^* \|^2 = \| A \|^2. \tag{5}$$

Take an arbitrary element $f \in H$, and let $f = \sum\limits_{k=1}^{\infty} f_k e_k$ be its Fourier series expansion. Since $Af \in H$, its Fourier coefficients

$$(Af)_j = (Af, e_j) = \Big(A \sum_{i=1}^{\infty} f_i e_i, e_j\Big) = \sum_{i=1}^{\infty} f_i (Ae_i, e_j) = \sum_{i=1}^{\infty} f_i a_{ij}, \tag{6}$$

$j = 1, 2, \ldots$. The series on the right side of (6) converges absolutely, because its general term $f_i a_{ij}$ does not exceed the general term $\frac{1}{2} (| f_i |^2 + | a_{ij} |^2)$ of a convergent series. Substituting the values of the Fourier coefficients in the Fourier series $Af = \sum\limits_{j=1}^{\infty} (Af)_j e_j$, we obtain

$$Af = \sum_{j=1}^{\infty} \Big(\sum_{i=1}^{\infty} a_{ij} f_i \Big) e_j. \tag{7}$$

Thus, for any $f \in H$, the element $Af \in H$ can be determined corresponding to f by means of only the matrix (a_{ij}). This means that the matrix (a_{ij}) completely defines the operator A.

If (a_{ij}) is the matrix representation of A in the basis $e_1, e_2, \ldots$, and (a^*_{ij}) is the corresponding representation of the adjoint operator A^*, then

$$a^*_{ij} = (A^* e_i, e_j) = (e_i, Ae_j) = \bar{a}_{ji} \quad \text{for all } i \geqslant 1, \ j \geqslant 1.$$

Operator A is designated as *finite-dimensional* (*n-dimensional*) if it maps a Hilbert space H into an n-dimensional subspace of it.

Suppose that H_n is a subspace of H spanned by the elements $e_1, \ldots, e_n$. For the bounded linear operator A to map H into H_n it is necessary and sufficient that $a_{ij} = 0$ for $j > n$, $i \geqslant 1$. This statement is an immediate consequence of (6) and (7).

5. Selfadjoint Operators. Operators of Orthogonal Projection. A bounded linear operator from a Hilbert space H into H which is defined on H is termed selfadjoint if $A = A^*$.

With a selfadjoint operator A we can associate a Hermitian bilinear form $W(f, g) = (Af, g)$ and the corresponding quadratic form (Af, f). These forms are called, respectively, *bilinear* and *quadratic forms of operator A*. The quadratic form of a selfadjoint operator is real-valued. A selfadjoint operator A is said to be *nonnegative* if $(Af, f) \geqslant 0$ for all $f \in H$. A nonnegative operator A is *positive* if $(Af, f) = 0$ only for $f = o$.

The matrix representation (a_{ij}) of a selfadjoint operator satisfies (when H is separable) the property: $a_{ij} = \bar{a}_{ji}$, $i, j = 1, 2, \ldots$.

Suppose that $e_1, e_2, \ldots$ is an orthonormal basis for a separable Hilbert space H and $e_{i_1}, \ldots, e_{i_k} \ldots$ is a countable (or finite) subset of it, while $e_{j_1}, \ldots, e_{j_k}, \ldots$, a subset of the basis, is complement of the chosen subset. Let $\mathfrak{R}'$, $\mathfrak{R}''$ denote, respectively, the subspaces spanned by the elements e_{i_k}, $k = 1, 2, \ldots$, and e_{j_k}, $k = 1, 2, \ldots$. Subspace $\mathfrak{R}'$ ($\mathfrak{R}''$) is the aggregate of elements of H that are orthogonal to all the elements e_{j_k}, $k = 1, 2, \ldots$ (e_{i_k}, $k = 1, 2, \ldots$). Equivalently, the subspace $\mathfrak{R}'$ ($\mathfrak{R}''$) is the set of all the elements of H having the property that in their Fourier expansions with respect to the basis e_k, $k = 1, 2, \ldots$, the Fourier coefficients of the elements e_{j_k}, $k = 1, 2, \ldots$ (e_{i_k}, $k = 1, 2, \ldots$) vanish (that is, the corresponding terms do not appear in the expansions). The subspaces $\mathfrak{R}'$ and $\mathfrak{R}''$ are orthogonal, $\mathfrak{R}' \perp \mathfrak{R}''$.

With an arbitrary $f \in H$, whose Fourier expansion is of the form $\sum f_k e_k$, we associate the elements

$$f' = P'f = \sum_{k=1}^{\infty} f_{i_k} e_{i_k}, \quad f'' = P''f = \sum_{k=1}^{\infty} f_{j_k} e_{j_k}. \tag{8}$$

Since, by Bessel's inequality and Lemma 1 (Sec. 2.6), the series in (8) converge in the norm of H, they define on H two operators P' and P''. These are linear with range $R_{P'} = \mathfrak{N}'$, $R_{P''} = \mathfrak{N}''$.

Operators P' and P'' are known as operators of *orthogonal projection* of H onto the subspaces $\mathfrak{N}'$ and $\mathfrak{N}''$, respectively (for the sake of brevity, these operators will be referred to as *projection* operators).

A projection operator is bounded and its norm is unity. In fact, since for all $f \in H$ $\|P'f\|^2 = \|f'\|^2 = \sum_{k=1}^{\infty} |f_{i_k}|^2 \leqslant \sum_{k=1}^{\infty} |f_k|^2 = \|f\|^2$, we have $\|P'\| \leqslant 1$. But $P'e_{i_1} = e_{i_1}$ which implies $\|P'\| = 1$.

A projection operator is selfadjoint, because $(P'f, h) = \left(\sum_{k=1}^{\infty} f_{i_k} e_{i_k}, h\right) = \sum_{k=1}^{\infty} f_{i_k} (e_{i_k}, h) = \sum_{k=1}^{\infty} f_{i_k} \bar{h}_{i_k} = (f, P'h)$ for any f and $h \in H$.

From equality (8) it follows that for any $f \in H$

$$f = If = P'f + P''f, \quad I = P' + P'', \tag{9}$$

where $P'f \in \mathfrak{N}'$, $P''f \in \mathfrak{N}''$. Furthermore,

$$\|f\|^2 = \|P'f + P''f\|^2 = \|P'f\|^2 + \|P''f\|^2 + (P'f, P''f) + (P''f, P'f) = \|P'f\|^2 + \|P''f\|^2, \tag{10}$$

because $\mathfrak{N}' \perp \mathfrak{N}''$.

6. Compact Sets. Let H be a Hilbert space. A set $\mathscr{M} \subset H$ is called *compact* in H if any (infinite) sequence of its elements contains a subsequence that is fundamental in H.

Lemma 1. *A compact set is bounded.*

Proof. Suppose $\mathscr{M}$ is unbounded. We claim that it cannot be compact. Taking any one of its elements f^1, we denote by S_{f^1} a ball of radius 1 with centre at f^1, that is, the set of those $f \in H$ for which $\| f - f' \| < 1$. Because $\mathscr{M}$ is unbounded, the set $\mathscr{M}_1 = \mathscr{M} \setminus S_{f^1}$ is nonempty. We take any $f^2 \in \mathscr{M}_1$ ($\| f^2 - f^1 \| \geqslant 1$). Since $\mathscr{M}_2 = \mathscr{M}_1 \setminus S_{f^2}$ is also nonempty, there is an element $f^3 \in \mathscr{M}$ such that $\| f^3 - f^1 \| \geqslant 1$, $\| f^3 - f^2 \| \geqslant 1$. Continuing in this manner, we obtain a sequence f^k, $k = 1, 2, \ldots$, of elements of $\mathscr{M}$ satisfying the inequality $\| f^i - f^j \| \geqslant 1$ for all i, j, $i \neq j$. This sequence does not contain any fundamental subsequence. Hence $\mathscr{M}$ cannot be compact. ▮

Lemma 2. *For a set $\mathscr{M}$ of a finite-dimensional (n-dimensional) Hilbert space H to be compact it is necessary and sufficient that it be bounded.*

That the condition of boundedness is necessary follows from Lemma 1. We shall show that it is sufficient also.

Since $\mathscr{M}$ is bounded, $\|f\| \leqslant C$ for all $f \in \mathscr{M}$. Consequently, the Fourier coefficients $f_i = (f, e_i)$, $i = 1, \ldots, n$, in the expansion

$f = f_1e_1 + \ldots + f_ne_n$ of an element $f \in \mathcal{M}$* satisfy the inequalities $|f_i| = |(f, e_i)| \leqslant \|f\| \, \|e_i\| = \|f\| \leqslant C$. Hence for any sequence f^k, $k = 1, 2, \ldots$, of elements of $\mathcal{M}$ the sequence $(f_1^k, \ldots, f_n^k)$, $k = 1, 2, \ldots$, of n-dimensional vectors, where $f_i^k = (f^k, e_i)$, is bounded. From it, by the Bolzano-Weirstrass theorem, one can choose a fundamental subsequence $(f_1^{k_s}, \ldots, f_n^{k_s})$, $s = 1, 2, \ldots$:

$$|f_1^{k_s} - f_1^{k_p}|^2 + \ldots + |f_n^{k_s} - f_n^{k_p}|^2 \to 0 \text{ as } s,\ p \to \infty.$$

The corresponding sequence $f^{k_s} = f_1^{k_s}e_1 + \ldots + f_n^{k_s}e_n$, $s = 1, 2, \ldots$, is fundamental in H, because

$$\|f^{k_s} - f^{k_p}\|^2 = |f_1^{k_s} - f_1^{k_p}|^2 + \ldots + |f_n^{k_s} - f_n^{k_p}|^2 \to 0 \text{ as } s,\ p \to \infty. \blacksquare$$

7. A Theorem on Compactness of Sets in a Separable Hilbert Space. Suppose that H is an infinite-dimensional separable Hilbert space and $e_1, \ldots, e_n, \ldots$ an orthonormal basis for it.

First of all we note that not every bounded set in H is compact. For instance, any bounded set containing the orthonormal basis is noncompact because no fundamental subsequence can be selected from the sequence e_k, $k = 1, 2, \ldots$, since $\|e_i - e_j\| = \sqrt{2}$, $i \neq j$. In particular, the set $\{\|f\| \leqslant 1\}$ (the closed unit ball) is noncompact in the infinite-dimensional space.

Let P'_n denote the projection operator which maps H onto the n-dimensional subspace H_n spanned by the elements $e_1, \ldots, e_n$, and put $P''_n = I - P'_n$. For any $f \in H$ and arbitrary $n \geqslant 1$ we have (see (9))

$$f = P'_nf + P''_nf, \tag{11}$$

where $P'_nf = \sum_{k=1}^{n} f_ke_k$, $P''_nf = \sum_{k=n+1}^{\infty} f_ke_k$. Then (11) yields

$$\|f\|^2 = \|P'_nf\|^2 + \|P''_nf\|^2, \tag{12}$$

where $\|P'_nf\|^2 = \sum_{k=1}^{n} |f_k|^2$, $\|P''_nf\|^2 = \sum_{k=n+1}^{\infty} |f_k|^2$, which implies that for any $f \in H$ the number sequence $\|P''_nf\|^2$, $n = 1, 2, \ldots$, being monotone nonincreasing, tends to zero as $n \to \infty$.

Theorem 2. *For the set $\mathcal{M} \subset H$ (H is a separable Hilbert space) to be compact it is necessary and sufficient that it be bounded and for any $\varepsilon > 0$ there be an $n = n(\varepsilon)$ such that $\|P''_nf\| \leqslant \varepsilon$ for all $f \in \mathcal{M}$.*

In other words, for the compactness of $\mathcal{M}$ it is necessary and sufficient that it be bounded and "almost finite-dimensional".

Sufficiency. Let $\|f\| \leqslant C$ for all $f \in \mathcal{M}$. We consider an arbitrary sequence f^k, $k = 1, 2, \ldots$, of elements of $\mathcal{M}$. Setting $\varepsilon = 1$, we have $\| P''_{n_1} f^k \| \leqslant 1$ for all k, where $n_1 = n(1)$. Since $\| P'_{n_1} f^k \| \leqslant \leqslant \| f^k \| \leqslant C$ for all k (P'_n is defined in (11)), the set $P'_{n_1} f^k$, $k = 1, 2, \ldots$, is bounded in the n_1-dimensional space H_{n_1}. From this space, according to Lemma 2 (Subsec. 6), one can choose a fundamental subsequence and from the latter a subsequence $P'_{n_1} f^{1,s}$, $s = 1, 2, \ldots$, having the property that $\| P'_{n_1} f^{1,s} - P'_{n_1} f^{1,p} \| \leqslant 1$ for all s and $p \geqslant 1$. Then, taking into account (12), for the subsequence $f^{1,1}, \ldots, f^{1,s}, \ldots$ we have the inequalities

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* With respect to some orthonormal basis $e_1, \ldots, e_n$.

$$\| f^{1,s} - f^{1,p} \|^2 = \| P'_{n_1} f^{1,s} - P'_{n_1} f^{1,p} \|^2 + \| P''_{n_1} f^{1,s} - P''_{n_1} f^{1,p} \|^2$$
$$\leqslant 1 + (\| P''_{n_1} f^{1,s} \| + \| P''_{n_1} f^{1,p} \|)^2 \leqslant 5,$$

true for all p and s.

Take a number $n_2 = n(1/2)$ corresponding to $\varepsilon = 1/2$. The sequence $P'_{n_2} f^{1,1}, \ldots, P'_{n_2} f^{1,s}, \ldots$ belongs to H_{n_2} and is bounded; hence from it a subsequence $P'_{n_2} f^{2,s}$, $s = 1, 2, \ldots$, can be selected for which $\| P'_{n_2} f^{2,s} - P'_{n_2} f^{2,p} \| \leqslant 1/2$ for all p and s.

In view of (12), we have $\| f^{2,s} - f^{2,p} \|^2 = \| P'_{n_2} f^{2,s} - P'_{n_2} f^{2,p} \|^2 + \| P''_{n_2} f^{2,s} - P''_{n_2} f^{2,p} \|^2 \leqslant \frac{1}{4} + \frac{4}{4} = \frac{5}{4}$ for all s, p; and so on. For $\varepsilon = 1/i$ we can find $n_i = n(1/i)$ and from the subsequence $P'_{n_i} f^{i-1,s}$, $s = 1, 2, \ldots$, we can choose a subsequence $P'_{n_i} f^{i,s}$, $s = 1, 2, \ldots$, such that $\| P'_{n_i} f^{i,s} - P'_{n_i} f^{i,p} \| \leqslant 1/i$ for all s, p. Noting (12), for the subsequence $f^{i,s}$, $s = 1, 2, \ldots$, we have $\| f^{i,s} - f^{i,p} \|^2 \leqslant \leqslant 1/i^2 + 4/i^2 = 5/i^2$ for all s, p.

The diagonal sequence $f^{s,s}$, $s = 1, 2, \ldots$, is a subsequence of the initial sequence and is such that $\| f^{p,p} - f^{s,s} \| \leqslant 5/i^2$ for all p, $s \geqslant i$, that is, it is fundamental.

Necessity. The necessity of boundedness of the set $\mathscr{M}$ was proved in Lemma 1 (Subsec. 6). We shall establish the necessity of the second condition in the hypothesis.

Suppose that $\mathscr{M}$ is compact but nevertheless there is an $\varepsilon_0 > 0$ such that for any n $\| P''_n f^n \| \geqslant \varepsilon_0$ for some $f^n \in \mathscr{M}$.

Taking arbitrary n_1, we find a corresponding $f^{n_1} \in \mathscr{M}$ such that $\| P''_{n_1} f^{n_1} \| \geqslant \varepsilon_0$. On the basis of f^{n_1} we choose a number $n_2 > n_1$ such that $\| P''_{n_2} f^{n_1} \| < \varepsilon_0/2$ (this is possible, since for any fixed $f \in H$, $\| P''_k f \| \to 0$ as $k \to \infty$). Corresponding to n_2 we choose $f^{n_2} \in \mathscr{M}$ such that $\| P''_{n_2} f^{n_2} \| \geqslant \varepsilon_0$, and on the basis of f^{n_2} we choose an n_3 such that $\| P''_{n_3} f^{n_2} \| < \varepsilon_0/2$, and so on. Thus, we obtain a sequence f^{n_k}, $k = 1, 2, \ldots$, of elements of $\mathscr{M}$ for which the inequalities

$$\| P''_{n_k} f^{n_k} \| \geqslant \varepsilon_0, \quad \| P''_{n_{k+1}} f^{n_k} \| \leqslant \varepsilon_0/2$$

hold.

We show that this sequence cannot contain a fundamental subsequence. Indeed, in view of (12) and the fact that $\| P''_n f \|$ is monotone with respect to n, we have for any $k > s$

$$\| f^{n_k} - f^{n_s} \|^2 = \| P'_{n_k} (f^{n_k} - f^{n_s}) \|^2 + \| P''_{n_k} (f^{n_k} - f^{n_s}) \|^2$$

$$\geqslant \| P''_{n_k} (f^{n_k} - f^{n_s}) \|^2 \geqslant (\| P''_{n_k} f^{n_k} \| - \| P''_{n_k} f^{n_s} \|)^2$$

$$\geqslant (\| P''_{n_k} f^{n_k} \| - \| P''_{n_{s+1}} f^{n_s} \|)^2 > (\varepsilon_0 - \varepsilon_0/2)^2 = \varepsilon_0^2/4.$$

Corollary. Let $\mathscr{M}$ be a set in the separable Hilbert space H. Consider a family of sets $\mathscr{M}_\varepsilon \subset H$, $\varepsilon > 0$, having the following property: for any $f \in \mathscr{M}$ an element $f' = f'(\varepsilon)$ can be found in each $\mathscr{M}_\varepsilon$, $\varepsilon > 0$, such that $\| f' - f \| \leqslant \varepsilon$. *If for some sequence* $\varepsilon_k \to 0$ *as* $k \to \infty$, $\varepsilon_k > 0$, *all the sets* $\mathscr{M}_{\varepsilon_k}$ *are compact, then* $\mathscr{M}$ *is compact.*

Consider an arbitrary ε_k of this sequence. Since $\mathscr{M}_{\varepsilon_k}$ is compact, we can find an $n = n(\varepsilon_k)$ such that $\| P''_n f \| \leqslant \varepsilon_k$ for all $f \in \mathscr{M}_{\varepsilon_k}$. But then for any $f \in \mathscr{M}$, $\| P''_n f \| = \| P''_n (f - f') + P''_n f' \| \leqslant \leqslant \| P''_n (f - f') \| + \| P''_n f' \| \leqslant \| f - f' \| + \varepsilon_k \leqslant 2\varepsilon_k$ if f' is an element of $\mathscr{M}_{\varepsilon_k}$ such that $\| f - f' \| \leqslant \varepsilon_k$. Since $\varepsilon_k \to 0$, by Theorem 2 the set $\mathscr{M}$ is compact.

8. Weak Compactness. A set $\mathscr{M}$ in the Hilbert space H is called *weakly compact* if from any (infinite) sequence of its elements one can construct a subsequence that converges weakly to an element of H (not necessarily belonging to $\mathscr{M}$).

Theorem 3. *Any bounded subset of a Hilbert space is weakly compact.*

In fact, boundedness is not only sufficient but also necessary for the set to be weakly compact; however, we shall not prove the necessity and confine only to the proof of sufficiency in the case of a separable Hilbert space.

Proof. Let e_k, $k = 1, 2, \ldots$, be an orthonormal basis for H and $\mathscr{M}$ a bounded set in H: $\| f \| \leqslant C$ for all $f \in \mathscr{M}$. Consider an arbitrary sequence f^k, $k = 1, 2, \ldots$, of elements of $\mathscr{M}$. Since for all k $\|f^k\| \leqslant \leqslant C$, the number sequence (f^k, e_1), $k = 1, 2, \ldots$, is bounded: $| (f^k, e_1) | \leqslant \| f^k \| \| e_1 \| \leqslant C$; hence from the sequence f^k, $k = = 1, 2, \ldots$, one can select a subsequence $f^{1,k}$, $k = 1, 2, \ldots$, such that the number sequence $(f^{1,k}, e_1)$ converges to some σ_1: $(f^{1,k}, e_1) \to \sigma_1$ as $k \to \infty$. The number sequence $(f^{1,k}, e_2)$ is also bounded; this means that from the sequence $f^{1,k}$, $k = 1, 2, \ldots$, a subsequence $f^{2,k}$, $k = 1, 2, \ldots$, may be chosen such that $(f^{2,k}, e_2)$ converges to a σ_2 as $k \to \infty$, and so on.

We shall show that the diagonal sequence $f^{k,k}$, $k = 1, 2, \ldots$, is weakly convergent. First, let us note that for any $s \geqslant 1$ $(f^{k,k}, e_s) \to$

$\to \sigma_s$ as $k \to \infty$. Hence for any $m \geqslant 1$

$$(f^{k,k}, \sum_{i=1}^{m} \sigma_i e_i) = \sum_{i=1}^{m} \bar{\sigma}_i (f^{k,k}, e_i) \to \sum_{i=1}^{m} |\sigma_i|^2 \text{ as } k \to \infty.$$

Since $|(f^{k,k}, \sum_{i=1}^{m} \sigma_i e_i)|^2 \leqslant \| f^{k,k} \|^2 \sum_{i=1}^{m} |\sigma_i|^2 \leqslant C^2 \sum_{i=1}^{m} |\sigma_i|^2$, we have $\sum_{i=1}^{m} |\sigma_i|^2 \leqslant C^2$ for any $m \geqslant 1$. Consequently, $\sum_{i=1}^{\infty} |\sigma_i|^2 \leqslant C^2$. By Lemma 1, (Sec. 2.6) the series $\sum_{i=1}^{\infty} \sigma_i e_i$ converges to an element $f \in H$ and $\| f \|^2 = \sum_{i=1}^{\infty} |\sigma_i|^2$. It will be shown that the sequence $f^{k,k}$, $k = 1, 2, \ldots$, converges weakly to f.

Let g be any element of H. Taking an $\varepsilon > 0$, we choose $s = s(\varepsilon)$ such that $\sum_{i=s+1}^{\infty} |g_i|^2 \leqslant \varepsilon^2$. By the generalized Parseval-Steklov equality (equality (5′) in Sec. 2.7), we obtain

$$|(f^{k,k} - f, g)| = |\sum_{i=1}^{\infty} ((f^{k,k}, e_i) - \sigma_i) \bar{g}_i| \leqslant \sum_{i=1}^{s} |(f^{k,k}, e_i) - \sigma_i| \cdot |g_i| + \sum_{i=s+1}^{\infty} |(f^{k,k}, e_i)| \cdot |g_i| + \sum_{i=s+1}^{\infty} |\sigma_i| \cdot |g_i|. \tag{13}$$

Furthermore,

$$\Big(\sum_{i=s+1}^{\infty} |\sigma_i| \cdot |g_i|\Big)^2 \leqslant \sum_{i=s+1}^{\infty} |\sigma_i|^2 \cdot \sum_{i=s+1}^{\infty} |g_i|^2 \leqslant \| f \|^2 \cdot \varepsilon^2,$$

$$\Big(\sum_{i=s+1}^{\infty} |(f^{k,k}, e_i)|\, |g_i|\Big)^2 \leqslant \sum_{i=s+1}^{\infty} |(f^{k,k}, e_i)|^2 \sum_{i=s+1}^{\infty} |g_i|^2 \leqslant C^2 \varepsilon^2.$$

By the definition of numbers σ_i, the first term in the right side of (13) can also be made $< \varepsilon$ if $k \geqslant k_0(\varepsilon)$ for some $k_0(\varepsilon)$. Thus $|(f^{k,k} - f, g)| \leqslant \varepsilon + \varepsilon (C + \| f \|)$ for $k \geqslant k_0(\varepsilon)$. ▌

9. Completely Continuous Operators. Let H bea Hilbert space. A linear operator A acting from H into H and defined on H is said to be *completely continuous* if it maps a bounded set into a compact set.

If A_1 and A_2 are completely continuous operators, then so is the operator $c_1 A_1 + c_2 A_2$ with arbitrary constants c_1, c_2. If the operator A is completely continuous and the operator B defined on H is bounded, then the operators AB and BA are also completely continuous.

From Lemma 1 (Subsec. 6) it follows that a completely continuous operator is bounded. However, not every bounded operator is completely continuous. For instance, the identity operator I acting in an infinite-dimensional Hilbert space cannot be completely continuous, for it maps a noncompact set—the orthonormal basis—into itself.

A finite-dimensional bounded operator is completely continuous; this fact is a consequence of Lemma 2 (Subsec. 6). The following result is a direct generalization of this assertion.

Theorem 4. *For a bounded linear operator A defined on a separable Hilbert space H and acting from H into H to be completely continuous, it is necessary and sufficient that for any $\varepsilon > 0$ it is possible to find an integer $n = n(\varepsilon)$ and linear operators A_1 and A_2, where A_1 is n-dimensional and $\| A_2 \| \leqslant \varepsilon$, such that*

$$A = A_1 + A_2. \tag{14}$$

Thus completely continuous operators are those operators which are "almost finite-dimensional".

Necessity. In view of (11) (see Subsec. 7), for any $f \in H$ and any $n > 0$ we have the representation

$$Af = P_n'Af + P_n''Af \qquad (A = P_n'A + P_n''A). \tag{15}$$

Since A is completely continuous, given an $\varepsilon > 0$ an $n = n(\varepsilon)$ can be found such that $\| P_n''A \| \leqslant \varepsilon$. Indeed, by Theorem 2, $\| P_n''Af \| = \| f \| \, \| P_n''A(f/\| f \|)\| \leqslant \varepsilon \, \| f \|$, because the boundedness of $\{f/\| f \|\}$ implies the compactness of $\{A\,(f/\| f \|)\}$. Since $P_n'A$ is n-dimensional, the necessity is established.

Sufficiency. Let f^k, $k = 1, 2, \ldots$, be an arbitrary bounded sequence in H, $\| f^k \| \leqslant C$, $k = 1, 2, \ldots$. We shall demonstrate that fundamental subsequence can be extracted from the sequence Af^k, $k = 1, 2, \ldots$. Choosing an $\varepsilon > 0$ from the set $\{1, 1/2, \ldots, 1/i, \ldots\}$, for $\varepsilon = 1/i$ we can find $n_i = n\,(1/i)$ and operators A_1^i and A_2^i such that $A = A_1^i + A_2^i$, where A_1^i is n_i-dimensional and $\| A_2^i \| \leqslant 1/i$. Then $\| A_1^i \| = \| A - A_2^i \| \leqslant \| A \| + \| A_2^i \| \leqslant \| A \| + 1$. The set $A_1^1 f^k$, $k = 1, 2, \ldots$, is bounded in the n_1-dimensional space, therefore (Lemma 2, Subsec. 6) from it a fundamental subsequence $A_1^1 f^{1,k}$, $k = 1, 2, \ldots$, can be chosen. The sequence $f^{1,k}$, $k = 1, 2, \ldots$, has the property: $\| A_2^1 f^{1,k} \| \leqslant \| A_2^1 \| \, \| f^{1,k} \| \leqslant 1 \cdot C$. The sequence $A_1^2 f^{1,k}$, $k = 1, 2, \ldots$, is a bounded sequence of the n_2-dimensional space, therefore there is a corresponding fundamental subsequence $A_1^2 f^{2,k}$, $k = 1, 2, \ldots$. This subsequence satisfies the inequality $\| A_2^2 f^{2,k} \| \leqslant \| A_2^2 \| \, \| f^{2,k} \| \leqslant \frac{1}{2} \cdot C$, and so on.

The diagonal sequence $f^{1,1}$, $f^{2,2}$, $\ldots$ has the following obvious properties: the sequence $A_1^i f^{k,k}$, $k = 1, 2, \ldots$, is fundamental for any i, because for $k \geqslant i$ $f^{k,k}$ are elements of the sequence $f^{i,k}$, $k = 1, 2, \ldots$. Further, $\| A_2^i f^{k,k} \| \leqslant C/i$ for all i. Let us show that the sequence $Af^{k,k}$, $k = 1, 2, \ldots$, is fundamental. Take an

$\varepsilon > 0$ and fix $i > 1/\varepsilon$. Since $A_1^i f^{k,k}$, $k = 1, 2, \ldots$, is a fundamental sequence, for sufficiently large k, s we have

$$\| Af^{k,k} - Af^{s,s} \| \leqslant \| A_1^i (f^{k,k} - f^{s,s}) \| + \| A_2^i (f^{k,k} - f^{s,s}) \|$$

$$\leqslant \varepsilon + \| A_2^i f^{k,k} \| + \| A_2^i f^{s,s} \| \leqslant \varepsilon + 2C/i \leqslant (1 + 2C)\,\varepsilon.$$

as required. ▮

Theorem 4 yields, in particular, the following result.

Suppose that the linear operator A acting from a separable Hilbert space H into H and defined on the whole H is completely continuous. Then its adjoint A^* is also completely continuous.

Indeed, the representation (14) generates the representation $A^* = A_1^* + A_2^*$, where $\| A_2^* \| = \| A_2 \| \leqslant \varepsilon$. Accordingly, the above assertion will have been established if it is shown that A_1^* is a finite-dimensional operator.

Let R_{A_1} be an n-dimensional subspace of H and $e_1, \ldots, e_n$ an orthonormal basis for it. Then for any $f \in H$ $A_1 f = \sum_{i=1}^{n} (A_1 f, e_i)\, e_i = \sum_{i=1}^{n} (f, A_1^* e_i)\, e_i$. Consequently, for any f, $g \in H$ we have

$$(A_1 f, g) = \sum_{i=1}^{n} (f, A_1^* e_i)\, \bar{g}_i = \Big(f, \sum_{i=1}^{n} g_i A_1^* e_i\Big),$$

that is,

$$(f, A_1^* g) = (A_1 f, g) \Big(f, \sum_{i=1}^{n} g_i A_1^* e_i\Big).$$

Hence for all $g \in H$ $A_1^* g = \sum_{i=1}^{n} g_i A_1^* e_i$. This means that $R_{A_1^*}$ is a subspace spanned by the elements $A_1^* e_1, \ldots, A_1^* e_n$, that is, A_1^* is finite-dimensional.

§ 4. LINEAR EQUATIONS IN A HILBERT SPACE

The results of this section are true for any Banach space; nonetheless, we shall confine our discussion only to a separable Hilbert space H.

1. Contracting Linear Operator. A linear operator A acting from H into H and defined on H is called a *contraction* if $\| A \| < 1$.

Lemma 1. *If A is a contraction from H into H, then there is an operator $(I - A)^{-1}$ from H into H which is defined on H, and* $\| (I - A)^{-1} \| \leqslant \frac{1}{1 - \| A \|}$.

To prove this, we consider the equation

$$(I - A) f = g \tag{1}$$

and show that for any $g \in H$ it has a unique solution given by the series $f = \sum_{k=0}^{\infty} A^k g$ $(A^0 = I)$ which converges in H.

The last series converges, because H is complete and the partial sums $g_m = \sum_{k=0}^{m} A^k g$ constitute a fundamental sequence: for, $p > m$

$$\| g_p - g_m \| = \| A^p g + \ldots + A^{m+1} g \| \leqslant \| A^p g \| + \ldots + \| A^{m+1} g \|$$
$$\leqslant \| g \| \left(\| A \|^{m+1} + \ldots \right) = \| g \| \frac{\| A \|^{m+1}}{1 - \| A \|} \to 0 \text{ as } m,\ p \to \infty.$$

The element $f \in H$ is a solution of (1), since $(I - A)f = (g + Ag + \ldots) - (Ag + A^2 g + \ldots) = g$.

This solution is unique. In fact, assume that there are two solutions f_1 and f_2 of Eq. (1). Then the element $f = f_1 - f_2$ is the solution of the homogeneous equation $f = Af$; accordingly, it satisfies $\| f \| = \| Af \| \leqslant \| A \| \| f \|$. Hence $f = o$, that is $f_1 = f_2$.

Thus the operator $(I - A)^{-1}$ exists, is defined on the whole H, is bounded, because for all $g \in H$ $\| (I - A)^{-1} g \| = \| g + Ag + \ldots + A^m g + \ldots \| \leqslant \| g \| (1 + \| A \| + \ldots) = \frac{\| g \|}{1 - \| A \|}$, and $\| (I - A)^{-1} \| \leqslant \frac{1}{1 - \| A \|}$. ∎

Remark. Under the hypothesis of Lemma 1 the bounded operator $(I - A^*)^{-1}$ also exists, since $\| A^* \| = \| A \| < 1$, and $(I - A^*)^{-1} = [(I - A)^{-1}]^*$.

To prove this relation, we take arbitrary $f', g' \in H$ and construct (Lemma 1) corresponding $f, g \in H$ such that $(I - A) f = f'$, $(I - A^*) g = g'$.

Since $f = (I - A)^{-1} f'$ and $g = (I - A^*)^{-1} g'$, the relation $((I - A) f, g) = (f, (I - A^*) g)$ can be written as $(f', (I - A^*)^{-1} g') = ((I - A)^{-1} f', g')$, whence the desired relation follows.

2. Equation with a Completely Continuous Operator. We consider Eq. (1) without the assumption that the norm of A is small. Instead we assume that A is completely continuous.

By Theorem 4, Sec. 3.9, (1) can be written as $(I - A_2) f - A_1 f = g$, where A_1 is an n-dimensional operator and $\| A_2 \| \leqslant \varepsilon < 1$. Put $h = (I - A_2) f$. By Lemma 1, the operator $I - A_2$ has a bounded inverse $(I - A_2)^{-1}$ defined on H;

$$(I - A_2) f = h, \quad f = (I - A_2)^{-1} h. \tag{2}$$

Eq. (1) for h can be expressed in the form

$$h - A_1 (I - A_2)^{-1} h = g. \tag{3}$$

Let A^* be adjoint of A. The equation

$$(I - A^*) f^* = g^* \tag{1*}$$

is referred to as *adjoint* of Eq. (1). The relation $A = A_1 + A_2$ implies $A^* = A_1^* + A_2^*$. In view of the Remark to Lemma 1, the operator $(I - A_2^*)$ has a bounded inverse $(I - A_2^*)^{-1} = [(I - A_2)^{-1}]^*$ defined on H,

$$(I - A_2^*)^{-1} g^* = z^*, \qquad g^* = (I - A_2^*) z^*. \tag{2*}$$

Eq. (1*) can be written in the form

$$(I - A_2^*) f^* - A_1^* f^* = g^*.$$

Applying to it the operator $(I - A_2^*)^{-1}$, we obtain the equivalent equation

$$f^* - [(I - A_2)^{-1}]^* A_1^* f^* = z^*, \tag{3*}$$

where the operator $[(I - A_2)^{-1}]^* A_1^*$ is adjoint of the operator $A_1(I - A_2)^{-1}$ (in Eq. (3)).

The operator $A_1(I - A_2)^{-1}$ is clearly n-dimensional, therefore, its matrix representation (a_{ij}) in the corresponding orthonormal basis e_k, $k = 1, 2, \ldots$ (the subspace spanned by the elements $e_1, e_2, \ldots, e_n$ coincides with $R_{A_1(I-A_2)^{-1}}$, satisfies the property that $a_{ij} = 0$ for $i \geqslant 1$, $j \geqslant n + 1$ (see Sec. 3.4), and formula (5) of Sec. 3.4 yields, for any j,

$$\sum_{i=1}^{\infty} |a_{ij}|^2 \leqslant \| A_1 (I - A_2)^{-1} \|^2.$$

By formula (7), Sec. 3.4, Eq. (3) can be expressed in the form $\sum_j h_j e_j - \sum_j \sum_i h_i a_{ij} e_j = \sum_j g_j e_j$ which, in view of the linear independence of the system $e_1, e_2, \ldots$, is equivalent to a system of algebraic equations for the Fourier coefficients $h_1, \ldots, h_m, \ldots$ of the desired element h:

$$h_j - \sum_{i=1}^{\infty} a_{ij} h_i = g_j, \qquad j \leqslant n; \quad h_j = g_j, \quad j > n.$$

Since the coefficients h_j are known when $j > n$:

$$h_j = g_j, \qquad j > n, \tag{4}$$

the last system reduces to the system of algebraic equations

$$h_j - \sum_{i=1}^{n} a_{ij} h_i = g_j + \sum_{i=n+1}^{\infty} a_{ij} g_i, \quad j = 1, \ldots, n, \tag{5}$$

for h_j, $j \leqslant n$.

Similarly, Eq. (3*) can be replaced by an equivalent algebraic system of equations for determining Fourier coefficients f_j^*, $j = 1, 2, \ldots$, of the element f^* in terms of the Fourier coefficients z_j^*, $j = 1, 2, \ldots$, of the element $z^* = (I - A_2^*)^{-1} g^*$. For f_j^*, $j \leqslant n$, we obtain the linear algebraic system

$$f_j^* - \sum_{i=1}^{n} \bar{a}_{ji} f_i^* = z_j^*, \qquad j = 1, \ldots, n, \tag{5*}$$

while f_j^*, $j > n$, are determined uniquely in terms of f_j^*, $j \leqslant n$, by the formulas

$$f_j^* = z_j^* + \sum_{i=1}^{n} \bar{a}_{ji} f_i^*, \qquad j > n. \tag{4*}$$

3. Fredholm's First Theorem. The matrices of systems (5) and (5*) are Hermitian conjugate, therefore moduli of their determinants are equal. Hence if one of these systems is solvable with any free term, that is, the corresponding determinant does not vanish, then the same property is possessed by the second of these systems, and the solutions of these systems are determined uniquely. In particular, the corresponding homogeneous systems have only trivial solutions. Or if one of the homogeneous systems (5) or (5*) has only a trivial solution (accordingly, the corresponding determinant does not vanish), then so has the other, and the systems (5) and (5*) are solvable (uniquely) with any free term.

Eqs. (1) and (1*) also have the same property.

In fact, assume that Eq. (1) (or Eq. (1*)) is solvable with any g (or g^*) of H; or, in view of (2) (or (2*)) what amounts to the same thing, Eq. (3) (or (3*)) is solvable with any g (or z^*) of H. In particular, it has a solution for any g (or z^*) of the subspace spanned by the elements $e_1, \ldots, e_n$. Consequently, the system of Eqs. (5) (or (5*)) is solvable with any right-hand side. That is, the determinant of the system does not vanish, and the homogeneous systems (5) and (5*) have only trivial solutions. Then, in view of (4) and (4*), the homogeneous equations (1) and (1*) have only trivial solutions.

Conversely, suppose that one of the homogeneous equations (1) or (1*) has only a trivial solution. Then the corresponding homogeneous system (5) or (5*) has only a trivial solution. Consequently, the determinants of both the systems do not vanish. That is, the nonhomogeneous systems (5) and (5*) have (unique) solutions for any free term. Then, in view of (4) and (4*), Eqs. (1) and (1*) are (uniquely) solvable with any free term belonging to H. This implies that the inverse operators $(I - A)^{-1}$ and $(I - A^*)^{-1}$ exist and are defined on H.

Let us show that these operators are bounded.

Suppose that the system (5) is uniquely solvable (the determinant of matrix in (5) is nonvanishing) and that $(h_1, \ldots, h_n)$ is its solution. From Cramer's rule it follows that there exists a constant $C > 0$, independent of the free term in (5), such that

$$\sum_{f=1}^{n} |h_j|^2 \leqslant C^2 \sum_{j=1}^{n} \Big| g_j + \sum_{i=n+1}^{\infty} a_{ij} g_i \Big|^2. \tag{6}$$

Since

$$\sum_{j=1}^{n} \Big| g_j + \sum_{i=n+1}^{\infty} a_{ij} g_i \Big|^2 \leqslant 2 \sum_{j=1}^{n} \Big(|g_j|^2 + \sum_{i=n+1}^{\infty} |a_{ij}|^2 \cdot \sum_{i=n+1}^{\infty} |g_i|^2 \Big)$$
$$\leqslant \| g \|^2 (2 + 2n \| A_1 (I - A_2)^{-1} \|^2) = C_1^2 \| g \|^2,$$

we have

$$\sum_{j=1}^{n} |h_j|^2 \leqslant (CC_1)^2 \| g \|^2$$

and

$$\| h \|^2 = \sum_{j=1}^{n} |h_j|^2 + \sum_{j=n+1}^{\infty} |g_j|^2 \leqslant (1 + C^2 C_1^2) \| g \|^2 = C_2^2 \| g \|^2.$$

Consequently, in view of (2)

$$\| f \| \leqslant C_3 \| g \|, \tag{7}$$

where the constant $C_3 > 0$ does not depend on g. This in turn means that the operator $(I - A)^{-1}$ and, hence also, the operator $(I - A^*)^{-1}$ are bounded: $\| (I - A)^{-1} = \| (I - A^*)^{-1} \| \leqslant C_3$.

Thus we have established the following assertion.

Theorem 1 (Fredholm's First Theorem). *Let A be a completely continuous linear operator from H into H which is defined on H. If one of the Eqs.* (1) *or* (1*) *has a solution for any free term, then the other also has a solution for any free term, and these solutions are unique, that is, the homogeneous equations* (1) ($g = o$) *and* (1*) ($g^* = o$) *have only trivial solutions.*

If one of the homogeneous equations (1) ($g = o$) *or* (1*) ($g^* = o$) *has only a trivial solution, then so has the other equation. Eqs.* (1) *and* (1*) *are uniquely solvable with any free terms, that is, the operators $(I - A)^{-1}$ and $(I - A^*)^{-1}$ exist and are defined on H, and these operators are bounded.*

4. **Fredholm's Second Theorem.** Note that the ranks of matrices $B = \| b_{ij} \|$, where $b_{ij} = \delta_{ij} - a_{ij}$, $i, j = 1, \ldots, n$ ($\delta_{ij} = 0$ when $i \neq j$, $\delta_{ii} = 1$), and $B^* = \| \bar{b}_{ji} \|$ in the systems (5) and (5*) are the same. Therefore the homogeneous systems (5) and (5*) have always the same number $k \leqslant n$ of linearly independent solutions.

Then, by (2), (4) and (4*), the sets of all solutions of homogeneous equations (1) and (1*) also contain exactly k linearly independent elements.

Thus we have the following result.

Theorem 2 (Fredholm's Second Theorem). *If the homogeneous equation* (1) (*A is a completely continuous operator from H into H and defined on H*) *has nontrivial solutions, then only a finite number of them are linearly independent. The homogeneous equation* (1*) *has also the same number of linearly independent solutions.*

5. **Fredholm's Third Theorem.** We now examine the question of solvability of Eq. (1) when the homogeneous equation (1) may have nontrivial solutions. By Fredholm's Second Theorem, the homogeneous equation (1) has only a finite number of linearly independent solutions: $f^1, \ldots, f^k$. The same number of linearly independent solutions has also the homogeneous equation (1*): $f^{1*}, \ldots, f^{k*}$. The system $f^1, \ldots, f^k$ (as also the system $f^{1*}, \ldots, f^{k*}$) can be assumed to be orthonormal.

Theorem 3 (Fredholm's Third Theorem.) *For Eq.* (1) *with a completely continuous operator A from H into H and defined on H to have a solution, it is necessary and sufficient that the element g be orthogonal to all the solutions of the homogeneous equation* (1*).

Among all the solutions of Eq. (1) *there is a unique solution f that is orthogonal to all the solutions of the homogeneous equation* (1). *Any other solution of Eq.* (1) *is expressed as a sum of this solution and some solution of the homogeneous equation* (1). *Solution f satisfies the inequality* (7) *in which the constant does not depend on g.*

Proof. Suppose that the solution of Eq. (1) exists, then, by (2), Eq. (3) as well as the system (5) also have solutions.

Let the rank of the matrix $B = \| b_{ij} \|$, where $b_{ij} = \delta_{ij} - a_{ij}$, $i, j = 1, \ldots, n$, be $n - k$. Then the subspace R_{n-k} of the n-dimensional vector space spanned by the vectors $B_i = (b_{i1}, \ldots, b_{in})$, $i = 1, \ldots, n$, forming columns of B, is of dimension $n - k$. Since the homogeneous system (5*): $\sum_{i=1}^{n} \overline{b}_{ji} f_i^* = 0$, $j = 1, \ldots, n$, can be written in the form $(\tilde{f}^*, B_i) = 0$, $i = 1, \ldots, n$, it follows that the solutions of the homogeneous system (5*) constitute a k-dimensional subspace, denoted by $R_{n-k}^{\perp}$, orthogonal to the subspace R_{n-k}.

By the Kronecker-Kapelli theorem, for the system (5) to have a solution it is necessary and sufficient that the ranks of B and the augmented matrix obtained by adjoining to B the column consisting of free terms of (5) coincide, that is, the vector constituted by free terms should belong to the space R_{n-k} or, what is the same, should be orthogonal to $R_{n-k}^{\perp}$.

Noting that any solution f^* of the homogeneous equation (1*) is of the form

$$f^* = f_1^* e_1 + \ldots + f_n^* e_n + f_{n+1}^* e_{n+1} + \ldots,$$

where the vector $\tilde{f}^* = (f_1^*, \ldots, f_n^*)$ is the solution of the homogeneous system (5*) and $f_j^* = \sum_{i=1}^{n} \bar{a}_{ji} f_i^*$ when $j > n$, and expressing the condition of orthogonality of vectors $\tilde{f}^*$ and the right side of (5) as follows

$$0 = \sum_{j=1}^{n} \left(g_j + \sum_{i=n+1}^{\infty} a_{ij} g_i\right) \bar{f}_j^* = \sum_{j=1}^{n} g_j \bar{f}_j^* + \sum_{i=n+1}^{\infty} g_i \bar{f}_i^* = (g, f^*),$$

we find that if the solution of the nonhomogeneous equation (1) exists, then the element g must be orthogonal to all the solutions of the homogeneous equation (1*).

Conversely, if the element g is orthogonal to all the solutions f^* of the homogeneous equation (1*), then the vector with components $g_j + \sum_{i=n+1}^{\infty} a_{ij} g_i$, $j = 1, \ldots, n$, is orthogonal to all the solutions $\tilde{f}^*$ of the homogeneous system (5*). Consequently, system (5) and together with it Eq. (1) have solutions.

Let f_0 be a solution of the nonhomogeneous equation (1) and $f^1, \ldots, f^k$ be an orthonormal system of solutions of the homogeneous equation (1). Then the element $f = f_0 - (f_0, f^1) f^1 - \ldots - (f_0, f^k) f^k$ is also a solution of Eq. (1), and it is orthogonal to all the solutions of the homogeneous equation (1). Such a solution is unique: for if there were one more such solution $\tilde{f}$, then their difference $f - \tilde{f}$, being a solution of the homogeneous equation (1), would be orthogonal to all the solutions of the homogeneous equation (1) including itself, that is, $f - \tilde{f} = o$.

If f' is any solution of the nonhomogeneous equation (1), then $f' - f = f''$ is a solution of the homogeneous equation (1), that is, $f' = f + f''$.

We shall now establish inequality (7). Suppose that h is an element of H corresponding, according to (2), to the element f. This means that h is a solution of Eq. (3) which satisfies k equations:

$$0 = (f, f^i) = ((I - A_2)^{-1} h, f^i) = (h, (I - A_2^*)^{-1} f^i), \qquad i = 1, \ldots, k. \tag{8}$$

Since the augmented matrix of system (5) has the same rank $n - k$ as the matrix B, some of k equations in the system (5) are linear combinations of the remaining $n - k$ equations. Therefore if we delete these k equations, the resulting system will be equivalent to the system (5).

Thus n-dimensional vector $(h_1, \ldots, h_n)$ is a solution of the linear system of n equations ($n - k$ equations of (5) and k equations of (8)) whose coefficients do not depend on the right-hand side of (5). Moreover, since the element f is unique, it follows that $(h_1, \ldots, h_n)$ is a unique solution of this system, that is, the determinant of the resulting system is not zero. Then the vector $(h_1, \ldots, h_n)$ can be obtained by Cramer's rule. Therefore inequality (6) holds for it, which readily yields the desired inequality (7). $\blacksquare$

6. Eigenvalues and Eigenelements of a Completely Continuous Operator. A quantity λ is called an *eigenvalue* of a linear operator A acting from H into H if there is an element $f \in H$ such that $f \neq o$ and $Af = \lambda f$; the element f is termed *eigenelement* of the operator A. The quantity $\mu = 1/\lambda$, $\lambda \neq 0$, is called the *characteristic value.* Since together with f the element cf for any constant $c \neq 0$ is also an eigenelement corresponding to the eigenvalue λ, the eigenelements can be assumed normalized by, for instance, the condition, $\| f \| = 1$.

The maximum number of linearly independent eigenelements corresponding to the given characteristic value (eigenvalue) is called the *multiplicity* of this characteristic value of the eigenvalue. If there is an infinite number of linearly independent eigenelements corresponding to a characteristic value (eigenvalue), the multiplicity of the characteristic value (of the eigenvalue) is infinite.

Suppose that the operator A defined on the whole H is completely continuous. Then the operator μA, where μ is a complex number, is also completely continuous. The following assertions are a consequence of Theorems 1, 2 and 3.

For the equation

$$f - \mu A f = g \tag{1'}$$

to have a solution for all $g \in H$, *it is necessary and sufficient that* μ *should not be a characteristic value of* A (*that is,* $1/\mu$ *should not be an eigenvalue*). *If* μ *is a characteristic value, then it is of finite multiplicity and* $\bar{\mu}$ *is characteristic value of the operator* A^* *with the same multiplicity. In order that Eq.* (1′) *may be solvable in this case, it is necessary and sufficient that the element g be orthogonal to all the eigenelements of* A^* *that correspond to the eigenvalue* $1/\bar{\mu}$. *In this case the solution of Eq.* (1′) *is unique, and this solution is orthogonal to all the eigenelements of* A *corresponding to the eigenvalue* $1/\mu$.

These are precisely the assertions that are usually referred to as Fredholm's Theorems.

7. Fredholm's Fourth Theorem. We shall establish some properties of characteristic values of a completely continuous operator.

Theorem 4 (Fredholm's Fourth Theorem). *For any $M > 0$ the disc $\{|\mu| < M\}$ of the complex plane can contain only a finite number of characteristic values of a completely continuous operator acting from H into H with domain of definition H or, equivalently, only a finite number of eigenvalues can lie outside the disc $\{|\lambda| < 1/M\}$.*

Proof. Assume the contrary, that is, let the disc $\{|\mu| < M\}$ contain an infinite number of characteristic values $\mu_1, \ldots, \mu_n, \ldots$, $\mu_i \neq \mu_j$, $i \neq j$. Let e_i be some eigenelement corresponding to the characteristic value μ_i, $i = 1, 2, \ldots$.

We shall demonstrate that for any $n \geqslant 1$ the system $e_1, \ldots, e_n$ is linearly independent. When $n = 1$, this assertion is trivial. Let it be also true for $n-1$ elements. Assume that $e_1, \ldots, e_n$ are linearly dependent. Then $e_n = c_1 e_1 + \ldots + c_{n-1} e_{n-1}$ for some constants $c_1, \ldots, c_{n-1}$, not all zero. But $Ae_n = \frac{e_n}{\mu_n} = c_1 \frac{e_1}{\mu_1} + \ldots$ $\ldots + c_{n-1} \frac{e_{n-1}}{\mu_{n-1}}$; accordingly, $c_1\left(1 - \frac{\mu_n}{\mu_1}\right) e_1 + \ldots + c_{n-1}\left(1 - \frac{\mu_n}{\mu_{n-1}}\right) \times$ $\times e_{n-1} = o$, which is impossible, since $1 - \frac{\mu_n}{\mu_k} \neq 0$, $k = 1, \ldots, n-1$.

Let $\mathfrak{N}_n$ be the subspace spanned by the elements $e_1, \ldots, e_n$. From what has been shown above, it follows that $\mathfrak{N}_1 \subset \mathfrak{N}_2 \subset \ldots$ $\ldots \subset \mathfrak{N}_n \subset \ldots$ and $\mathfrak{N}_n \neq \mathfrak{N}_{n-1}$ for any n. Therefore, for any n an element $f_n \in \mathfrak{N}_n$ can be found such that $f_n \perp \mathfrak{N}_{n-1}$, $\| f_n \| = 1$. Since the set $f_1, f_2, \ldots, f_n, \ldots$ is bounded and the operator A is completely continuous, a fundamental subsequence can be chosen from the set $Af_1, \ldots, Af_n, \ldots$.

We shall demonstrate that this, in fact, cannot be done, which will be the contradiction required for the proof of the theorem.

For arbitrary integers m, n, $m < n$, $Af_n - Af_m = \frac{1}{\mu_n} f_n +$ $+ \frac{1}{\mu_n}(\mu_n Af_n - f_n) - Af_m = \frac{1}{\mu_n} f_n + \sigma_n$, where $\sigma_n \in \mathfrak{N}_{n-1}$, because $Af_m \in \mathfrak{N}_m \subset \mathfrak{N}_{n-1}$ and $\mu_n Af_n - f_n = \mu_n A\,(c_1 e_1 + \ldots + c_n e_n) - (c_1 e_1 +$ $\ldots + c_n e_n) = c_1\left(\frac{\mu_n}{\mu_1} - 1\right) e_1 + \ldots + c_{n-1}\left(\frac{\mu_n}{\mu_{n-1}} - 1\right) e_{n-1} \in \mathfrak{N}_{n-1}$. Therefore $\| Af_n - Af_m \| = \left\| \frac{1}{\mu_n} f_n + \sigma_n \right\| \geqslant \frac{1}{|\mu_n|} \| f_n \| = \frac{1}{|\mu_n|} \geqslant \frac{1}{M}$, that is, the sequence $Af_1, Af_2, \ldots, Af_n, \ldots$ cannot contain a fundamental subsequence. ▮

Theorem 4 implies that the *set of characteristic values of a completely continuous operator is at most countable (it may even be empty!). The characteristic values, if they exist, can be arranged in the order of non-decreasing moduli*

$$\mu_1, \mu_2, \ldots, \tag{9}$$

$|\mu_i| \leqslant |\mu_{i+1}|$, $i = 1, 2, \ldots$; *moreover, every characteristic value in the sequence* (9) *is counted as many times as its multiplicity. The* *set* (9) *may contain a finite number of elements* (*in particular, it may be empty*) *or infinitely-many elements. In the latter case*

$$|\mu_k| \to \infty \text{ as } k \to \infty. \tag{10}$$

With the sequence (9) *is associated the sequence of corresponding eigenelements*

$$e_1, e_2, \ldots, \tag{11}$$

which is linearly independent.

It will be proved in the following section that for a selfadjoint completely continuous operator $A \neq O$ the sets (9) and (11) are nonempty.

§ 5. SELFADJOINT COMPLETELY CONTINUOUS OPERATORS

1. Eigenvalues and Eigenelements of a Selfadjoint Completely Continuous Operator. Let A be a bounded selfadjoint linear operator from H into H. Since for all f, $\|f\| = 1$, $|(Af, f)| \leqslant \|A\|$, it follows that on the unit sphere $\|f\| = 1$ there exist exact upper and lower bounds of the quadratic form (Af, f) associated with the operator A: $m = \inf\limits_{\|f\|=1} (Af, f)$, $M = \sup\limits_{\|f\|=1} (Af, f)$; furthermore, $|m| \leqslant \|A\|$, $M \leqslant \|A\|$, $m \leqslant (Af, f) \leqslant M$.

If f is an arbitrary nonzero element of H, then the element $f/\|f\|$ belongs to the unit sphere; consequently, $m = \inf\limits_{f \in H} \frac{(Af, f)}{\|f\|^2}$, $M = \sup\limits_{f \in H} \frac{(Af, f)}{\|f\|^2}$, and hence the inequalities $m\|f\|^2 \leqslant (Af, f) \leqslant M\|f\|^2$ hold for all $f \in H$.

Since the quadratic form of operator A is real-valued, *all its eigenvalues* (*characteristic values*) *are real*: if λ is an eigenvalue and f the corresponding eigenelement, that is, $Af = \lambda f$, then $\lambda = (Af, f)/\|f\|^2$. Therefore, $m \leqslant \lambda \leqslant M$.

The *eigenelements* f_1 and f_2 of operator A *corresponding to different eigenvalues* λ_1 *and* λ_2 *are orthogonal.* In fact, scalar multiplication of the relations $Af_1 = \lambda_1 f_1$ and $Af_2 = \lambda_2 f_2$ by f_2 and f_1, respectively, and subsequent subtraction yield $(Af_1, f_2) - (f_1, Af_2) = (\lambda_1 - \lambda_2)(f_1, f_2)$. This implies $(f_1, f_2) = 0$, since $(Af_1, f_2) = (f_1, Af_2)$ and $\lambda_1 \neq \lambda_2$.

Lemma 1. *For the number* $M = \sup\limits_{\|f\|=1} (Af, f)$ *to be an eigenvalue and* f_0 *the corresponding eigenelement* (*it is assumed that* $\|f_0\| = 1$) *of a selfadjoint bounded operator* A *from* H *into* H, *it is necessary and sufficient that* $(Af_0, f_0) = M$.

Similarly, for the number $m = \inf\limits_{\|f\|=1} (Af, f)$ *to be an eigenvalue and* f_0 *the corresponding eigenelement (it is assumed that* $\|f_0\| = 1$) *of the operator* A, *it is necessary and sufficient that* $(Af_0, f_0) = m$.

Proof. If M is an eigenvalue and f_0 any corresponding eigenelement of the operator A, then $Af_0 = Mf_0$. Therefore $(Af_0, f_0) = M(f_0, f_0) = M$, which proves the necessity part of the lemma.

To prove the sufficiency, let $(Af_0, f_0) = M$ for some f_0, $\|f_0\| = 1$, or, equivalently, $(Mf_0 - Af_0, f_0) = 0$. Since for all f in H $0 \leqslant M\|f\|^2 - (Af, f) = (Mf - Af, f)$, it follows that for an arbitrary φ in H and any complex t $(M(f_0 + t\varphi) - A(f_0 + t\varphi), f_0 + t\varphi) \geqslant 0$, that is, $\bar{t}(Mf_0 - Af_0, \varphi) + t\overline{(Mf_0 - Af_0, \varphi)} + |t|^2(M\varphi - A\varphi, \varphi) \geqslant 0$. Putting in this inequality $t = -\sigma e^{i\alpha}$, where $\alpha = \arg(Mf_0 - Af_0, \varphi)$ and σ is real, we obtain the inequality $-2\sigma\,|(Mf_0 - Af_0, \varphi)| + \sigma^2(M\varphi - A\varphi, \varphi) \geqslant 0$, true for all real σ. This inequality implies $(Mf_0 - Af_0, \varphi) = 0$, and, since φ is arbitrary, $Mf_0 - Af_0 = 0$.

The second part of the lemma follows from the first part if instead of A one considers the operator $-A$. ▮

Lemma 2. *If the operator* A *acting from* H *into* H *is selfadjoint and completely continuous, then the quantity* $M = \sup\limits_{\|f\|=1}(Af, f)$ *(similarly, the quantity* $m = \inf\limits_{\|f\|=1}(Af, f)$), *if different from zero, is an eigenvalue of this operator.*

Proof. Suppose $M \neq 0$. We consider the Hermitian bilinear from $(Mf - Af, g)$, $f, g \in H$, and the corresponding quadratic form $(Mf - Af, f)$. For all f in H $(Mf - Af, fs) \geqslant 0$.

We shall demonstrate that there is a nonzero element f_0 such that $(Mf_0 - Af_0, f_0) = 0$. Then Lemma 2 will follow from Lemma 1.

Assume that no such f_0 exists. Then $(Mf - Af, f)$, $f \in H$, can vanish only when $f = o$. Therefore the bilinear form $(Mf - Af, g)$ can be taken as a new scalar product in H. This means that for any f, g in H Bunyakovskii's inequality

$$|(Mf - Af, g)|^2 \leqslant (Mf - Af, f)(Mg - Ag, g) \tag{1}$$

holds.

From the definition of M as the exact upper bound of the quadratic form (Af, f) on the unit sphere $\|f\| = 1$ it follows that there is a sequence $f_1, f_2, \ldots$, $\|f_i\| = 1$, $i = 1, 2, \ldots$, for which $(Af_n, f_n) \to M$ or

$$(Mf_n - Af_n, f_n) \to 0, \quad n \to \infty.$$

Putting in (1) $f = f_n$, $g = Mf_n - Af_n$, we obtain $\|Mf_n - Af_n\|^4 \leqslant (Mf_n - Af_n, f_n)\ (M^2f_n - 2MAf_n + A^2f_n,\ Mf_n - Af_n) \leqslant (Mf_n - Af_n, f_n)(|M| + \|A\|)^3$. Therefore, in view of (2), it follows that the sequence $Mf_n - Af_n \to 0$ as $n \to \infty$. Since the operator A is completely continuous and the sequence $f_1, f_2, \ldots$

($\| f_i \| = 1$) is bounded, the sequence $Af_1, Af_2, \dots$ is compact. This means that a convergent subsequence can be chosen from it; we assume that it coincides with $Af_1, Af_2, \dots$. Then the sequence $Mf_1, Mf_2, \dots$, and together with it ($M \neq 0$) also the sequence $f_1, f_2, \dots$, converges. If the limit of the sequence $f_1, f_2, \dots$ is denoted by f_0, then, evidently, $\| f_0 \| = 1$, and by (2) $(Mf_0 - Af_0, f_0) = 0$.∎

Theorem 1. *For every nonzero selfadjoint completely continuous operator A from H into H (H is a Hilbert space) one of the quantities $\pm 1/\| A \| = \pm 1/\sup\limits_{\|f\|=1} |(Af, f)|$ is the first (smallest in absolute value) characteristic value μ_1, and $\mu_1 = 1/M$ if $M > |m|$, where $M = \sup\limits_{\|f\|=1} (Af, f)$, $m = \inf\limits_{\|f\|=1} (Af, f)$, $\mu_1 = 1/m$ if $M < |m|$. If $M = |m|$, then both the quantities $1/m$ and $1/M$ are characteristic values, smallest in absolute values, of the operator A.*

All the elements f_0 for which $(Af_0, f_0)/\| f_0 \|^2 = M$ when $M > |m|$ or $(Af_0, f_0)/\| f_0 \|^2 = m$ when $M < |m|$, and only they, are eigenelements corresponding to μ_1. If $M = |m|$, then eigenelements corresponding to the characteristic value $1/M$ are only those f_0 for which $(Af_0, f_0)/\| f_0 \|^2 = M$, while eigenelements corresponding to the characteristic value $1/m$ are only those f_0 for which $(Af_0, f_0)/\| f_0 \|^2 = m$.

In particular, if the operator A is nonnegative, then

$$\mu_1 = \frac{1}{\| A \|} = \frac{1}{\sup\limits_{\|f\|=1} (Af, f)} = \inf_{\|f\|=1} \frac{1}{(Af, f)} = \inf_{f \in H} \frac{\| f \|^2}{(Af, f)}$$

and the eigenelements corresponding to μ_1, normalized by the condition $\| f \| = 1$, are only those elements f_0, $\| f_0 \| = 1$, where the quadratic form (Af, f) attains its upper bound on the unit sphere.

Proof. To establish Theorem 1, it suffices, in view of Lemmas 1 and 2, to demonstrate that $\| A \| = N$, where $N = \sup\limits_{\|f\|=1} |(Af, f)| = \max(|m|, M)$. As shown above, $N \leqslant \| A \|$, so that it remains only to establish the reverse inequality $\| A \| \leqslant N$.

Since for any $g \in H$ $|(Ag, g)| \leqslant N \| g \|^2$ and since $(A(f_1 \pm f_2), f_1 \pm f_2) = (Af_1, f_1) + (Af_2, f_2) \pm 2\mathrm{Re}\,(Af_1, f_2)$, we have for any $f_1, f_2 \in H$

$$\begin{aligned}
|\mathrm{Re}\,(Af_1, f_2)| &= \frac{1}{4} |(A(f_1 + f_2), f_1 + f_2) - (A(f_1 - f_2), f_1 - f_2)| \\
&\leqslant \frac{1}{4} (|(A(f_1 + f_2), f_1 + f_2)| + |(A(f_1 - f_2), f_1 - f_2)|) \\
&\leqslant \frac{N}{4} (\| f_1 + f_2 \|^2 + \| f_1 - f_2 \|^2) = \frac{N}{2} (\| f_1 \|^2 + \| f_2 \|^2).
\end{aligned}$$

Putting in this inequality $f_1 = \sqrt{N} f$, $f_2 = \frac{1}{\sqrt{N}} Af$, where f is an arbitrary element of H, we have $\|Af\|^2 \leqslant \frac{N}{2}\left(N\|f\|^2 + \frac{1}{N}\|Af\|^2\right)$, whence it follows that $\|Af\| \leqslant N\|f\|$. Accordingly, $\|A\| \leqslant N$. ∎

Thus the sets

$$\mu_1, \mu_2, \ldots, \tag{3}$$

$$e_1, e_2, \ldots \tag{4}$$

of characteristic values and corresponding eigenelements of a selfadjoint completely continuous operator $A \neq O$ are not empty. In this case all the characteristic values are real and the system of eigenelements can be assumed to be orthonormal, since eigenelements corresponding to different characteristic values are orthogonal and the finite number of linearly independent eigenelements corresponding to a given characteristic value can be orthonormalized.

Let A be a completely continuous selfadjoint operator acting from H into H. By H_n we denote the subspace of H consisting of elements f that are orthogonal to the first n eigenelements of the operator A: $(f, e_i) = 0$, $i = 1, \ldots, n$.

For any f in H_n, the element Af is also in H_n, since $(Af, e_i) = (f, Ae_i) = \frac{1}{\mu_i}(f, e_i) = 0$ for all $i = 1, \ldots, n$. This means that the operator A can be regarded as an operator acting from the Hilbert space H_n into H_n. And it is, of course, selfadjoint and completely continuous. Its characteristic values and the corresponding eigenelements coincide with the characteristic values $\mu_{n+1}, \mu_{n+2}, \ldots$ and the corresponding eigenelements $e_{n+1}, e_{n+2}, \ldots$ of the operator A acting from H into H. Therefore Theorem 1, applied to the operator A from H_n into H_n, yields

$$|\mu_{n+1}| = \frac{1}{\sup\limits_{\substack{\|f\|=1\\ f\in H_n}} |(Af, f)|} = \frac{1}{\sup\limits_{\substack{\|f\|=1\\ (f, e_i)=0\\ i=1, \ldots, n}} |(Af, f)|}.$$

If the operator A is nonnegative, then

$$\mu_{n+1} = \frac{1}{\sup\limits_{\substack{\|f\|=1\\ (f, e_i)=0\\ i=1, \ldots, n}} (Af, f)} = \inf_{\substack{(f, e_i)=0\\ i=1, \ldots, n}} \frac{\|f\|^2}{(Af, f)}. \tag{5}$$

2. Fourier Expansion with Respect to Eigenelements of a Selfadjoint Completely Continuous Operator. Consider the orthonormal system (4) which consists of the eigenelements of a selfadjoint completely continuous operator A from H into H, $A \neq O$. Let P_n be the operator of orthogonal projection onto the subspace spanned by the elements $e_1, \ldots, e_n$, and let the operator $A_n = A - AP_n = A(I - P_n)$.

The operator A_n is linear and bounded: $\| A_n \| \leqslant \| A \|$.

The operator P_n commutes with the operator A, because for any $f \in H$

$$AP_nf = A(f_1e_1 + \ldots + f_ne_n) = f_1Ae_1 + \ldots + f_nAe_n$$
$$= \frac{f_1}{\mu_1}e_1 + \ldots + \frac{f_n}{\mu_n}e_n = \left(f, \frac{e_1}{\mu_1}\right)e_1 + \ldots + \left(f, \frac{e_n}{\mu_n}\right)e_n$$
$$= (f, Ae_1)e_1 + \ldots + (f, Ae_n)e_n = (Af, e_1)e_1 + \ldots + (Af, e_n)e_n = P_nAf.$$

Since the operators A and P_n commute and are selfadjoint, the operator AP_n is selfadjoint: $(AP_n)^* = P_n^*A^* = P_nA = AP_n$. Accordingly, the operator A_n is also selfadjoint.

Furthermore, the operator A_n as the sum of two completely continuous operators, the operator A and the finite-dimensional operator $AP_n = P_nA$, is completely continuous.

For any $f \in H$

$$A_nf = Af - \sum_{k=1}^{n} \frac{f_k}{\mu_k} e_k. \tag{6}$$

The quantities $\mu_{n+1}, \ldots$ and the elements $e_{n+1}, \ldots$ are characteristic values and corresponding eigenelements of the operator A_n. Indeed, since $(e_p, e_k) = 0$, $k \neq p$, using (6) we have $A_ne_p = = Ae_p - \sum_{k=1}^{n} \frac{(e_p, e_k)}{\mu_k} e_k = \frac{e_p}{\mu_p}$ for $p \geqslant n + 1$.

The operator A_n has no other characteristic values. Assume, on the contrary, that μ and e are the characteristic value and eigenelement, $\mu A_ne = e$, $\mu \neq \mu_p$, $p \geqslant n + 1$. Scalar multiplication of the last equality by e_k, $k \leqslant n$, yields $(e, e_k) = \mu(A_ne, e_k) = \mu(e, A_ne_k) = 0$, since $A_ne_k = o$ when $k \leqslant n$. Therefore, noting (6), $A_ne = Ae$, that is, $\mu Ae = e$. Thus μ is the characteristic value and e is the eigenelement of the operator A. But all the characteristic values of A are contained in sequence (3) and since $e \perp e_k$, $k = = 1, \ldots, n$ it follows that μ coincides with one of μ_k, $k \geqslant n + 1$.

Since μ_{n+1} is characteristic value, smallest in absolute value, of the operator A_n, Theorem 1 yields

$$|\mu_{n+1}| = \frac{1}{\| A_n \|}. \tag{7}$$

If the sequences (3) and (4) are finite and contain m elements, then, according to Theorem 1, $A_m = O$, since it has no characteristic values. In this case $A = AP_m$ is a finite-dimensional operator, that is, for any $f \in H$

$$Af = \sum_{k=1}^{m} \frac{f_k}{\mu_k} e_k = \sum_{k=1}^{m} (Af)_k e_k. \tag{8}$$

When the sequences (3) and (4) are infinite, (7) and relation (10) of the preceding section imply that $\|A_n\| \to 0$ as $n \to \infty$. This means that for any $f \in H$ $\|A_n f\| \leqslant \|A_n\| \, \|f\| \to 0$ as $n \to \infty$, that is, for any $f \in H$

$$Af = \lim_{n\to\infty} AP_n f = \sum_{k=1}^{\infty} \frac{f_k}{\mu_k} e_k = \sum_{k=1}^{\infty} (Af)_k e_k. \tag{9}$$

We have thus established the following important theorem.

Theorem 2 (Hilbert-Schmidt). *If A is the selfadjoint completely continuous operator from H into H and f is an arbitrary element of H, then the element Af has Fourier expansion* (9) (*or* (8)) *with respect to the system* (4).

In our later discussions we shall require some corollaries of the Hilbert-Schmidt theorem.

According to Lemma 2, Sec. 2.6, the Fourier series $\sum_{k=1}^{\infty} f_k e_k$ of any element $f \in H$ with respect to the orthonormal system (4) converges in H; consequently, $A \sum_{k=1}^{\infty} f_k e_k = \sum_{k=1}^{\infty} f_k A e_k$. But $f_k A e_k = f_k \frac{e_k}{\mu_k} = (f, Ae_k)\, e_k = (Af, e_k)\, e_k = (Af)_k\, e_k$. Therefore by (9) we have

$$A\Big(f - \sum_{k=1}^{\infty} f_k e_k\Big) = o. \tag{10}$$

If the operator A has an inverse A^{-1}, then (10) yields

$$f = \sum_{k=1}^{\infty} f_k e_k$$

for any element $f \in H$. This means that in this case the system (4) is an orthonormal basis for the space H. Thus we have proved the following result.

Corollary 1. *If the selfadjoint completely continuous operator A from H into H has an inverse, then the system* (4) *is an orthonormal basis for the space H.*

In the general case, from (10) we can only conclude that for any element $f \in H$ there exists an element $e_0 \in H$, $Ae_0 = 0$, such that

$$f = e_0 + \sum_{k=1}^{\infty} f_k e_k. \tag{11}$$

The set $\mathfrak{N}$ containing the elements $g \in H$ for which $Ag = 0$ is a subspace of H; any nonzero element of $\mathfrak{N}$ is an eigenelement of the operator A corresponding to the zero eigenvalue. If the space H is assumed separable, a countable orthonormal basis e'_1, e'_2, . . . (consisting of eigenelements of the operator A corresponding to the zero eigenvalue) can be constructed in $\mathfrak{N}$. Then, by (11), for any $f \in H$ we have the expansion

$$f = \sum f'_k e'_k + \sum f_k e_k,$$

where $f'_k = (f, e'_k)$.

Thus we have proved

Corollary 2. *For any selfadjoint completely continuous operator A from the separable (Hilbert) space H into H there exists an orthonormal basis for H whose elements are the eigenelements of the operator A.*

CHAPTER III

FUNCTION SPACES

In the preceding chapter we introduced the notions of Banach and Hilbert spaces. These notions were based only on the relationships between elements: it was enough to introduce the operations of addition of elements and multiplication by numbers, norm or corresponding scalar product that satisfy some definite axioms. The nature of elements of these spaces was not at all important, and the general results obtained in the last chapter are applicable to all the spaces irrespective of the elements they are composed of. However, for the theory of differential equations these general properties are far from sufficient. In the investigation of partial differential equations it is natural to consider function spaces, that is, spaces whose elements are, for our purposes, functions of n real variables. In the present chapter we shall introduce some function spaces and obtain assertions regarding mutual relationships between them that enable us to conclude from some properties of their elements various other properties.

§ 1. SPACES OF CONTINUOUS AND CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

1. Normed Spaces $C(\bar{Q})$ and $C^k(\bar{Q})$. We consider the set $C(\bar{Q})$ of all the functions that are continuous in $\bar{Q}$ ($\bar{Q}$ is a bounded region in the space R_n). Let us first note that this set is a linear space. It can be directly verified that the functional $\max\limits_{x\in\bar{Q}} |f(x)|$ defined on $C(\bar{Q})$ satisfies all the axioms for being a norm (see Sec. 2.2., Chap. II): $\max\limits_{x\in\bar{Q}} |cf| = |c| \max\limits_{x\in\bar{Q}} |f|$; $|f_1(x) + f_2(x)| \leqslant |f_1(x)| + |f_2(x)|$ for all $x \in \bar{Q}$, therefore $\max\limits_{x\in\bar{Q}} |f_1(x) + f_2(x)| \leqslant$

$\leqslant \max\limits_{x \in \overline{Q}} |f_1(x)| + \max\limits_{x \in \overline{Q}} |f_2(x)$; $\max\limits_{x \in \overline{Q}} |f(x)| \geqslant 0$ and $\max\limits_{x \in \overline{Q}} |f(x)| = 0$ only when $f(x) \equiv 0$. Thus in $C(\overline{Q})$ we may introduce the norm

$$\|f\|_{C(\overline{Q})} = \max_{x \in \overline{Q}} |f(x)|. \tag{1}$$

The convergence in norm (1) coincides with uniform convergence in $\overline{Q}$.

The space $C(\overline{Q})$ with norm (1) is Banach space, because, by the Cauchy criterion, any sequence of functions in $C(\overline{Q})$ that is fundamental in norm (1) converges uniformly to a function belonging to $C(\overline{Q})$.

Since, by Weirstrass's theorem, every function continuous in $\overline{Q}$ is the limit of a sequence of polynomials that converges uniformly in $\overline{Q}$ (that is, in the norm (1)), the set of all the polynomials is everywhere dense in $C(\overline{Q})$. But, in turn, an arbitrary polynomial can be expressed as the limit of the sequence of polynomials with rational coefficients that converges uniformly in $\overline{Q}$. Therefore the countable set of all polynomials with rational coefficients is also everywhere dense in $C(\overline{Q})$. This means that the space $C(\overline{Q})$ is separable.

We consider in $C(\overline{Q})$ the set $\overset{\circ}{C}(\overline{Q})$ containing all the functions that vanish on the boundary ∂Q of Q. Clearly, $\overset{\circ}{C}(\overline{Q})$ is a linear manifold in $C(\overline{Q})$. This manifold is closed (in the norm (1)), because the limit of a sequence of functions in $\overset{\circ}{C}(\overline{Q})$ converging uniformly in $\overline{Q}$ is a function belonging to $\overset{\circ}{C}(\overline{Q})$; accordingly, $\overset{\circ}{C}(\overline{Q})$ is a subspace of the space $C(\overline{Q})$.

Next, in $C(\overline{Q})$ consider the subsets $C^k(\overline{Q})$, $k = 1, 2, \ldots$, consisting of the functions whose all the derivatives up to order k are continuous in $\overline{Q}$. The set $C^k(\overline{Q})$ is a linear space. What is more, in $C^k(\overline{Q})$ one may introduce the norm

$$\|f\|_{C^k(\overline{Q})} = \sum_{|\alpha| \leqslant k} \max_{x \in \overline{Q}} |D^\alpha f(x)|. \tag{2}$$

Convergence in this norm means the uniform convergence in $\overline{Q}$ of the functions and all their derivatives up to order k. Evidently, $C^k(\overline{Q})$ is a Banach space (with norm (2)).

Let $\omega_h(|x-y|)$ be some averaging kernel (see Chap. I, Introduction) and $f \in C(\overline{Q})$. With $h > 0$, consider the function

$$f_h(x) = \int_Q f(y)\, \omega_h(|x-y|)\, dy, \qquad x \in R_n. \tag{3}$$

Functions $f_h(x)$, $h > 0$, are called the *average functions* for the function $f(x)$ (the *averaging functions* for $f(x)$). From the Property (a) of an averaging kernel and Theorem 7, Sec. 1.7, Chap. II, it follows that $f_h(x) \in C^\infty(R_n)$ for any $h > 0$. Further, $f_h(x) \equiv 0$ outside Q^h (Q^h is the union of balls $\{|x - x^0| < h\}$ taken over all $x^0 \in Q$).

Let us show that *if* $f \in C(\overline{Q})$, *then* $f_h(x)$ *converges to* $f(x)$, *as* $h \to 0$, *uniformly on any strictly interior subregion* Q' *of* Q, $Q' \Subset Q$. In fact, for sufficiently small h (less than the distance between $\partial Q'$ and ∂Q) by Properties (b), (c) and (a) of the averaging kernel we have, for $x \in Q'$,

$$|f_h(x) - f(x)| = \left| \int\limits_{|x-y|<h} f(y)\,\omega_h(|x-y|)\,dy - f(x) \int\limits_{|x-y|<h} \omega_h(|x-y|)\,dy \right|$$

$$\leqslant \max_{|x-y|\leqslant h} |f(y) - f(x)| \int\limits_{|x-y|<h} \omega_h(|x-y|)\,dy = \max_{|x-y|\leqslant h} |f(y) - f(x)|.$$

Therefore, by uniform convergence of $f(x)$ in Q, we obtain

$$\|f_h - f\|_{C(\overline{Q}')} \to 0 \text{ as } h \to 0.$$

Since for $f \in C^k(\overline{Q})$, with $x \in Q'$ and sufficiently small h,

$$D_x^\alpha f_h(x) = \int\limits_Q f(y)\,D_x^\alpha \omega_h(|x-y|)\,dy$$

$$= (-1)^{|\alpha|} \int\limits_Q f(y)\,D_y^\alpha \omega_h(|x-y|)\,dy$$

$$= \int\limits_Q D_y^\alpha f(y) \cdot \omega_h(|x-y|)\,dy, \qquad |\alpha| \leqslant k,$$

the above assertion implies that

if $f \in C^k(\overline{Q})$, *then for any* $Q' \Subset Q$

$$\|f_h - f\|_{C^k(\overline{Q}')} \to 0 \textit{ as } h \to 0.$$

2. Formulas of Integration by Parts. Suppose that in the region Q (the boundary $\partial Q \in C^1$) there is defined a vector $A(x) = (A_1(x), \ldots, A_n(x))$, whose components $A_i(x) \in C(\overline{Q}) \cap C^1(Q)$, $i = 1, \ldots, n$. From the Course of Analysis it is known that if the function $\operatorname{div} A(x) \equiv \frac{\partial A_1}{\partial x_1} + \ldots + \frac{\partial A_n}{\partial x_n}$ is continuous in $\overline{Q}$, or even integrable over Q, then the following formula, known as *Ostrogradskii's formula*, holds:

$$\int\limits_Q \operatorname{div} A(x)\,dx = \int\limits_{\partial Q} A(x)\,n(x)\,dS, \tag{4}$$

where n is the unit vector normal to the boundary ∂Q and directed outwards with respect to Q.

Let $u(x) \in C^2(Q) \cap C^1(\bar{Q})$, $v \in C^1(\bar{Q})$, and let the function $\Delta u = \operatorname{div}(\nabla u)$ be integrable over Q. Since $v\,\Delta u = v \cdot \operatorname{div}(\nabla u) = \operatorname{div}(v\nabla u) - \nabla u \nabla v$ $(\nabla u \nabla v = u_{x_1} v_{x_1} + \ldots + u_{x_n} v_{x_n})$, by Ostrogradskii's formula (4) we have

$$\int_Q v\Delta u\, dx = \int_{\partial Q} v \frac{\partial u}{\partial n}\, dS - \int_Q \nabla u\, \nabla v\, dx, \tag{5}$$

because $\nabla u \cdot n|_{\partial Q} = \left.\frac{\partial u}{\partial n}\right|_{\partial Q}$.

If both the functions u and v belong to $C^2(Q) \cap C^1(\bar{Q})$ and the functions Δu and Δv are integrable over Q, then, apart from formula (5), we also have

$$\int_Q u\Delta v\, dx = \int_{\partial Q} u \frac{\partial v}{\partial n}\, dS - \int_Q \nabla v \nabla u\, dx. \tag{5'}$$

Subtraction of (5′) termwise from (5) yields

$$\int_Q (v\Delta u - u\Delta v)\, dx = \int_{\partial Q} \left(v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n}\right) dS. \tag{6}$$

Formulas (5) and (6) are known as *Green's formulas.*

§ 2. SPACES OF INTEGRABLE FUNCTIONS

As shown above, the set of continuous functions in $\bar{Q}$ is a Banach space with the norm $\max_{x \in \bar{Q}} |f(x)|$. Nevertheless, it often proves convenient to consider on this set integral norms, for example, $\int_Q |f(x)|\, dx$ or $(\int_Q |f(x)|^2\, dx)^{1/2}$ (it is easy to see that they satisfy all the axioms of a norm). We examine the space with norm $\int_Q |f(x)|\, dx$ whose elements are continuous functions in $\bar{Q}$. This normed space is incomplete. In fact, from the definition of Lebesgue integral it follows that for any function $f(x)$ integrable over Q there is a sequence $f_m(x)$ of functions continuous in $\bar{Q}$ which converges to $f(x)$ in this norm: $\int_Q |f_m(x) - f(x)|\, dx \to 0$ as $m \to \infty$. This means that if we wish to obtain a complete normed (Banach) space with the norm $\int_Q |f(x)|\, dx$ which would contain all the functions continuous (or even infinitely differentiable) in $\bar{Q}$, we must include all those functions which are integrable over Q. But then the functional $\int\limits_Q |f(x)|\,dx$ ceases to be a norm—it does not satisfy the last axiom (see Sec. 2.2, Chap. II) of a norm, since $\int\limits_Q |f(x)|\,dx = 0$ for all $f(x) = 0$ a.e. in Q. However, according to Theorem 2, Sec. 1.4, Chap. II, $\int\limits_Q |f(x)|\,dx = 0$ only for functions $f(x)$ that vanish a.e. in Q. Therefore, for the last axiom of the norm to be satisfied, we must *identify* all the functions that vanish a.e. in Q. For this we may consider as the elements of the space either classes of functions each of which contains all the functions that are equal a.e. or else, equivalently, may introduce a new definition of equality of functions: *functions are equal if their values coincide almost everywhere.* Since it is more convenient to operate with functions than with classes of functions, in the sequel we shall regard functions equal if their values coincide for almost all (not necessarily for all) x in Q. Since in such a definition of equality of functions the functions remain unchanged when their values change arbitrarily on any fixed set of measure zero, in this case it is natural to assume that the functions are defined almost everywhere. If a function f vanishes almost everywhere, we take it as the zero function. Similarly, if a function coincides almost everywhere with an everywhere defined continuous function, we regard it as a continuous function, while f is continuously differentiable up to order k if it coincides almost everywhere with a function that is everywhere defined and is continuously differentiable up to order k. In accordance with the abovementioned notion of equality, we shall assume that the space $C^k(\bar{Q})$, $k \geqslant 0$, contains also the functions defined almost everywhere in Q and continuously differentiable up to order k. That is, a function $f(x)$ belongs to $C^k(\bar{Q})$ if it coincides almost everywhere with a function which is defined at all points of $\bar{Q}$ and is continuous in $\bar{Q}$ together with all its derivatives up to order k. Further, by the value of an element of the space $C(\bar{Q})$ (and, more so, of functions in $C^k(\bar{Q})$) at some point we shall mean the value at this point of the continuous function defined everywhere that coincides with this element almost everywhere in Q.

1. Spaces $L_1(Q)$ and $L_2(Q)$. Consider the set of complex-valued functions that are integrable over Q. Obviously, it is a linear space (also in the new sense of equality of functions), and the functional $\int\limits_Q |f(x)|\,dx$ satisfies all the axioms of a norm. This linear space will be denoted by $L_1(Q)$:

$$\|f\|_{L_1(Q)} = \int_Q |f(x)|\,dx. \tag{1}$$

The set of complex-valued measurable functions (recall that functions coinciding a. e. are identified) the squares of whose moduli are integrable over Q will be denoted by $L_2(Q)$. Let us demonstrate that $L_2(Q)$ is a linear space. Let c_1, c_2 be arbitrary numbers, and $f_1(x)$, $f_2(x)$ arbitrary functions in $L_2(Q)$. Since the measurable function $c_1f_1(x) + c_2f_2(x)$ satisfies the inequality $|c_1f_1(x) + c_2f_2(x)|^2 \leqslant 2|c_1|^2|f_1(x)|^2 + 2|c_2|^2|f_2(x)|^2$, by Theorem 5, Sec. 1.6, Chap. II, the function $|c_1f_1(x) + c_2f_2(x)|^2$ is integrable, which implies that $c_1f_1(x) + c_2f_2(x) \in L_2(Q)$.

The function $f_1(x)\,f_2(x)$, where $f_1(x), f_2(x) \in L_2(Q)$, is integrable, because it is measurable and $|f_1(x)\,f_2(x)| \leqslant \frac{1}{2}(|f_1(x)|^2 + |f_2(x)|^2)$. Therefore with the pair of functions f_1 and f_2 one can associate the quantity

$$(f_1, f_2)_{L_2(Q)} = \int_Q f_1(x)\,\overline{f_2(x)}\,dx. \tag{2}$$

It is easily checked that formula (2) defines a scalar product in $L_2(Q)$. The norm generated by this scalar product is of the form

$$\|f\|_{L_2(Q)} = \left(\int_Q |f(x)|^2\,dx\right)^{1/2}. \tag{3}$$

Since $|f| = |f|\cdot 1 \leqslant \frac{1}{2}(|f|^2 + 1)$, we find that when the region Q is bounded the function $f(x)$ belonging to $L_2(Q)$ also belongs to $L_1(Q)$. This means that $L_2(Q) \subset L_1(Q)$ for a bounded region. In this case it is also evident that $C(\overline{Q}) \subset L_2(Q) \subset L_1(Q)$.

Theorem 1. *$L_1(Q)$ is a Banach space with norm* (1). *$L_2(Q)$ is a Hilbert space with scalar product* (2).

To prove this theorem, it suffices to establish that the spaces $L_1(Q)$ and $L_2(Q)$ are complete in the respective norms.

1. Let the sequence f_k, $k = 1, 2, \ldots$, of functions in $L_1(Q)$ be fundamental in $L_1(Q)$, that is, for any $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that $\|f_k - f_m\|_{L_1(Q)} < \varepsilon$ for all $k, m \geqslant N(\varepsilon)$. Choose $\varepsilon = 2^{-k}$ for some integer k, and let N_k denote the number $N(2^{-k})$ such that $N_k \leqslant N_{k+1}$. Then for $m \geqslant N_k$ we have

$$\|f_{N_k} - f_m\|_{L_1(Q)} < 2^{-k}, \tag{4}$$

and, in particular, $\|f_{N_k} - f_{N_{k+1}}\|_{L_1(Q)} < 2^{-k}$. Hence the series $\sum_{k=1}^{\infty} \|f_{N_k} - f_{N_{k+1}}\|_{L_1(Q)}$ converges. This, in view of Corollary to

Sec. 1.6, Chap. II, means that the series $\sum\limits_{k=1}^{\infty}(f_{N_{k+1}}-f_{N_k})$ converges a. e. in Q to a function belonging to $L_1(Q)$, which, in turn, means that the sequence f_{N_k}, $k=1, 2, \ldots$, converges, as $k\to\infty$, a.e. in Q to a function $f\in L_1(Q)$:

$$f_{N_k}(x)\to f(x), \qquad k\to\infty.$$

We shall show that $\|f_m-f\|_{L_1(Q)}\to 0$ as $m\to\infty$. Indeed, for $m\geqslant N_r$, $k\geqslant r$ the inequality (4) implies

$$| f_m-f_{N_k}\|_{L_1(Q)}\leqslant\|f_m-f_{N_r}\|_{L_1(Q)}+\|f_{N_r}-f_{N_k}\|_{L_1(Q)}\leqslant 2\cdot 2^{-r}=2^{1-r}.$$

Letting $k\to\infty$ in this inequality, we obtain, by Fatou's lemma (Theorem 4, Sec. 1.6, Chap. II), the inequality $\|f_m-f\|_{L_1(Q)}\leqslant$ $\leqslant 2^{1-r}$ which is true for all $m\geqslant N_r$. For sufficiently large m the number r can be chosen large enough, therefore $\|f_m-f\|_{L_1(Q)}\to 0$ as $m\to\infty$. Thus the space $L_1(Q)$ is complete.

2. Suppose that the members of the sequence $f_k(x)$, $k=1, 2, \ldots$, belong to $L_2(Q)$ and that the sequence is fundamental in the norm $L_2(Q)$. As in the proof of the first part of this theorem, a number sequence $N_1\leqslant N_2\leqslant\ldots\leqslant N_k\leqslant\ldots$ can be found such that

$$\|f_{N_k}-f_m\|_{L_2(Q)}<2^{-k} \tag{4'}$$

for all $m\geqslant N_k$, and, in particular, $\|f_{N_{k+1}}-f_{N_k}\|_{L_2(Q)}<2^{-k}$. An application of Bunyakovskii's inequality gives $\|f_{N_{k+1}}-$ $-f_{N_k}\|_{L_1(Q)}\leqslant\sqrt{|Q|}\,\|f_{N_{k+1}}-f_{N_k}\|_{L_2(Q)}<\sqrt{|Q|}\,2^{-k}$, therefore there is a function $f(x)\in L_1(Q)$ such that $f_{N_k}(x)\to f(x)$, as $k\to\infty$, a.e. in Q. This means that $|f_{N_k}|^2\to|f|^2$, as $k\to\infty$, a.e. in Q. Furthermore,

$$\|f_{N_k}\|_{L_2(Q)}\leqslant\|f_{N_k}-f_{N_1}\|_{L_2(Q)}+\|f_{N_1}\|_{L_2(Q)}\leqslant\frac{1}{2}+\|f_{N_1}\|_{L_2(Q)}.$$

Hence, by Fatou's lemma, $f(x)\in L_2(Q)$.

Let us show that $\|f_m-f\|_{L_2(Q)}\to 0$ as $m\to\infty$. For $m\geqslant N_r$ and $k\geqslant r$ inequality (4′) implies

$$\|f_m-f_{N_k}\|_{L_2(Q)}\leqslant\|f_m-f_{N_r}\|_{L_2(Q)}+\|f_{N_r}-f_{N_k}\|_{L_2(Q)}\leqslant 2^{1-r}.$$

Letting $k\to\infty$ in this inequality and again using Fatou's lemma, we conclude that $\|f_m-f\|_{(L_2Q)}\leqslant 2^{1-r}$ for all $m\geqslant N_r$. And since with sufficiently large m, r can be chosen sufficiently large, we obtain $\|f_m-f\|_{L_2(Q)}\to 0$ as $m\to\infty$. ▮

Remark. We remark that in proving Theorem 1 we have at the same time established the following assertion.

From every sequence of functions converging to a function f in $L_1(Q)$ *or in* $L_2(Q)$ *a subsequence can be chosen that converges to f a.e.*

2. Denseness of the Set $C(\overline{Q})$ in $L_1(Q)$ and $L_2(Q)$. Separability of $L_1(Q)$ and $L_2(Q)$. Continuity in the Mean of Elements of $L_1(Q)$ and $L_2(Q)$.

Theorem 2. *The set of functions continuous in* $\overline{Q}$ *is everywhere dense in* $L_1(Q)$ *and* $L_2(Q)$.

1. Let $f(x) \in L_1(Q)$. With no loss of generality this function can be assumed real-valued and nonnegative. Then by the definition of integrability of $f(x)$, there exists a sequence $f_k(x)$, $k = 1, 2, \ldots$, of functions continuous in $\overline{Q}$ having the property that $f_k(x) \uparrow f(x)$ a.e. in Q and $\int\limits_Q f_k\, dx \to \int\limits_Q f\, dx$ as $k \to \infty$. Since $\int\limits_Q |f - f_k|\, dx =$ $= \int\limits_Q (f - f_k)\, dx$, it follows that $\|f - f_k\|_{L_1(Q)} \to 0$ as $k \to \infty$, as required.

2. Let $f(x) \in L_2(Q)$ which is again assumed real-valued and nonnegative. Since $f(x) \in L_1(Q)$, we can find a monotone nondecreasing sequence f_k, $k = 1, 2, \ldots$, of functions belonging to $C(\overline{Q})$ that converges to f a.e. in Q. The functions $f_k(x)$ can be additionally assumed to be nonnegative, for, if necessary, this sequence can be replaced by the sequence $f_k^+(x)$, $k = 1, 2, \ldots$. But then $f_k^2(x) \uparrow$ $\uparrow f^2(x)$, as $k \to \infty$, a.e. in Q. By the definition of the integral of $f^2(x)$ $\int\limits_Q f_k^2\, dx \to \int\limits_Q f^2\, dx$, that is, $\|f_k\|^2_{L_2(Q)} \to \|f\|^2_{L_2(Q)}$. Since $f_k f \leqslant$ $\leqslant f^2$, by Lebesgue's theorem (Theorem 6, Sec. 1.7, Chap. II) $\lim\limits_{k\to\infty} (f_k, f)_{L_2(Q)} = \|f\|^2_{L_2(Q)}$, which implies that $\|f_k - f\|^2_{L_2(Q)} =$ $= \|f_k\|^2_{L_2(Q)} - 2\,(f_k, f)_{L_2(Q)} + \|f\|^2_{L_2(Q)} \to 0$ as $k \to \infty$. ∎

Note that if a sequence of functions in $C(\overline{Q})$ converges to a function in the norm of $C\,(\overline{Q})$, it also converges to it in the norms of $L_1(Q)$ and $L_2(Q)$. Consequently, any function continuous in $\overline{Q}$ can be approximated by a sequence of polynomials with rational coefficients in the norms of $L_1(Q)$ and $L_2(Q)$. Then Theorem 2 implies that the countable set of polynomials with rational coefficients is everywhere dense in $L_1(Q)$ and $L_2(Q)$. We have the following result.

Theorem 3. $L_1(Q)$ *and* $L_2(Q)$ *are separable spaces.*

A function $f(x)$ belonging to $L_2(Q)$ (and extended outside Q by assigning to it the value zero) is called *continuous in the mean (square) or in the norm of* $L_2(Q)$ if for a given $\varepsilon > 0$ a $\delta > 0$ can be found such that $\|f(x + z) - f(x)\|_{L_2(Q)} < \varepsilon$ for all z, $|z| < \delta$.

A function $f(x)$ belonging to $L_1(Q)$ (and extended as being equal to zero outside Q) is called *continuous in the mean or in the norm of*

$L_1(Q)$ if for a given $\varepsilon > 0$ a $\delta > 0$ can be found such that $\| f(x+z) - f(x) \|_{L_1(Q)} < \varepsilon$ for all z, $|z| < \delta$.

Theorem 2 implies the following result.

Theorem 4. *Any function belonging to $L_2(Q)$ is continuous in the mean (square). Any function belonging to $L_1(Q)$ is continuous in the mean.*

Proof. Let $f \in L_2(Q)$ (the proof is exactly the same when $f \in L_1(Q)$). Consider a large number $a > 0$ so that $Q \Subset S_a$, where S_a is the ball $\{|x| < a\}$. Since $f(x) \in L_2(Q)$, the function $F(x)$, which is equal to $f(x)$ when $x \in Q$ and vanishes when $x \in S_{2a} \diagdown Q$, belongs to $L_2(S_{2a})$. Take an arbitrary $\varepsilon > 0$. By Theorem 2, there is a function $\tilde{F}(x)$ which is continuous in $\bar{S}_{2a}$ and satisfies the inequality $\| F(x) - \tilde{F}(x) \|_{L_2(S_{2a})} < \varepsilon/3$. By multiplying $\tilde{F}(x)$ by a properly chosen slicing function for the region S_a, it can be assumed that $\tilde{F}(x) \equiv 0$ for $x \in S_{2a} \diagdown S_a$. Therefore for $|z| \leqslant a$, $\| F(x+z) - \tilde{F}(x+z) \|_{L_2(S_{2a})} = \| F(x) - \tilde{F}(x) \|_{L_2(S_a)} \leqslant \varepsilon/3$. Since the function $\tilde{F}(x)$ is uniformly continuous in $\bar{S}_{2a}$, there is a $\delta > 0$ ($\delta < a$) such that $\| \tilde{F}(x+z) - \tilde{F}(x) \|_{L_2(S_{2a})} \leqslant \varepsilon/3$ whenever $|z| < \delta$. This means that for $|z| < \delta$

$$
\begin{aligned}
\| f(x+z) - f(x) \|_{L_2(Q)} &= \| F(x+z) - F(x) \|_{L_2(S_{2a})} \\
&\leqslant \| F(x+z) - \tilde{F}(x+z) \|_{L_2(S_{2a})} + \| \tilde{F}(x+z) - \tilde{F}(x) \|_{L_2(S_{2a})} \\
&\quad + \| \tilde{F}(x) - F(x) \|_{L_2(S_{2a})} \leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \blacksquare
\end{aligned}
$$

3. Averaging of Functions Belonging to $L_1(Q)$ and $L_2(Q)$. As in the case of functions belonging to $C(\bar{Q})$, averaging functions can also be defined for functions belonging to $L_1(Q)$ and $L_2(Q)$.

Let $\omega_h(|x-y|)$ be some averaging kernel (Chap. I, Introduction), and let $f(x) \in L_1(Q)$. The function

$$
f_h(x) = \int_Q f(y)\, \omega_h(|x-y|)\, dy, \qquad h > 0, \tag{5}
$$

is called the *average function for the function f (averaging function for f)*.

By the Property (a) of an averaging kernel and Theorem 7, Sec. 1.7, Chap. II, $f_h(x) \in C^\infty(R_n)$ for $h > 0$. Further, $f_h(x) \equiv 0$ beyond Q^h.

Theorem 5. *If $f(x) \in L_1(Q)$ ($L_2(Q)$), then $\| f_h - f \|_{L_1(Q)} \to 0$ ($\| f_h - f \|_{L_2(Q)} \to 0$) as $h \to 0$.*

Proof. Both the assertions are proved exactly in the same way, therefore we examine, for example, the case when $f \in L_2(Q)$. The function f is assumed extended outside Q by assigning to it the value zero. Successive applications of Properties (b) and (c) of an averaging kernel, Bunyakovskii's inequality and Property (d) of the averaging kernel yield

$$|f_h(x)-f(x)|^2$$
$$=\int\limits_{|x-y|<h} f(y)\,\omega_h(|x-y|)\,dy - f(x)\int\limits_{|x-y|<h}\omega_h(|x-y|)\,dy$$
$$\leqslant \int\limits_{|x-y|<h}\omega_h^2(|x-y|)\,dy\int\limits_{|x-y|<h}|f(y)-f(x)|^2\,dy$$
$$\leqslant\frac{\text{const}}{h^n}\int\limits_{|z|<h}|f(x+z)-f(x)|^2\,dz.$$

By the Corollary of Fubini's theorem (Sec. 1.11, Chap. II)

$$\|f_h-f\|^2_{L_2(Q)}\leqslant\frac{\text{const}}{h^n}\int\limits_Q dx\int\limits_{|z|<h}|f(x+z)-f(x)|^2\,dx$$
$$=\frac{\text{const}}{h^n}\int\limits_{|z|<h}dz\int\limits_Q|f(x+z)-f(x)|^2\,dx. \qquad (6)$$

Take any $\varepsilon>0$. By the theorem on continuity in the mean (Theorem 4), a $\delta>0$ can be found such that $\|f(x+z)-f(x)\|_{L_2(Q)}<\varepsilon$ whenever $|z|\leqslant h<\delta$. Therefore for such h the inequality (6) implies $\|f_h-f\|_{L_2(Q)}\leqslant\text{const}\cdot\varepsilon$. ■

Remark. It should be noted that the proof of Theorem 5 nowhere uses the fact that the averaging kernel is nonnegative. Consequently, if the averaging function $f_h(x)$ for $f(x)$ is defined by the formula (5), where $\omega_h(|x-y|)=\frac{1}{h^n}\,\omega_1\left(\frac{|x-y|}{h}\right)$ and $\omega_1(t)$, $-\infty<t<\infty$, is an infinitely differentiable even function that vanishes for $|t|\geqslant 1$ and is such that $\int\limits_{R_n}\omega_1(|x|)\,dx=1$ (compare with the definition of the averaging kernel given in Introduction, Chap. I), then Theorem 5 remains valid in this case also.

Theorem 6. *The set $\dot{C}^\infty(\overline{Q})$ is everywhere dense in $L_1(Q)$ and in $L_2(Q)$.*

Proof. Let $f(x)\in L_2(Q)$ (the case $f\in L_1(Q)$ is disposed of similarly), and fix any $\varepsilon>0$. By the theorem on absolute continuity of a Lebesgue integral (Theorem 9, Sec. 1.10, Chap. II), there is a $\delta>0$ such that $\int\limits_{Q\setminus Q_\delta}|f|^2\,dx<\varepsilon^2/4$. This means that the function $F(x)$ with compact support in Q which belongs to $L_2(Q)$ and equals $f(x)$ when $x\in Q_\delta$ and vanishes when $x\in Q\setminus Q_\delta$, satisfies the inequality $\|F-f\|_{L_2(Q)}\leqslant\varepsilon/2$. By Theorem 5, a $h_0>0$ can be found such that $\| F_h - F \|_{L_2(Q)} \leqslant \varepsilon/2$ for all $0 < h \leqslant h_0$. The averaging function F_h for the function F with compact support belongs to $\dot{C}^\infty(\overline{Q})$ for sufficiently small h and

$$\| f - F_h \|_{L_2(Q)} \leqslant \| f - F \|_{L_2(Q)} + \| F - F_h \|_{L_2(Q)} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \quad \blacksquare$$

Since for any $k \geqslant 0$ $\dot{C}^\infty(\overline{Q}) \subset \dot{C}^k(\overline{Q}) \subset L_2(Q)$, it follows that for all $k \geqslant 0$ $\dot{C}^k(\overline{Q})$ are everywhere dense in $L_2(Q)$.

4. **Linear Spaces $L_{1,\,\mathrm{loc}}$, $L_{2,\,\mathrm{loc}}$.** The set of functions that are integrable over any strictly interior subregion Q' of Q, $Q' \Subset Q$, are denoted by $L_{1,\,\mathrm{loc}}(Q)$.

The set of functions that are measurable in Q and the squares of whose moduli are integrable over any strictly interior subregion Q' of Q, $Q' \Subset Q$, is denoted by $L_{2,\,\mathrm{loc}}(Q)$.

It is clear that $L_{1,\,\mathrm{loc}}(Q)$ and $L_{2,\,\mathrm{loc}}(Q)$ are linear spaces. Further, $L_1(Q) \subset L_{1,\,\mathrm{loc}}(Q)$ and $L_2(Q) \subset L_{2,\,\mathrm{loc}}(Q)$. The function $\frac{1}{(1-|x|)^m}$, for example, belongs to $L_{1,\,\mathrm{loc}}(|x| < 1)$ and $L_{2,\,\mathrm{loc}}(|x| < 1)$ for any m, and at the same time it belongs to $L_1(|x| < 1)$ only when $m < 1$ and to $L_2(|x| < 1)$ only when $m < 1/2$.

§ 3. GENERALIZED DERIVATIVES

1. **Simplest Properties of Generalized Derivatives.** Suppose that a continuous function $f(x)$ in Q has continuous derivative $f_{x_i}(x)$ in Q. Then for any function $g(x) \in \dot{C}^1(\overline{Q})$

$$\int_Q f\bar{g}_{x_i}\,dx = -\int_Q f_{x_i}\bar{g}\,dx.$$

It turns out that the derivative f_{x_i} of f is completely determined by the last equality: it can be easily shown that if for a continuous function $f(x)$ there exists a continuous function $h_i(x)$ such that

$$\int_Q f\bar{g}_{x_i}\,dx = -\int_Q h_i\bar{g}\,dx \tag{1}$$

for any $g(x) \in \dot{C}^1(\overline{Q})$, then the function $f(x)$ has in Q the derivative f_{x_i} and $f_{x_i} = h_i$ for all $x \in Q$. Thus by means of identity (1) a definition of the derivative of $f(x)$ can be given that is equivalent (in the class of continuous functions) to the usual definition. If in (1) the continuity condition of $f(x)$ and $h_i(x)$ is dropped and instead it is required that they be integrable or their squares be integrable (the latter is more convenient for us) and the integrals in (1) are understood in Lebesgue sense, then we enlarge the class of functions for which the notion of derivative can be introduced; the function h_i is called the generalized derivative of f with respect to x_i in Q.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a vector with nonnegative integral components. A function $f^{\alpha}(x) \in L_{2,\,\mathrm{loc}}(Q)$ is called *αth generalized derivative* (*g.d.*) *in Q* of a function $f(x) \in L_{2,\,\mathrm{loc}}(Q)$ if

$$\int\limits_Q f(x)\,\overline{D^{\alpha} g(x)}\,dx = -(1)^{|\alpha|} \int\limits_Q f^{\alpha}(x)\,\overline{g(x)}\,dx \tag{2}$$

for any $g(x) \in \dot{C}^{|\alpha|}(\overline{Q})$.

We shall first show that a function $f(x)$ can have only one g.d. $f^{\alpha}(x)$ (recall that functions are considered equal if they coincide a.e.).

In fact, let $f_1^{\alpha}(x)$ and $f_2^{\alpha}(x)$ be two g.d. of $f(x)$. For an arbitrarily fixed subregion Q', $Q' \Subset Q$, and an arbitrary function $g(x) \in \dot{C}^{|\alpha|}(\overline{Q}')$ the identity (2) yields $\int\limits_{Q'} (f_1^{\alpha} - f_2^{\alpha})\,\overline{g}\,dx = 0$. But $f_1^{\alpha} - f_2^{\alpha} \in L_2(Q')$, therefore, by Theorem 6, Subsec. 3 of the previous section, $f_1^{\alpha} - f_2^{\alpha} = 0$ a.e. in Q', which means that this holds a.e. also in Q.

Let $f(x) \in \dot{C}^{|\alpha|}(\overline{Q})$. By Ostrogradskii's formula we have

$$\int\limits_Q f(x)\,\overline{D^{\alpha} g(x)}\,dx = (-1)^{|\alpha|} \int\limits_Q D^{\alpha} f(x)\,\overline{g(x)}\,dx \tag{3}$$

for any $g(x) \in \dot{C}^{|\alpha|}(\overline{Q})$. That is, the function $f(x)$ has g.d. $f^{\alpha}(x)$ equal to $D^{\alpha} f(x)$. In particular, the function $f(x)$ which is equal to a constant (a.e.) in Q has each g.d. $f^{\alpha}(x) = 0$, $|\alpha| > 0$.

In the sequel g.d. f^{α} of a function f will be denoted by $D^{\alpha} f$. The generalized derivatives, primarily the first-and second-order derivatives, will also be denoted by $f_{x_i}, f_{x_i x_j}, \ldots$ and $\frac{\partial f}{\partial x_i}, \frac{\partial^2 f}{\partial x_i \partial x_j}, \ldots$.

Since for smooth functions $g(x)$ the derivative $\frac{\partial^{|\alpha|} g}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ does not depend on the order of differentiation, it follows from the uniqueness of a generalized derivative and formula (2) that a *generalized derivative* also *does not depend on the order of differentiation.*

The definition of g.d. also implies that if the functions $f_i(x)$, $i = 1, 2$, have g.d. $D^{\alpha} f_i$, then the function $c_1 f_1 + c_2 f_2$, with any constants c_i, has g.d. $D^{\alpha}(c_1 f_1 + c_2 f_2) = c_1 D^{\alpha} f_1 + c_2 D^{\alpha} f_2$.

Example 1. The function $f(x) = |x_1|$ has in the ball $Q = \{|x| < 1\}$ first generalized derivatives $f_{x_1} = \operatorname{sign} x_1$, $f_{x_i} = 0$, $i = 2, \ldots, n$.

In fact, for any $g(x) \in \dot{C}^1(\overline{Q})$

$$\int_Q |x_1| \bar{g}_{x_1}\, dx = \int_{Q^+} x_1 \bar{g}_{x_1}\, dx - \int_{Q^-} x_1 \bar{g}_{x_1}\, dx,$$

where $Q^+ = Q \cap (x_1 > 0)$, $Q^- = Q \cap (x_1 < 0)$. Applying Ostrogradskii's formula, we have ($x_1 \bar{g} = 0$ on ∂Q and also for $x_1 = 0$)

$$\int_Q |x_1| \bar{g}_{x_1}\, dx = -\int_{Q^+} \bar{g}\, dx + \int_{Q^-} \bar{g}\, dx = -\int_Q \operatorname{sign} x_1 \cdot \bar{g}\, dx.$$

Therefore g.d. with respect to x_1 of the function $|x_1|$ exists and is equal to the function sign x_1. Since for $i \geqslant 2$

$$\int_Q |x_1| \bar{g}_{x_i}\, dx = \int_Q (|x_1| \bar{g})_{x_i}\, dx = 0 = -\int_Q 0 \cdot \bar{g}\, dx,$$

the function $|x_1|$ has generalized derivatives with respect to x_i, $i = 2, \ldots, n$, equal to zero.

Note that the function $|x_1|$ has no classical derivative with respect to x_1 in Q (the derivative does not exist for $x_1 = 0$).

Example 2. The function $f(x) = \operatorname{sign} x_1$ has in the ball $Q = \{|x| < 1\}$ first generalized derivatives $f_{x_i} = 0$, $i = 2, \ldots, n$, but has no generalized derivative f_{x_1}. The existence of g.d. f_{x_i}, $i = 2, \ldots, n$, is established in the same manner as in Example 1. Let us show that f has no g.d. with respect to x_1. Suppose, on the contrary, that there is a function $\omega \in L_{2,\mathrm{loc}}(Q)$ which is a generalized derivative of f with respect to x_1. Then for any $g(x) \in \dot{C}^1(\overline{Q})$

$$\int_Q \omega \bar{g}\, dx = -\int_Q (\operatorname{sign} x_1)\, \bar{g}_{x_1}\, dx = -\int_{Q^+} \bar{g}_{x_1}\, dx + \int_{Q^-} \bar{g}_{x_1}\, dx$$

$$= 2 \int_{Q \cap \{x_1 = 0\}} \bar{g}\, dx_2 \ldots dx_n. \qquad (4)$$

This equality, first of all, implies that $\omega = 0$ (a.e.) in Q. In fact, substituting in (4) an arbitrary $g(x) \in \dot{C}^1(\overline{Q})$ vanishing in Q^-, we have $\int_{Q^+} \omega \bar{g}\, dx = 0$, which implies that $\omega = 0$ (a.e.) in Q^+. Analogously, it can be shown that $\omega = 0$ (a.e.) in Q^-. Accordingly, for any $g(x) \in \dot{C}^1(\overline{Q})$, $\int_Q \omega \bar{g}\, dx = 0$, that is, $\int_{Q \cap \{x_1 = 0\}} \bar{g}(x)\, dx_2 \ldots dx_n = 0$, but this cannot hold for an arbitrary function $g(x) \in \dot{C}^1(\overline{Q})$.

In contrast to the corresponding classical derivative, the generalized derivative $D^\alpha f$ is defined by (2) *globally*, at once in all of Q.

However, in *every subregion* $Q' \subset Q$ also *the function* $D^{\alpha}f$ *will be g.d. of the function* f, since the function $g(x)$, belonging to $\dot{C}^{|\alpha|}(\overline{Q}')$ and extended outside Q' by assigning to it the value zero, belongs to $\dot{C}^{|\alpha|}(\overline{Q})$ (this property was, in fact, used while proving the uniqueness of a g.d.). Therefore if $f(x)$ has in Q g.d. $D^{\alpha}f$ and $f(x) = c$ (a.e.) in $Q' \subset Q$, then $D^{\alpha}f = 0$ (a.e.) in Q'. In particular, g.d. (if it exists) of a function $f(x)$ having compact support in Q (that is, for some Q'', $Q'' \Subset Q$, $f(x) = 0$ a.e. in $Q \setminus Q''$) has compact support in Q, and therefore belongs to $L_2(Q)$.

Suppose that the function $f(x)$ belonging to $L_{2,\,\mathrm{loc}}(Q)$ has g.d. $D^{\alpha}f = F$ and the function $F(x)$ has g.d. $D^{\beta}F = G$. Then *there exists g.d.* $D^{\alpha+\beta}f$ *and* $D^{\alpha+\beta}f = G$.

Indeed, suppose $g(x) \in \dot{C}^{|\alpha+\beta|}(\overline{Q})$. Since $D^{\beta}g \in \dot{C}^{|\alpha|}(\overline{Q})$, we have

$$\int_Q f\overline{D^{\alpha+\beta}g}\,dx = (-1)^{|\alpha|}\int_Q D^{\alpha}f\overline{D^{\beta}g}\,dx$$

$$= (-1)^{|\alpha|}\int_Q F\overline{D^{\beta}g}\,dx = (-1)^{|\alpha|+|\beta|}\int_Q D^{\beta}F\bar{g}\,dx = (-1)^{|\alpha+\beta|}\int_Q G\bar{g}\,dx,$$

as required.

In contrast to the classical derivative, the generalized derivative $D^{\alpha}f$ is defined at once for order $|\alpha|$ without assuming the existence of corresponding derivatives of lower orders. Let us show that the derivatives of lower orders may not, in fact, exist.

Example 3. In the ball $Q = \{|x| < 1\}$ consider the function $f(x) = \varphi(x_1) + \varphi(x_2)$, where $\varphi(x_1) = \operatorname{sign} x_1$. The results of Example 2 show that $f(x)$ has no generalized derivatives f_{x_1} and f_{x_2}.

Nevertheless, we shall now show that the generalized derivative $f_{x_1x_2}$ exists. Taking an arbitrary $g(x) \in \dot{C}^2(\overline{Q})$, we have

$$\int_Q \bar{g}_{x_1x_2}f\,dx = \int_Q \varphi(x_1)\,\bar{g}_{x_1x_2}\,dx + \int_Q \varphi(x_2)\,\bar{g}_{x_1x_2}\,dx.$$

Since

$$\int_Q \varphi(x_1)\,\bar{g}_{x_1x_2}\,dx = -\int_{Q\cap\{x_1<0\}} \bar{g}_{x_1x_2}\,dx + \int_{Q\cap\{x_1>0\}} \bar{g}_{x_1x_2}\,dx = 0$$

and, similarly, $\int_Q \varphi(x_2)\bar{g}_{x_1x_2}\,dx = 0$, it follows that

$$\int_Q f\bar{g}_{x_1x_2}\,dx = 0 = \int_Q 0\cdot\bar{g}\,dx.$$

Thus the generalized derivative $f_{x_1x_2}$ exists and is equal to zero.

2. Generalized Derivatives and Averaging Functions. A Criterion for the Existence of Generalized Derivatives. Let $f(x) \in L_2(Q)$, and ω_h be some averaging kernel, and let

$$f_h(x) = \int_Q \omega_h(|x-y|)\, f(y)\, dy, \qquad h > 0,$$

be the averaging function for the function $f(x)$, $f_h(x) \in \dot{C}^\infty(R_n)$.

Lemma 1. *If the function* $f(x) \in L_2(Q)$ *has a generalized derivative* $D^\alpha f \in L_2(Q)$, *then for any point* $y \in Q$

$$(D^\alpha f)_h(y) = D^\alpha f_h(y) \tag{5}$$

with sufficiently small $h > 0$, *and for any subregion* $Q' \Subset Q$

$$\| D^\alpha f_h - D^\alpha f \|_{L_2(Q')} \to 0 \tag{6}$$

as $h \to 0$.

If in addition to the above, the function $f(x)$ *has compact support in* Q (*and is extended outside* Q *by assigning to it the value zero*), *then formula* (5) *holds for all* $y \in \bar{Q}$ *for sufficiently small* $h > 0$, *and*

$$\| D^\alpha f_h - D^\alpha f \|_{L_2(Q)} \to 0 \text{ as } h \to 0. \tag{7}$$

Proof. Taking in (2) for $g(x)$ the averaging kernel $\omega_h(|x-y|)$, $y \in Q$, with sufficiently small $h > 0$ (h is less than the distance between y and the boundary ∂Q) and applying Theorem 7, Sec. 1.7, Chap. II, we obtain formula (5)

$$(D^\alpha f)_h(y) = (-1)^{|\alpha|} \int_Q f(x) D_x^\alpha \omega_h(|x-y|)\, dx$$
$$= \int_Q f(x)\, D_y^\alpha \omega_h(|x-y|)\, dx = D_y^\alpha f_h(y).$$

If $Q' \Subset Q$, there exists a $h_0 > 0$ such that when $h \leqslant h_0$ formula (5) holds for all $y \in \bar{Q}'$. If $f(x)$ has compact support (in this case $D^\alpha f$ also has compact support and belongs to $L_2(Q)$), again there is a $h_0 > 0$ such that when $h \leqslant h_0$ formula (5) holds for all $y \in \bar{Q}$. Therefore relations (6) and (7) follow from Theorem 5, Sec. 2.3.

Corollary. *If all the first-order g.d. of a function* f *are zero, then* $f = \text{const}$.

In fact, in a subregion $Q' \Subset Q$ $(f_{x_i})_h = 0$, $i = 1, \ldots, n$, for sufficiently small h. By (5), $(f_h)_{x_i} = 0$, $i = 1, \ldots, n$, that is, $f_h = \text{const} = c(h)$ in Q' for such h. Since $\| f_h - f \|_{L_2(Q')} = \| c(h) - f \|_{L_2(Q')} \to 0$ as $h \to 0$ (Theorem 5, Sec. 2.3), it follows that $\| c(h_1) - c(h_2) \|_{L_2(Q')} = | c(h_1) - c(h_2) | \sqrt{|Q'|} \to 0$ as $h_1, h_2 \to 0$. Consequently, $c(h) = f_h$ converges uniformly in $\bar{Q}'$ (and more so in $L_2(Q')$) to some constant, that is, $f = \text{const}$, in Q', and therefore also in Q.

By means of Lemma 1 the following criterion for the existence of generalized derivative of $f \in L_2(Q)$ is established.

Theorem 1. *For the existence of g.d.* $D^\alpha f$ *of a function* $f \in L_2(Q)$ *it is necessary and sufficient that for any subregion* $Q' \Subset Q$ *there exist constants* $C(Q')$ *and* $h_0(Q')$ *such that* $\| D^\alpha f_h \|_{L_2(Q')} \leqslant C(Q')$ *for all* $h < h_0(Q')$.

Proof. The necessity has been proved in Lemma 1.

Sufficiency. Consider a system of regions $Q_1 \Subset Q_2 \Subset \ldots \Subset Q_m \Subset \ldots \Subset Q$ such that any point $x \in Q$ belongs to some Q_i (and therefore to all Q_j, $j > i$). Since for $h < h_0(Q_1)$ $\| D^\alpha f_h \|_{L_2(Q_1)} \leqslant C(Q_1)$, the set $\{D^\alpha f_h\}$ is weakly compact for such h (Theorem 3, Sec. 3.8, Chap. II). Therefore a sequence of values of h, $h_{1,1}, \ldots, h_{1,k}, \downarrow 0$ as $k \to \infty$, can be found such that the sequence of functions $D^\alpha f_{h_{1,k}}$, $k = 1, 2, \ldots$, converges weakly in $L_2(Q_1)$. Similarly, from the sequence $h_{1,k}$, $k = 1, 2, \ldots$, one may choose a subsequence $h_{2,k}$, $k = 1, 2, \ldots$ such that the sequence of functions $D^\alpha f_{h_{2,k}}$, $k = 1, 2, \ldots$, converges weakly in $L_2(Q_2)$; and the weak limit of this sequence in Q_1 coincides, of course, with the weak limit of the sequence $D^\alpha f_{h_{1,k}}$, $k = 1, 2, \ldots$, and so forth. The diagonal sequence $D^\alpha f_{h_{k,k}}$, $k = 1, 2, \ldots$, converges weakly to some function $\omega(x) \in L_{2,\text{loc}}(Q)$ in the space $L_2(Q_i)$ for any $i = 1, 2, \ldots$. Then for any $Q' \Subset Q$ $D^\alpha f_{h_{k,k}}$ converges weakly to ω in $L_2(Q')$.

Consider an arbitrary function $g \in \dot{C}^{|\alpha|}(\bar{Q})$, and let Q' be the region beyond which $g(x) = 0$, $Q' \Subset Q$. For all $k = 1, 2, \ldots$, we have

$$\int\limits_Q D^\alpha f_{h_{k,k}} \bar{g}\, dx = (-1)^{|\alpha|} \int\limits_Q f_{h_{k,k}} D^\alpha \bar{g}\, dx,$$

where the integration is, in fact, not over entire Q but over Q'. Since the sequence $D^\alpha f_{h_{k,k}}$, $k = 1, 2, \ldots$, converges weakly in $L_2(Q')$ to the function ω and the sequence $f_{h_{k,k}}$, $k = 1, 2, \ldots$, converges strongly (and therefore also weakly) to the function f, we may pass to the limit as $k \to \infty$ in the last identity:

$$\int\limits_Q \omega \bar{g}\, dx = (-1)^{|\alpha|} \int\limits_Q f D^\alpha \bar{g}\, dx.$$

This means that f has generalized derivative $D^\alpha f$ equal to the function ω. ▮

3. Existence of Generalized Derivative in the Union of Regions. As noted in Subsec. 1, if $D^\alpha f$ is g.d. of a function f in Q, then it is g.d. of this function in any subregion $Q' \subset Q$ also. In the present subsection we shall prove the following assertion.

Theorem 2. *If a function f has g.d. $D^\alpha f$ in regions Q_1 and Q_2 and if $Q_1 \cup Q_2 = Q$ is also a region (that is, a connected set), then g.d. $D^\alpha f$ exists in Q.*

Proof. Take an arbitrary point $x \in Q$. Let $S_\rho(x)$ be the ball of radius $\rho > 0$ with centre at x, and let $\rho_1 = \min\limits_{y \in \partial Q_1} |x - y|$, $\rho_2 = \min\limits_{y \in \partial Q_2} |x - y|$. If $x \in Q_1 \diagdown Q_2$, then $S_{\rho_1/2}(x) \Subset Q_1$, while if $x \in Q_2 \diagdown Q_1$, then $S_{\rho_2/2}(x) \Subset Q_2$; however, if $x \in Q_1 \cap Q_2$ and $\rho = \min(\rho_1, \rho_2)$, then $S_{\rho/2}(x) \Subset Q_1$ and $S_{\rho/2}(x) \Subset Q_2$.

Let all the points of Q be divided into two classes: the first class contains all the points of $Q_1 \diagdown Q_2$ and those points of $Q_1 \cap Q_2$ for which $\rho_1 < \rho_2$, $\rho = \rho_1$, and the second class contains the remaining points, that is, all the points of $Q_2 \diagdown Q_1$ as well as those points of $Q_1 \cap Q_2$ for which $\rho_2 \leqslant \rho_1$, $\rho = \rho_2$.

In this way, Q is covered by the balls $S_{\rho/2}(x)$: if x belongs to the first class, then $\rho = \rho_1$, and if x belongs to the second class, then $\rho = \rho_2$.

Let Q' be any strictly interior subregion of Q, $Q' \Subset Q$. From the cover of $\overline{Q}'$ by the balls $S_{\rho/2}(x)$ one can choose a finite subcover. A part of the balls of this subcover having centres at the points of the first class constitutes an open set $Q_1' \Subset Q_1$, while the remaining ones constitute an open set $Q_2' \Subset Q_2$. Thus for Q' there are two open sets Q_1' and Q_2' having the following properties: (a) Q_i', $i = 1, 2$, is the sum of a finite number of balls, (b) Q' belongs to the region $Q_1' \cup Q_2'$ and $Q_1' \Subset Q_1$, $Q_2' \Subset Q_2$. Since g.d. $D^\alpha f$ exists in Q_1 and Q_2, by Theorem 1 there are constants $C(Q_1')$, $C(Q_2')$, $h_0(Q_1')$ and $h_0(Q_2')$ such that for $h < h_0 = \min(h_0(Q_1'), h_0(Q_2'))$, $\| D^\alpha f_h \|_{L_2(Q_1')} \leqslant C(Q_1')$, $\| D^\alpha f_h \|_{L_2(Q_2')} \leqslant C(Q_2')$, where f_h is the averaging function for f in Q. Hence

$$\| D^\alpha f_h \|^2_{L_2(Q')} \leqslant \| D^\alpha f_h \|^2_{L_2(Q_1')} + \| D^\alpha f_h \|^2_{L_2(Q_2')} \leqslant C_1^2(Q_1') + C_2^2(Q_2') = C_2(Q')$$

for all $h < h_0$. Therefore, by Theorem 1, the function f has g.d. $D^\alpha f$ of αth order in Q that coincides, of course, with $D^\alpha f$ in Q_1 and Q_2. ∎

4. A Connection Between Generalized Derivatives and Finite-Difference Ratios. Let $f(x)$ have compact support in Q and belong to $L_2(Q)$. We extend this function outside Q by assigning to it the value zero and consider for $h \neq 0$ the difference ratio

$$\delta_h^k f(x) = \frac{f(x_1, \ldots, x_{k-1}, x_k + h, x_{k+1}, \ldots, x_n) - f(x)}{h}, \tag{8}$$

$k = 1, \ldots, n$. Clearly, for all $h \neq 0$, $\delta_h^k f(x) \in L_2(Q)$. If the function $g(x) \in L_2(Q)$ (and is extended as being equal to zero outside

Q), then for sufficiently small $|h|$ (less than the distance between the boundaries of Q and Q', outside which $f = 0$) the formula of "integration by parts" holds:

$$\begin{aligned}(\delta_h^k f, g)_{L_2(Q)} &\\ &= \frac{1}{h}\int_Q (f(x_1, \dots, x_{k-1}, x_k + h, x_{k+1}, \dots, x_n) - f(x))\, \bar g(x)\, dx \\ &= \frac{1}{h}\int_Q f(x)\, (\bar g(x_1, \dots, x_{k-1}, x_k - h, x_{k+1}, \dots, x_n) - \bar g(x))\, dx \\ &= -(f, \delta_{-h}^k g)_{L_2(Q)} \qquad (9)\end{aligned}$$

Theorem 3. *Let the function $f(x)$ with compact support in Q belong to $L_2(Q)$.*

(a) *If g.d. f_{x_k} exists for some $k = 1, \dots, n$, then for all sufficiently small $|h|$, $h \neq 0$, $\| \delta_h^k f \|_{L_2(Q)} \leqslant \| f_{x_k} \|_{L_2(Q)}$ and*

$$\| \delta_h^k f - f_{x_k} \|_{L_2(Q)} \to 0 \text{ as } h \to 0. \qquad (10)$$

(b) *If there is a constant $C > 0$ such that for all sufficiently small $|h|$, $h \neq 0$, $\| \delta_h^k f \|_{L_2(Q)} \leqslant C$, then the function f has g.d. f_{x_k} in Q, and the inequality $\| f_{x_k} \|_{L_2(Q)} \leqslant C$ as well as relation* (10) *hold.*

Proof of (*a*). Suppose first $f \in \dot C^1(\bar Q)$. With no loss of generality one can take $k = n$. Then $\delta_h^n f = \frac{1}{h}\int\limits_{x_n}^{x_n+h} \frac{\partial f(x', \xi_n)}{\partial \xi_n}\, d\xi_n$, where, as usual, $x' = (x_1, \dots, x_{n-1})$. Therefore (suppose $h > 0$, for definiteness)

$$|\delta_h^n f(x)|^2 \leqslant \frac{1}{h^2}\left(\int\limits_{x_n}^{x_n+h} \left|\frac{\partial f(x', \xi_n)}{\partial \xi_n}\right| d\xi_n\right)^2 \leqslant \frac{1}{h}\int\limits_{x_n}^{x_n+h} \left|\frac{\partial f(x', \xi_n)}{\partial \xi_n}\right|^2 d\xi_n,$$

whence it follows that

$$\int\limits_{-\infty}^{+\infty} |\delta_h^n f(x)|^2\, dx_n \leqslant \frac{1}{h}\int\limits_{-\infty}^{+\infty} dx_n \int\limits_{x_n}^{x_n+h} \left|\frac{\partial f(x', \xi_n)}{\partial \xi_n}\right|^2 d\xi_n = \int\limits_{-\infty}^{+\infty} \left|\frac{\partial f(x', x_n)}{\partial x_n}\right|^2 dx_n.$$

Integration with respect to $x' \in R_{n-1}$ yields

$$\| \delta_h^n f \|_{L_2(Q)} \leqslant \| f_{x_n} \|_{L_2(Q)}. \qquad (11)$$

Further,

$$\delta_h^n f(x) - f_{x_n}(x) = \frac{1}{h} \int\limits_{x_n}^{x_n+h} \frac{\partial f(x', \xi_n)}{\partial \xi_n}\, d\xi_n - \frac{\partial f(x)}{\partial x_n}$$

$$= \frac{1}{h} \int\limits_{x_n}^{x_n+h} \left(\frac{\partial f(x', \xi_n)}{\partial \xi_n} - \frac{\partial f(x', x_n)}{\partial x_n} \right) d\xi_n .$$

Hence

$$\int\limits_{-\infty}^{+\infty} (\delta_h^n f(x) - f_{x_n}(x))^2\, dx_n$$

$$\leqslant \frac{1}{h} \int\limits_{-\infty}^{+\infty} dx_n \int\limits_{x_n}^{x_n+h} \left| \frac{\partial f(x', \xi_n)}{\partial \xi_n} - \frac{\partial f\,(x', x_n)}{\partial x_n} \right|^2 d\xi_n$$

$$= \frac{1}{h} \int\limits_0^h d\eta \int\limits_{-\infty}^{+\infty} \left(\frac{\partial f(x', x_n + \eta)}{\partial x_n} - \frac{\partial f\,(x', x_n)}{\partial x_n} \right)^2 dx_n .$$

Integrating with respect to $x' \in R_{n-1}$, we have

$$\| \delta_h^n f - f_{x_n} \|^2_{L_2(Q)} \leqslant \frac{1}{h} \int\limits_0^h d\eta \int\limits_Q \left(\frac{\partial f(x', x_n + \eta)}{\partial x_n} - \frac{\partial f(x', x_n)}{\partial x_n} \right)^2 dx. \quad (12)$$

Inequalities (11) and (12) established for the time being only for functions $f \in \dot{C}^1(\bar{Q})$ are also true for functions in $L_2(Q)$ having compact support and g.d. f_{x_n} in Q. To show this, it suffices to approximate $f(x)$ by its averaging function with a sufficiently small averaging radius ρ, use for this last function inequalities (11) and (12) (the averaging function has compact support in Q) and then take the limit as $\rho \to 0$.

Thus the first inequality in (a), coinciding with (11), is proved.

To prove relation (10), we apply the theorem on continuity in the mean (square) of functions belonging to $L_2(Q)$ (Theorem 4, Sec. 2.2), which implies that for a given $\varepsilon > 0$ a $\delta = \delta(\varepsilon)$ can be found such that

$$\int\limits_Q \left(\frac{\partial f(x', x_n + \eta)}{\partial x_n} - \frac{\partial f(x', x_n)}{\partial x_n} \right)^2 dx \leqslant \varepsilon^2$$

whenever $|\eta| \leqslant |h| \leqslant \delta$. Therefore (12) yields the inequality $\|\delta_h^n f - f_{x_n}\|^2_{L_2(Q)} \leqslant \varepsilon^2$ whenever $|h| \leqslant \delta$. This proves Proposition (a).

Proof of (*b*). By Theorem 3, Sec. 3.8, Chap. II, the set $\{\delta_h^k f\}$ with small $|h|$ is weakly compact in $L_2(Q)$. Accordingly, a sequence

$\delta^k_{h_p} f$, $p = 1, 2, \ldots, h_p \to 0$ as $p \to \infty$, can be chosen that converges weakly to some function $\omega \in L_2(Q)$; moreover, $\| \omega \|_{L_2(Q)} \leqslant C$. Further, in view of (9), $(\delta^k_{h_p} f, g)_{L_2(Q)} = -(f, \delta^k_{-h_p} g)_{L_2(Q)}$ for any $g(x) \in \dot{C}^1(\bar{Q})$. When $p \to \infty$, the left side of this identity tends to (ω, g) and the right side, by Lebesgue theorem, to $-(f, g_{x_k})$. Therefore g.d. f_{x_k} exists and $f_{x_k} = \omega$. ▌

In the sequel the following proposition will also be used.

Let Q be a simply connected region in R_n which contains the origin and is symmetrical with respect to the plane $x_n = 0$ (that is, if $x = (x', x_n)$ belongs to Q, then $(x', -x_n)$ also belongs to Q), and let $\delta > 0$ be a small number so that Q_δ is a region. We denote

$$Q^+ = Q \cap \{x_n > 0\}, \; Q^- = Q \cap \{x_n < 0\},$$
$$(Q_\delta)^+ = Q_\delta \cap \{x_n > 0\}.$$

Theorem 4. *Let* $f(x) \in L_2(Q^+)$ *and* $f(x) = 0$ *in* $Q^+ \setminus (Q_\delta)^+$.

(a) *If for some* $k < n$ *g.d.* f_{x_k} *exists in* Q^+, *then for all sufficiently small* $|h|$, $h \neq 0$,

$$\| \delta^k_h f \|_{L_2(Q^+)} \leqslant \| f_{x_k} \|_{L_2(Q^+)}$$

and

$$\| \delta^k_h f - f_{x_k} \|_{L_2(Q^+)} \to 0 \quad as \quad h \to 0. \tag{10'}$$

(b) *If there is a constant* $C > 0$ *such that for all sufficiently small* $|h|$, $h \neq 0$, $\| \delta^k_h f \|_{L_2(Q^+)} \leqslant C$, $k < n$, *then g.d.* f_{x_k} *exists in* Q^+, *and* $\| f_{x_k} \|_{L_2(Q^+)} \leqslant C$ *and relation* (10′) *holds.*

In Q we define a function $F(x)$ as follows: $F(x) = f(x)$ in Q^+ and $F(x) = f(x', -x_n)$ in Q^-. Clearly, $F \in L_2(Q)$ and $F(x) = 0$ outside Q_δ. Moreover, $\| \delta^k_h F \|^2_{L_2(Q)} = 2 \| \delta^k_h f \|^2_{L_2(Q^+)}$, $k < n$, $0 < |h| < \delta$.

Proof of (a). Suppose that the function f has g.d. f_{x_k} in Q^+. We shall first show that the function F has g.d. F_{x_k} in Q. In fact, consider an arbitrary function $g(x) \in \dot{C}^1(\bar{Q})$ and with any $\delta > 0$ the even function $\zeta_\delta(x_n) \in C^1(-\infty, +\infty)$, $\zeta_\delta(-x_n) = \zeta_\delta(x_n)$, satisfying the inequality $|\zeta_\delta(x_n)| \leqslant 1$ for all x_n, and equal to 1 when $x_n \geqslant \delta$ and to zero when $0 \leqslant x_n \leqslant \delta/2$.

The equality

$$\begin{aligned}
&\int_Q F(x)\, g_{x_k}(x)\, \zeta_\delta(x_n)\, dx \\
&\quad = \int_{Q^+} f(x)\, g_{x_k}(x)\, \zeta_\delta(x_n)\, dx + \int_{Q^-} f(x', -x_n)\, g_{x_k}(x)\, \zeta_\delta(x_n)\, dx \\
&\qquad\qquad = \int_{Q^+} f(x) \frac{\partial}{\partial x_k} (\zeta_\delta(x_n) [g(x', x_n) + g(x', -x_n)])\, dx
\end{aligned}$$

and definition of g.d. of f in Q^+ (the function $\zeta_\delta(x_n)\,(g(x', x_n) + g(x', -x_n)) \in \dot{C}^1(\overline{Q}^+)$) imply

$$\int_Q F(x)\, g_{x_k}(x)\, \zeta_\delta(x_n)\, dx$$

$$= -\int_{Q^+} f_{x_k}(x)\, \zeta_\delta(x_n)\, (g(x', x_n) + g(x', -x_n))\, dx$$

$$= -\int_{Q^+} f_{x_k}(x', x_n)\, \zeta_\delta(x_n)\, g(x)\, dx - \int_{Q^-} f_{x_k}(x', -x_n)\, \zeta_\delta(x_n)\, g(x)\, dx.$$

Letting here $\delta \to 0$, by Lebesgue theorem we find that the function which is equal to $f_{x_k}(x)$ in Q^+ and to $f_{x_k}(x', -x_n)$ in Q^- is g.d. F_{x_k} in Q of F, and $\| F_{x_k} \|^2_{L_2(Q)} = 2 \| f_{x_k} \|^2_{L_2(Q^+)}$.

By Theorem 3, $\| \delta^k_h F \|_{L_2(Q)} \leqslant \| F_{x_k} \|_{L_2(Q)}$, therefore $\| \delta^k_h f \|^2_{L_2(Q^+)} = \frac{1}{2} \| \delta^k_h F \|^2_{L_2(Q)} \leqslant \frac{1}{2} \| F_{x_k} \|^2_{L_2(Q)} = \| f_{x_k} \|_{L_2(Q^+)}$. Since $\| \delta^k_h F - F_{x_k} \|^2_{L_2(Q)} = \frac{1}{2} \| \delta^k_h f - f_{x_k} \|^2_{L_2(Q^+)}$ and $\| \delta^k_h F - F_{x_k} \|_{L_2(Q)} \to 0$ as $h \to 0$, we find that $\| \delta^k_h f - f_{x_k} \|_{L_2(Q^+)} \to 0$ as $h \to 0$. This proves Proposition (a).

Proof of (*b*). Suppose that $\| \delta^k_h f \|_{L_2(Q^+)} \leqslant C$, $k < n$, for all sufficiently small $| h |$, $h \neq 0$. Then for all such h $\| \delta^k_h F \|^2_{L_2(Q)} \leqslant 2 \cdot C^2$. By Theorem 3, g.d. F_{x_k} exists in Q and $\| F_{x_k} \|^2_{L_2(Q)} \leqslant 2 \cdot C^2$, which means that g.d. f_{x_k} exists in Q^+ and $\| f_{x_k} \|^2_{L_2(Q^+)} \leqslant C^2$ and (10′) holds. ∎

§ 4. SPACES $H^k(Q)$

1. Linear Space $H^k_{\text{loc}}(Q)$. Hilbert Space $H^k(Q)$. The set of functions belonging to $L_{2,\,\text{loc}}(Q)$ and having all generalized derivatives up to order k, $k \geqslant 1$, (belonging to $L_{2,\,\text{loc}}(Q)$) will be denoted by $H^k_{\text{loc}}(Q)$. By $H^k(Q)$ we shall denote a subset of $H^k_{\text{loc}}(Q)$ whose elements belong to $L_2(Q)$ together with all the generalized derivatives up to order k. When $k = 0$, $H^k_{\text{loc}}(Q)$ and $H^k(Q)$ will mean $L_{2,\,\text{loc}}(Q)$ and $L_2(Q)$, respectively: $H^0_{\text{loc}}(Q) = L_{2,\,\text{loc}}(Q)$, $H^0(Q) = L_2(Q)$.

It is clear that $H^k_{\text{loc}}(Q)$ and $H^k(Q)$ are linear spaces. Let us show that $H^k(Q)$ is a Hilbert space with the scalar product

$$(f, g)_{H^k(Q)} = \sum_{|\alpha| \leqslant k} \int_Q D^\alpha f D^\alpha \overline{g}\, dx. \tag{1}$$

To demonstrate this, it is enough to prove that $H^k(Q)$ is complete in the norm

$$\|f\|_{H^k(Q)}=\sqrt{\sum_{|\alpha|\leqslant k}\int_Q |D^\alpha f|^2\,dx}, \tag{2}$$

generated by scalar product (1).

Let f_m, $m = 1, 2, \ldots$, be a sequence of elements of $H^k(Q)$ that is fundamental in the norm (2):

$$\|f_s-f_m\|^2_{H^k(Q)}=\sum_{|\alpha|\leqslant k}\int_Q |D^\alpha f_s-D^\alpha f_m|^2\,dx\to 0 \text{ as } m,\ s\to\infty.$$

Then for any α, $|\alpha|\leqslant k$, when $m,\ s\to\infty$

$$\int_Q |D^\alpha f_s-D^\alpha f_m|^2\,dx\to 0, \tag{3}$$

and, in particular (when $\alpha = 0$),

$$\int_Q |f_s-f_m|^2\,dx\to 0. \tag{4}$$

Since $L_2(Q)$ is complete, (4) implies the existence of a function $f\in L_2(Q)$ to which the sequence f_m, $m = 1, 2, \ldots$, converges (in $L_2(Q)$), and (3) implies the existence, for any α, $|\alpha|\leqslant k$, of a function $f^\alpha\in L_2(Q)$ to which the sequence $D^\alpha f_m$, $m = 1, 2, \ldots$, converges (in $L_2(Q)$).

Since each of the functions $f_m(x)$ has all generalized derivatives up to order k belonging to $L_2(Q)$, it follows that for any α, $|\alpha|\leqslant k$,

$$(f_m, D^\alpha g)_{L_2(Q)}=(-1)^{|\alpha|}(D^\alpha f_m, g)_{L_2(Q)}$$

for any $g\in\dot{C}^k(\overline{Q})$. Letting $m\to\infty$ in this identity (strong convergence implies weak convergence), we find that the function f^α is αth g.d. of f. So $f\in H^k(Q)$ and $\|f_m-f\|_{H^k(Q)}\to 0$ as $m\to\infty$, which proves the statement.

Remark. Sometimes it becomes convenient to consider the set of all real-valued functions belonging to $H^k(Q)$, $k = 0, 1, \ldots$ ($H^0(Q) = L_2(Q)$). This set is, of course, a (real) Hilbert space with the scalar product (1). It will be referred to as the real $H^k(Q)$ space and the same notation will be used for it.

Let us note some of the properties of spaces $H^k(Q)$.

1. If the region $Q'\subset Q$ and $f\in H^k(Q)$, then $f\in H^k(Q')$.

2. If $f\in H^k(Q)$ and $a(x)\in C^k(\overline{Q})$, then $af\in H^k(Q)$. In this case any generalized derivative $D^\alpha(af)$, $|\alpha|\leqslant k$, is computed accord-

ing to the usual rule of differentiating the product of functions. In particular, $(af)_{x_i} = a_{x_i}f + af_{x_i}$, $i = 1, \ldots, n$.

3. If $f \in H^k(Q)$ and $f_h(x)$ is the averaging function for f, then for any subregion Q', $Q' \Subset Q$, $\| f_h - f \|_{H^k(Q')} \to 0$ as $h \to 0$. If, in addition, the function f has compact support in Q, then $\| f_h - f \|_{H^k(Q)} \to 0$ as $h \to 0$.

4. If $f \in H^k(Q)$ and has compact support in Q, then a function equal to f in Q and to zero outside Q belongs to $H^k(Q')$ for any Q', $Q' \supset Q$.

Properties 1-4 are a direct consequence of the definition of spaces $H^k(Q)$ and the properties of generalized derivatives.

5. Let the transformation $y = y(x)$ $(y_i = y_i(x_1, \ldots, x_n)$, $i = 1, \ldots, n)$ map one-to-one the region Q onto the region Ω, and let $x = x(y)$ $(x_i = x_i(y_1, \ldots, y_n)$, $i = 1, \ldots, n)$ be the corresponding inverse transformation. Suppose that for some $k \geqslant 1$ $y_i(x) \in C^k(\bar{Q})$, $x_i(y) \in C^k(\bar{\Omega})$, $i = 1, \ldots, n$. Then in order that the function $F(x) = f(y(x))$, where $f(y)$ is a function defined in Ω, may belong to the space $H^k(Q)$ it is necessary and sufficient that $f(y)$ should belong to $H^k(\Omega)$. Derivatives of $F(x)$ are calculated according to the usual rule of differentiating a composite function. For example, the first derivatives are given by the formulas

$$F_{x_i}(x) = \sum_{j=1}^{n} f_{y_i}(y(x)) \frac{\partial y_j(x)}{\partial x_i}, \qquad i = 1, \ldots, n. \tag{5}$$

Moreover, there are constants C_1 and C_2 depending on functions $y_i(x)$, $i = 1, \ldots, n$, such that (a) $\| F \|_{H^k(Q)} \leqslant C_1 \| f \|_{H^k(\Omega)}$, (b) $\| f \|_{H^k(\Omega)} \leqslant C_2 \| F \|_{H^k(Q)}$.

The inverse transformation $x = x(y)$ satisfies the same conditions as the transformation $y = y(x)$, therefore we confine our proof to the sufficiency part and the inequality (a).

Let $k = 1$, and $f(y) \in H^1(\Omega)$. By Remark to Theorem 8, Sec. 1.8, Chap. II, the function $F(x)$ and the functions $F_i(x) = \sum_{j=1}^{n} f_{y_j}(y(x)) \times \times \frac{\partial y_j}{\partial x_i}$, $i = 1, \ldots, n$, belong to $L_2(Q)$. If $f_h(y)$ is the averaging function for $f(y)$, then the function $F(h, x) = f_h(y(x))$ belongs to $C^1(\bar{Q})$, and $\frac{\partial F(h, x)}{\partial x_i} = \sum_{j=1}^{n} f_{hy_j}(y(x)) \frac{\partial y_j}{\partial x_i}$, $i = 1, \ldots, n$.

Let the subregion $Q' \Subset Q$ and Ω' be its image, then $\Omega' \Subset \Omega$. Since, as $h \to 0$, $\| f_h - f \|_{L_2(\Omega)} \to 0$ and $\| f_{hy_i} - f_{y_i} \|_{L_2(\Omega')} \to 0$,

$i = 1, \ldots, n$, by Remark to Theorem 8, Sec. 1.8, Chap. II, $\| F(h, x) - F(x) \|_{L_2(Q)} \to 0$ and $\| F_{x_i}(h, x) - F_i(x) \|_{L_2(Q')} \to 0$, $i = 1, \ldots, n$, as $h \to 0$, for any $Q' \Subset Q$. This means that in the equalities $(F(h, x), g_{x_i}(x))_{L_2(Q)} = -(F_{x_i}(h, x), g(x))_{L_2(Q)}$, $i = 1, \ldots, n$, where g is any function in $\dot{C}^1(\overline{Q})$ (Q' is chosen so that $g = 0$ in $Q \setminus Q'$) one can pass to the limit as $h \to 0$: $(F, g_{x_i})_{L_2(Q)} = -(F_i, g)_{L_2(Q)}$. Therefore F has all the first generalized derivatives belonging to $L_2(Q)$, that is, F belongs to $H^1(Q)$; the relations (5) hold, and therefore inequality (a) also holds when $k = 1$.

Suppose now $k = 2$. We have already proved that $F(x) \in H^1(Q)$ and formulas (5) hold. By Property 2, the right-hand sides of (5), being the function of y, belong to $H^1(\Omega)$. Then the functions $F_{x_i}(x)$ also belong to $H^1(Q)$. Consequently, $F \in H^2(Q)$ and inequality (a) holds for $k = 2$. Regarding third derivatives as the derivatives of second derivatives and so forth, we see that the assertion is true for any k.

The following property will be used in Subsec. 2.

6. If the region Q is a rectangular parallelepiped, then $C^\infty(\overline{Q})$ (and hence $C^k(\overline{Q})$) is an everywhere dense set in $H^k(Q)$.

It suffices to establish this assertion for the parallelepiped $\Pi_a = \{ | x_i | < a_i,\ i = 1, \ldots, n\}$, where $a \doteq (a_1, \ldots, a_n)$, $a_i > 0$, $i = 1, \ldots, n$.

Take any function $f \in H^k(\Pi_a)$ and any $\varepsilon > 0$. Whatever be α, $0 \leqslant | \alpha | \leqslant k$, the function $D^\alpha f \in L_2(\Pi_a)$, therefore, by Theorem 2, Sec. 2.2, there is a function $\varphi_\alpha(x) \in C(\overline{\Pi}_a)$ such that $\| D^\alpha f - \varphi_\alpha \|_{L_2(\Pi_a)} < \varepsilon$.

In the parallelepiped $\Pi_{a\sigma} = \{ | x_i | < a_i \sigma,\ i = 1, \ldots, n\}$ where $\sigma > 1$, $\Pi_a \Subset \Pi_{a\sigma}$, consider the function $F_\sigma(x) = f(x/\sigma)$. By Property 4, $F_\sigma \in H^k(\Pi_{a\sigma})$, and hence $F_\sigma \in H^k(\Pi_a)$. Since

$$\| D^\alpha F_\sigma(x) - \varphi_\alpha(x) \|_{L_2(\Pi_a)} \leqslant \| D^\alpha F_\sigma(x) - \varphi_\alpha(x/\sigma) \|_{L_2(\Pi_{a\sigma})} + \| \varphi_\alpha(x) - \varphi_\alpha(x/\sigma) \|_{L_2(\Pi_a)}$$

and by Theorem 8, Sec. 1.8, Chap. II,

$$\| D^\alpha F_\sigma(x) - \varphi_\alpha(x/\sigma) \|_{L_2(\Pi_{a\sigma})} = \left\| \frac{1}{\sigma^{|\alpha|}} D^\alpha f(x/\sigma) - \varphi_\alpha(x/\sigma) \right\|_{L_2(\Pi_{a\sigma})} \leqslant \left\| \left(1 - \frac{1}{\sigma^{|\alpha|}}\right) D^\alpha f(x/\sigma) \right\|_{L_2(\Pi_{a\sigma})}$$

$$+ \| D^\alpha f(x/\sigma) - \varphi_\alpha(x/\sigma) \|_{L_2(\Pi_{a\sigma})} \leqslant \sigma^{n/2} \left(1 - \frac{1}{\sigma^{|\alpha|}}\right) \| D^\alpha f \|_{L_2(\Pi_a)} + \sigma^{n/2} \varepsilon,$$

it follows that

$$\|D^{\alpha}F_{\sigma}(x)-\varphi_{\alpha}(x)\|_{L_2(\Pi_a)}$$
$$\leqslant\sigma^{n/2}\left(1-\frac{1}{\sigma^{|\alpha|}}\right)\|D^{\alpha}f\|_{L_2(\Pi_a)}+\sigma^{n/2}\varepsilon+\|\varphi_{\alpha}(x)-\varphi_{\alpha}(x/\sigma)\|_{L_2(\Pi_a)}.$$

Therefore for any α, $0\leqslant|\alpha|\leqslant k$,

$$\|D^{\alpha}f(x)-D^{\alpha}F_{\sigma}(x)\|_{L_2(\Pi_a)}\leqslant\|D^{\alpha}f-\varphi_{\alpha}\|_{L_2(\Pi_a)}+\|D^{\alpha}F_{\sigma}-\varphi_{\alpha}\|_{L_2(\Pi_a)}$$
$$\leqslant\varepsilon\,(1+\sigma^{n/2})+\sigma^{n/2}\left(1-\frac{1}{\sigma^{|\alpha|}}\right)\|D^{\alpha}f\|_{L_2(\Pi_a)}+\|\varphi_{\alpha}(x)-\varphi_{\alpha}(x/\sigma)\|_{L_2(\Pi_a)}.$$

The function $\varphi_{\alpha}(x)\in C(\overline{\Pi}_{\alpha})$, which means that $\|\varphi_{\alpha}(x)-\varphi_{\alpha}(x/\sigma)\|_{L_2(\Pi_a)}\to 0$ as $\sigma\to 1$. Therefore a $\sigma=\sigma_0>1$ can be found such that for all α, $0\leqslant|\alpha|\leqslant k$, $\|D^{\alpha}f(x)-D^{\alpha}F_{\sigma_0}(x)\|_{L_2(\Pi_a)}\leqslant 3\varepsilon$. Consequently,

$$\|f-F_{\sigma_0}\|_{H^k(\Pi_a)}\leqslant C\varepsilon.$$

Take now the averaging function $(F_{\sigma_0})_h(x)$ for the function $F_{\sigma_0}(x)\in H^k(\Pi_{a\sigma_0})$. By Property 3, $\|(F_{\sigma_0})_h-F_{\sigma_0}\|_{H^k(\Pi_a)}\to 0$ as $h\to 0$, thereby implying that a number $h=h_0$ can be found such that $\|(F_{\sigma_0})_{h_0}-F_{\sigma_0}\|_{H^k(\Pi_a)}\leqslant\varepsilon$. The function $(F_{\sigma_0})_{h_0}(x)\in C^{\infty}(\overline{\Pi}_a)$ and

$$\|(F_{\sigma_0})_{h_0}-f\|_{H^k(\Pi_a)}\leqslant\|(F_{\sigma_0})_{h_0}-F_{\sigma_0}\|_{H^k(\Pi_a)}$$
$$+\|F_{\sigma_0}-f\|_{H^k(\Pi_a)}\leqslant(C+1)\,\varepsilon. \quad ■$$

2. On Extension of Functions. Suppose that a function $f(x)$ is defined in a region Q and the region Q' contains Q. A function $F(x)$ defined in Q' and coinciding with $f(x)$ in Q is called *extension* of $f(x)$ *into* Q'. Note first that every function $f(x)$ has an extension. For example, $F(x)$ can be taken as zero in $Q'\setminus Q$. When $f(x)\in L_2(Q)$, we already used such an extension above. However, if $f(x)$ is a smooth function in Q, for example, $f\in H^k(Q)$ (or $f\in C^k(\overline{Q})$) for some $k\geqslant 1$, then it is natural to seek its extension $F(x)$ in the class of functions that are as much smooth in Q': belonging to $H^k(Q')$ (or to $C^k(\overline{Q}')$). We shall demonstrate that under definite conditions on the boundary of Q such extensions are possible.

Suppose first that Q' is the cube K_a with side $2a>0$, $K_a=\{|y_i|<a,\ i=1,\ldots,n\}$ (independent variables here will be denoted by $y_1,\ldots,y_n$) and Q is the parallelepiped $K_a^+=K_a\cap\{y_n>0\}$. The extension $Z(y)$ of a function $z(y)\in C^k(\overline{K_a^+})$ into $K_a^-=K_a\cap\{y_n<0\}$ is defined as follows:

$$Z(y)=\sum_{i=1}^{k+1}A_i z(y',\,-y_n/i), \tag{6}$$

where $y' = (y_1, \dots, y_{n-1})$ and $A_1, \dots, A_{k+1}$ is the solution of the linear algebraic system of equations

$$\sum_{i=1}^{k+1} (-1/i)^s A_i = 1, \quad s = 0, \dots, k. \tag{7}$$

Note that when $y \in K_a^-$, the points $(y', -y_n/i)$ in (6) lie in K_a^+ for all $i = 1, \dots, k+1$. The determinant (Vandermonde's determinant) of the system (7) does not vanish, therefore the system (7) has a unique solution $A_1, \dots, A_{k+1}$.

For any $y^0 = (y^{0\prime}, 0) \in K_a \cap \{y_n = 0\}$, the function $Z(y)$ is taken equal to $\lim\limits_{\substack{y\to y^0\\ y\in K_a^+}} z(y)$. Thus the function $Z(y)$ is defined on entire K_a. Since $z(y) \in C^k(\overline{K_a^+})$, by (6) $Z(y) \in C^k(\overline{K_a^-})$. We shall first show that $Z(y) \in C(\overline{K}_a)$.

Passing to the limit in (6) as $y \to y^0$, $y \in K_a^-$, and taking into account (7), we have

$$\lim_{\substack{y\to y^0\\ y\in K_a^-}} Z(y) = \sum_{i=1}^{k+1} A_i \lim_{\substack{y\to y^0\\ y\in K_a^+}} z(y) = \sum_{i=1}^{k+1} A_i Z(y^0) = Z(y^0),$$

which means that $Z(y) \in C(\overline{K}_a)$.

For any vector $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| \leqslant k$, with integer components, by (6) we have, for $y \in K_a^-$,

$$D^\alpha Z(y) = \sum_{i=1}^{k+1} A_i (-1/i)^{\alpha_n} D^\alpha z(y', -y_n/i). \tag{8}$$

Letting $y \to y^0$, $y \in K_a^-$, in (8), we obtain

$$\lim_{\substack{y\to y^0\\ y\in K_a^-}} D^\alpha Z(y) = \lim_{\substack{y\to y^0\\ y\in K_a^+}} D^\alpha Z(y)$$

for all possible α such that $|\alpha| = 1$. Then at the points in the plane $K_a \cap \{y_n = 0\}$ all the first derivatives of $Z(y)$ exist, and they coincide with corresponding limiting values. Therefore $Z(y) \in C^1(\overline{K}_a)$. Repeating these arguments and using (7), we find that $Z(y) \in C^l(\overline{K}_a)$ for all $l \leqslant k$.

For any α, $|\alpha| \leqslant k$, the relation (8) yields, for all $y \in K_a^-$,

$$|D^\alpha Z(y)|^2 \leqslant \sum_{i=1}^{k+1} A_i^2 \frac{1}{i^{2\alpha_n}} \cdot \sum_{i=1}^{k+1} |D^\alpha z(y', -y_n/i)|^2$$

$$= C_0 \sum_{i=1}^{k+1} |D^\alpha z(y', -y_n/i)|^2.$$

Integration with respect to $y \in K_a^-$ gives

$$\int_{K_a^-} |D^\alpha Z|^2\, dy \leqslant C_0 \sum_{i=1}^{k+1} \int_{K_a^-} |D^\alpha z(y', -y_n/i)|^2\, dy$$

$$= C_0 \sum_{i=1}^{k+1} i \int_{K_a^+ \cap \{y_n < a/i\}} |D^\alpha z(y)|^2\, dy \leqslant C' \int_{K_a^+} |D^\alpha z(y)|^2\, dy.$$

Since $Z(y) = z(y)$ when $y \in K_a^+$, it follows that

$$\int_{K_a} |D^\alpha Z(y)|^2\, dy = \int_{K_a^+} |D^\alpha Z(y)|^2\, dy + \int_{K_a^-} |D^\alpha Z(y)|^2\, dy$$

$$\leqslant C'' \int_{K_a^+} |D^\alpha z(y)|^2\, dy.$$

Summing these inequalities over all α, $|\alpha| \leqslant k$, we obtain the inequality

$$\|Z\|_{H^k(K_a)} \leqslant C_1 \|z\|_{H^k(K_a^+)}, \tag{9}$$

where the constant $C_1 > 0$ does not depend on the function $z(y)$.

Thus an extension $Z(y) \in C^k(\overline{K}_a)$ has been obtained for the function $z(y) \in C^k(\overline{K_a^+})$ and for this extension inequality (9) holds.

Suppose now that the function $z(y) \in H^k(K_a^+)$. By Property 6 of the previous subsection, there is a sequence $z_s(y)$, $s = 1, 2, \ldots$, of functions in $C^k(\overline{K_a^+})$ converging to $z(y)$ in the norm $H^k(K_a^+)$: $\|z_s - z\|_{H^k(K_a^+)} \to 0$ as $s \to \infty$. Denote by $Z_s(y)$ the extension of $z_s(y)$ into K_a obtained in abovementioned manner, $Z_s(y) \in C^k(\overline{K}_a)$. From (9) follows the inequality $\|Z_s - Z_p\|_{H^k(K_a)} \leqslant C_1 \|z_s - z_p\|_{H^k(K_a^+)}$, which shows that the sequence of functions Z_s, $s = 1, 2, \ldots$, is fundamental in the norm $H^k(K_a)$. This means that there is a function $Z(y) \in H^k(K_a)$ to which this sequence converges in the norm $H^k(K_a)$. Since $Z(y) = z(y)$ for $y \in K_a^+$, the function $Z(y)$ is an extension into K_a of the function $z(y)$. The function $Z(y)$ clearly satisfies inequality (9).

Thus the following result has been established.

Lemma 1. *For any function* $z(y) \in H^k(K_a^+)$ $(C^k(\overline{K_a^+}))$ *there is an extension* $Z(y) \in H^k(K_a)$ $(C^k(K_a))$, *and inequality* (9) *holds.*

Note that since equalities (6) hold for $Z_s(y)$, $s = 1, 2, \ldots$, and $z_s \to z$ in $H^k(K_a^+)$ and $Z_s \to Z$ in $H^k(K_a^-)$, it follows that this equality also holds for $Z(y)$.

Lemma 2. *Suppose that the function* $f(x) \in H^k(Q)$ *(or* $C^k(\overline{Q})$*) and for any point* $\xi \in \partial Q$ *there is a function* $F_\xi(x)$ *defined in the ball*

$S_r(\xi) = \{|x-\xi| < r\}$ *of radius* $r = r(\xi) > 0$ *such that* $F_\xi(x) = f(x)$ *for* $x \in Q \cap S_r(\xi)$ *and* $F_\xi(x) \in H^k(S_r(\xi))$ $(C^k(S_r(\xi)))$ *(the function* $F_\xi(x)$ *will be referred to as the extension of* $f(x)$ *into the ball* $S_r(\xi)$*). Suppose further that inequality*

$$\| F_\xi \|_{H^k(S_r(\xi))} \leqslant C_2 \| f \|_{H^k(Q)}, \tag{10}$$

where the constant C_2 *does not depend on* $f(x)$*, holds.*

Then for any $\rho > 0$ *there is an extension* $F(x)$ *into the region* Q^ρ* *of the function* $f(x)$ *having the properties:* $F(x) \in H^k(Q^\rho)$ $(C^k(\overline{Q}^\rho))$, $F(x) = 0$ *outside* $Q^{\rho/2}$*, there is a constant* $C_3 > 0$ *depending only on* Q *and the number* ρ *such that*

$$\| F \|_{H^k(Q^\rho)} \leqslant C_3 \| f \|_{H^k(Q)}. \tag{11}$$

Proof. According to the hypothesis, for any point $\xi \in \overline{Q}$ there exists a ball $S_r(\xi)$, $r = r(\xi)$, in which either the function $f(x) \in H^k(S_r(\xi))$ $(C^k(\overline{S}_r(\xi)))$ itself is defined if $\xi \in Q$ or else its extension of the same class. Assume that $r(\xi) < \rho$. The aggregate of balls $S_{r/3}(\xi)$ for all possible $\xi \in \overline{Q}$ covers the set $\overline{Q}$; accordingly (recall that the region Q is bounded), from this cover a finite subcover $S_{r_1/3}(x^1), \ldots, S_{r_N/3}(x^N)$, where $r_i = r(x^i)$, can be chosen.

Let the function $\theta_i(x) \in C^\infty(R_n)$, $\theta_i(x) = 1$ in $S_{r_i/3}(x^i)$ and $\theta_i(x) = 0$ outside the ball $S_{r_i/2}(x^i)$, $i = 1, \ldots, N$. We denote by $\sigma_i(x)$ the function $1 - \theta_i(x)$, $i = 1, \ldots, N$, and construct the functions

$$\gamma_1(x) = \theta_1(x), \quad \gamma_2(x) = \sigma_1(x)\,\theta_2(x), \ldots,$$
$$\gamma_i(x) = \sigma_1(x) \ldots \sigma_{i-1}(x)\,\theta_i(x), \quad i \leqslant N.$$

It is clear that $\gamma_i(x) \in C^\infty(R_n)$,

$$\gamma_i(x) = 0 \text{ in } \bigcup_{j<i} S_{r_j/3}(x^j) \tag{12}$$

and

$$\gamma_i(x) = 0 \text{ outside } S_{r_i/2}(x^i). \tag{13}$$

Further,

$$\gamma_1(x) + \ldots + \gamma_i(x) = (1 - \sigma_1(x)) + \sigma_1(x)\,(1 - \sigma_2(x))$$
$$+ \ldots + \sigma_1(x) \ldots \sigma_{i-1}(x)\,(1 - \sigma_i(x)) = 1 - \sigma_1(x) \ldots \sigma_i(x),$$

therefore

$$\gamma_1(x) + \ldots + \gamma_i(x) = 1 \tag{14}$$

for $x \in \bigcup_{j \leqslant i} S_{r_j/3}(x^j)$, and, in particular, for $x \in S_{r_i/3}(x^i)$.

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* Q^ρ is the union over all $x^0 \in Q$ of the balls $\{|x - x^0| < \rho\}$.

We define functions $f_i(x)$, $i = 1, \ldots, N$, for all $x \in R_n$ in the following manner: in $S_{r_i}(x^i)$ the function $f_i(x)$ coincides with either $f(x)$ or its extension $F_{xi}(x)$ into $S_{r_i}(x^i)$, outside $S_{r_i}(x^i)$ the function $f_i(x) = f(x)$ if $x \in Q$ or $f_i(x) = 0$ if $x \notin Q$.

By (13) and Properties 2 and 4 of the preceding subsection, the function $\gamma_i(x) f_i(x) \in H^k(Q^\rho)$ $(C^k(\bar{Q}^\rho))$. Therefore the function

$$F(x) = \sum_{i=1}^{N} f_i(x)\gamma_i(x) \qquad (15)$$

belongs to $H^k(Q^\rho)$ $(C^k(\bar{Q}^\rho))$.

Let x be a point of Q and $S_{r_l/3}(x^l)$ the first ball of the chosen finite subcover containing this point. As $f_i(x) = f(x)$ for all $i = 1, \ldots, N$ and by (12) $\gamma_i(x) f(x) = 0$ when $i > l$, so $F(x) = \sum_{i=1}^{l} \gamma_i(x) f(x) = f(x)$ in view of (14). This means that the function $F(x)$ given by (15) is an extension of $f(x)$. The relation $F(x) \equiv 0$ outside $Q^{\rho/2}$ is a consequence of (13) and (15), because $r_i < \rho$, $i = 1, \ldots, N$. Inequality (11) readily follows from (10) and (15). ∎

Theorem 1 (on extension). *Let Q and Q' be bounded regions, $Q \Subset Q'$, and $\partial Q \in C^k$. Then any function $f(x) \in H^k(Q)$ $(C^k(\bar{Q}))$ has an extension $F(x) \in H^k(Q')$ $(\dot{C}^k(\bar{Q}'))$ into Q' with compact support. Moreover,*

$$\|F\|_{H^k(Q')} \leqslant C \|f\|_{H^k(Q)}, \qquad (16)$$

where the constant $C > 0$ depends only on Q and Q'.

Proof. Take an arbitrary point $\xi \in \partial Q$. In some neighbourhood U_ξ of this point the equation of ∂Q can be expressed (if necessary, by redesignating the variables) in the form $x_n = \varphi(x_1, \ldots, x_{n-1})$ with $\varphi(x_1, \ldots, x_{n-1}) \in C^k(\bar{D})$, where $(n-1)$-dimensional region D is the projection of $\partial Q \cap U_\xi$ onto the plane $x_n = 0$. It is assumed that $x_n > \varphi$ in $Q \cap U_\xi$. The change of variables

$$y_i = x_i - \xi_i, \quad i = 1, \ldots, n-1, \quad y_n = x_n - \varphi(x_1, \ldots, x_{n-1}) \qquad (17)$$

maps U_ξ one-to-one onto some neighbourhood Ω of the origin which is expressed in terms of the variables $y_1, \ldots, y_n$. Let K_a be the cube $\{|y_i| < a, i = 1, \ldots, n\}$ lying in Ω and U'_ξ its original under the transformation (17). The image of $Q \cap U'_\xi$ is then the parallelepiped $K_a^+ = K_a \cap \{y_n > 0\}$ and the function $f(x)$ defined in $Q \cap U'_\xi$ becomes the function $z(y) = f(y_1 + \xi_1, \ldots, y_{n-1} + \xi_{n-1}, y_n + \varphi(y_1 + \xi_1, \ldots, y_{n-1} + \xi_{n-1}))$ belonging to $H^k(K_a^+)$ $(C^k(\bar{K}_a^+))$, by Property 5 of the preceding subsection.

By Lemma 1, there is an extension $Z(y)$ of $z(y)$ into the cube K_a. By the inverse transformation of (17)

$$x_i=y_i+\xi_i,\quad i=1,\ \ldots,\ n-1,\quad x_n=y_n+\varphi(y_1+\xi_1,\ \ldots,\ y_{n-1}+\xi_{n-1})$$

this extension generates extension $F_\xi(x)$ of $f(x)$ from $Q\cap U'_\xi$ into U'_ξ, and, more so, into the ball $S_r(\xi)$, contained in U'_ξ, of radius $r=r(\xi)>0$ with centre at the point ξ. Moreover (see Property 5, Subsec. 1),

$$\|F_\xi\|_{H^k(S_r(\xi))}\leqslant\|F_\xi\|_{H^k(U'_\xi)}\leqslant C_3\|Z\|_{H^k(K_a)},$$

$$\|z\|_{H^k(K_a^+)}\leqslant C_4\|f\|_{H^k(U'_\xi\cap Q)}\leqslant C_4\|f\|_{H^k(Q)},$$

where the constants C_3 and C_4 depend only on the function $\varphi(x_1,\ \ldots,\ x_{n-1})$ from (17) and its derivatives up to order k. These inequalities and (9) imply (10). The conclusion of the theorem now follows from Lemma 2 if ρ is taken less than the distance between the boundaries ∂Q and $\partial Q'$ of Q and Q'. ▮

Remark. The extension $F(x)$ into Q' of the function $f(x)$ belonging to $H^k(Q)$, obtained in the above proof, satisfies not only the inequality (16) but also the inequalities

$$\|F\|_{H^s(Q')}\leqslant C\|f\|_{H^s(Q)}$$

for all $s\leqslant k$.

So far the functions were extended from a given region into some wider region. In the sequel we shall have to use the smooth extension of a function from the boundary.

Suppose that a continuous function $f(x)$ is defined on the boundary ∂Q of the region Q. A function $F(x)$ continuous in $\bar{Q}$ is called *extension into* Q of the function $f(x)$ if for all $x\in\partial Q$ $F(x)=f(x)$. The following result holds.

Theorem 2. *If the boundary* $\partial Q\in C^k$ *for some* $k\geqslant 1$, *then any function* $f(x)\in C^k(\partial Q)$ *has an extension* $F(x)$ *into* Q *which belongs to* $C^k(\bar{Q})$. *Moreover,*

$$\|F\|_{C^k(\bar{Q})}\leqslant C\|f\|_{C^k(\partial Q)},$$

where the constant $C>0$ *does not depend on* f.

Proof. Since $\partial Q\in C^k$, for any point $\xi\in\partial Q$ there is a number $\rho=\rho(\xi)>0$ such that a portion of the boundary $\partial Q\cap S_\rho(\xi)$ ($S_\rho(\xi)$ denotes the ball with radius ρ and centre at the point ξ) is uniquely projected into a region D_ξ in some coordinate plane, the plane $x_n=0$, say, (it can be always achieved by redesignating the variables) and let the equation of the surface $\partial Q\cap S_\rho(\xi)$ have the form $x_n=\varphi(x')$, $x'\in D_\xi$, where $\varphi(x')\in C^k(D_\xi)$.

Choose a sufficiently small number $r=r(\xi)>0$ so that the $(n-1)$-dimensional ball $\{|x'-\xi'|<r\}\Subset D_\xi$. Then the function

$F_\xi(x)=f(x',\varphi(x'))$ (independent of x_n) of n variables is defined on the closed ball $\overline{S_r(\xi)}$, belongs to $C^k(\overline{S_r(\xi)})$ and coincides with f on $\partial Q\cap S_r(\xi)$. Moreover $\|F_\xi\|_{C^k(\overline{S_r(\xi)})}\leqslant C(\xi)\|f\|_{C^k(\partial Q)}$, where the constant $C(\xi)$ does not depend on f.

The set of balls $S_{r/3}(\xi)$ covers the boundary ∂Q for all $\xi\in\partial Q$. From this set we choose a finite cover of the boundary $S_{r_1/3}(x^1),\ldots,S_{r_N/3}(x^N)$, where $r_i=r(x^i)$.

For any $i=1,\ldots,N$, we define the function $f_i(x)$ as follows: in the ball $S_{r_i}(x^i)$ take it equal to $F_{x^i}(x)$, outside $S_{r_i}(x^i)$ take it equal to zero if $x\notin\partial Q$ and equal to $f(x)$ if $x\in\partial Q$. Then for all $i=1,\ldots,N$ the functions $f_i(x)\,\gamma_i(x)$, where $\gamma_i(x)$ is the function constructed in the proof of Lemma 2, belong to $C^k(R_n)$, and hence to $C^k(\overline{Q})$. Hence the function

$$F(x)=\sum_{i=1}^{N}\gamma_i(x)\,f_i(x)$$

also belongs to $C^k(\overline{Q})$.

Take an arbitrary $x\in\partial Q$ and assume that $S_{r_l/3}(x^l)$ is the first ball of the selected finite cover of the boundary containing this point. Since for all $i=1,\ldots,N$ $f_i(x)=f(x)$, relations (12) and (14) imply that $F(x)=\sum\limits_{i=1}^{l}\gamma_i(x)\,f(x)=f(x)$. Thus the function $F(x)$ belonging to $C^k(\overline{Q})$ is an extension of $f(x)$. The desired estimate is a consequence of corresponding inequalities for the functions $F_{x^i}(x)$. ∎

3. Denseness of $C^\infty(\overline{Q})$ in $H^k(Q)$. Spaces $\overset{\circ}{H}{}^k(Q)$. Let the boundary ∂Q of Q belong to the class C^k.

Theorem 3. *The set of functions $C^\infty(\overline{Q})$ (and hence $C^k(\overline{Q})$) is everywhere dense in the space $H^k(Q)$.*

Proof. Consider any region Q' for which Q is strictly interior, $Q\Subset Q'$. Let $f(x)$ be any function belonging to $H^k(Q)$. By Theorem 1 of the preceding subsection, there is an extension $F(x)$ belonging to $H^k(Q')$ of $f(x)$ from Q into Q'. By Property 3 (Subsec. 1),

$$\|F_h-f\|_{H^k(Q)}=\|F_h-F\|_{H^k(Q)}\to 0 \text{ as } h\to 0,$$

where $F_h(x)$ is the averaging function for $F(x)$. Since $F_h(x)\in C^\infty(\overline{Q})$, the conclusion of the theorem follows.

The set $C^k(\overline{Q})$ is a linear manifold in $H^k(Q)$. From Theorem 3 it follows that if the boundary $\partial Q\in C^k$, then the closure of the set $C^k(\overline{Q})$ in the norm $H^k(Q)$ coincides with $H^k(Q)$.

Let S be an $(n-1)$-dimensional surface lying in $\overline{Q}$. The subset $\overset{\circ}{C}{}^k_S(\overline{Q})$ of functions belonging to $C^k(\overline{Q})$ that vanish on the intersection of Q with some neighbourhood of S (every function has its own neighbourhood) is also a linear manifold in $H^k(Q)$. The closure of $\dot{C}^k_S(\overline{Q})$ in the norm of $H^k(Q)$ is a subspace of $H^k(Q)$; this will be denoted by $\overset{\circ}{H}{}^k_S(Q)$.

When $S = \partial Q$, the subspace $\overset{\circ}{H}{}^k_{\partial Q}(Q)$ will be denoted by $\overset{\circ}{H}{}^k(Q)$ (the norm in $\overset{\circ}{H}{}^k(Q)$ is the norm of $H^k(Q)$). Theorem 6, Sec. 2.3, implies that for $k = 0$ the subspace $\overset{\circ}{H}{}^k(Q) = \overset{\circ}{H}{}^0(Q)$ coincides with the space $H^0(Q) = L_2(Q)$. In Subsec. 1 of the next section it will be shown that $\overset{\circ}{H}{}^k(Q)$ does not coincide with $H^k(Q)$ when $k \geqslant 1$.

If the region Q is contained in Q', $Q \subset Q'$, then any function $f(x)$ in $\dot{C}^k(\overline{Q})$ extended as being equal to zero in $Q' \setminus Q$ belongs to $\dot{C}^k(\overline{Q'})$. Therefore from the definition of $\overset{\circ}{H}{}^k$ it follows that the function $f(x)$ belonging to $\overset{\circ}{H}{}^k(Q)$ and extended as being equal to zero into $Q' \setminus Q$ belongs to $\overset{\circ}{H}{}^k(Q')$.

4. Separability of Space $H^k(Q)$. It is assumed that the boundary ∂Q of Q belongs to the class C^k.

Theorem 4. *The space $H^k(Q)$ is separable.*

Proof. Consider first the cube $K = \{|x_i| < \pi,\ i = 1, \ldots, n\}$. The countable system of functions $(2\pi)^{-n/2}e^{i(m,x)}$, where $m = (m_1, \ldots, m_n)$, $m_i = 0, \pm 1, \pm 2, \ldots$, $i = 1, \ldots, n$, $(m, x) = m_1x_1 + \ldots + m_nx_n$, is orthonormal in $L_2(K)$. Any function $f(x) \in L_2(K)$ has a Fourier series expansion

$$\frac{1}{(2\pi)^{n/2}} \sum_m f_m e^{i(m,x)} = \frac{1}{(2\pi)^{n/2}} \sum_{s=0}^{\infty} \sum_{s \leqslant |m| < s+1} f_m e^{i(m,x)}, \tag{18}$$

where $f_m = \dfrac{(f(x), e^{i(m,x)})_{L_2(K)}}{(2\pi)^{n/2}}$ are Fourier coefficients of $f(x)$ and $|m|^2 = m_1^2 + \ldots + m_n^2$.

Let $f(x) \in \dot{C}^\infty(\overline{K})$. First note that for all m

$$|f_m| \leqslant (2\pi)^{n/2} \|f\|_{C(\overline{K})} = C_0. \tag{19}$$

Put $m' = (m_1, \ldots, m_{n-1})$, $x' = (x_1, \ldots, x_{n-1})$, $K' = \{|x_i| < \pi,\ i = 1, \ldots, n-1\} \subset R_{n-1}$. If $m_n \neq 0$, then

$$f_m = \frac{1}{(2\pi)^{n/2}} \int_K f(x)\, e^{i(m,x)}\, dx$$

$$= \frac{1}{(2\pi)^{n/2}} \int_{K'} e^{i(m',x')}\, dx' \left(\int_{-\pi}^{\pi} f(x', x_n)\, e^{ix_nm_n}\, dx_n \right)$$

$$= \frac{1}{(2\pi)^{n/2}} \left(-\frac{1}{im_n} \right)^p \int_K \frac{\partial^p f(x)}{\partial x_n^p}\, e^{i(m,x)}\, dx$$

for any natural p, whence it follows that $|f_m| \leqslant \frac{(2\pi)^{n/2} \|f\|_{C^p(\bar{K})}}{|m_n|^p}$; accordingly, by (19),

$$|f_m| \leqslant \frac{\|f\|_{C^p(\bar{K})} 2^p (2\pi)^{n/2}}{(1+|m_n|)^p} = \frac{C'_p}{(1+|m_n|)^p}$$

for any natural p.

Apart from this inequality, the inequalities

$$|f_m| \leqslant \frac{C'_p}{(1+|m_i|)^p}, \quad i = 1, \ldots, \quad n-1,$$

also hold, and therefore also the inequalities

$$|f_m| \leqslant C'_p \min_i \left\{\frac{1}{(1+|m_i|)^p}\right\} = \frac{C'_p}{(1+\max\limits_i |m_i|)^p}. \tag{20}$$

Since $\max\limits_i |m_i| \geqslant \frac{1}{\sqrt{n}} |m|$, from (20) there follow the inequalities

$$|f_m| \leqslant \frac{C'_p}{\left(1+\frac{1}{\sqrt{n}} |m|\right)^p} \leqslant \frac{C_p}{(1+|m|)^p} \tag{21}$$

true for all m and any natural p. Take $p = n + 2$. The number of terms in the summation $\sum\limits_{s \leqslant |m| < s+1} f_m e^{i(m,\, x)}$, which is equal to the number of points m with integer coordinates in the annular region $s \leqslant |m| < s + 1$, does not exceed the number of such points in the cube with side $2(s + 1)$, that is, does not exceed $(2s + 1))^n$. Therefore

$$\left|\sum_{s \leqslant |m| < s+1} f_m e^{i(m,\, x)}\right| \leqslant \sum_{s \leqslant |m| < s+1} |f_m| \leqslant \frac{C_{n+2} 2^n (1+s)^n}{(1+s)^{n+2}} = \frac{C_{n+2} 2^n}{(1+s)^2},$$

which means that the series (18) converges uniformly in $\bar{K}$.

Taking $p = n + 3$, we find that for any r, $1 \leqslant r \leqslant n$,

$$\left|\sum_{s \leqslant |m| < s+1} i m_r f_m e^{i(m,\, x)}\right| \leqslant \frac{C_{n+3} 2^n (1+s)^{n+1}}{(1+s)^{n+3}} = \frac{C_{n+3} 2^n}{(1+s)^2}.$$

Therefore the series obtained from (18) by termwise differentiation with respect to x_r, $r = 1, \ldots, n$, converges uniformly in $\bar{K}$. It can be similarly shown that the series obtained from (18) by termwise differentiation l times, $l = 2, 3, \ldots$, converge uniformly in $\bar{K}$.

Denote the sum of (18) by $g(x)$:

$$g(x) = \frac{1}{(2\pi)^{n/2}} \sum_m f_m e^{i(m,\, x)}.$$

We have shown that $g(x) \in C^\infty(\bar{K})$. This means that also the function $\varphi(x) = g(x) - f(x) \in C^\infty(\bar{K})$. Let us show that $\varphi(x) \equiv \equiv 0$ in $\bar{K}$.

Since $\left(g(x), \frac{e^{i(m,\,x)}}{(2\pi)^{n/2}}\right)_{L_2(K)} = f_m$, for all m

$$\int_K \varphi(x)\, e^{i(m,\,x)}\, dx = 0.$$

Having fixed an arbitrary $m' = (m_1, \ldots, m_{n-1})$, we write this equality in the form

$$\int_{-\pi}^{\pi} e^{im_n x_n} \left[\int_{K'} \varphi(x', x_n)\, e^{i(m',\,x')}\, dx' \right] dx_n = 0.$$

Since the function $\varphi_{m'}(x_n) = \int_{K'} \varphi(x', x_n)\, e^{i(m',x')}\, dx'$, which is infinitely differentiable with respect to x_n, $|x_n| \leqslant \pi$, is orthogonal in the space with the scalar product, $L_2(-\pi, \pi)$, to the functions $e^{im_n x_n}$ for all $m_n = 0, \pm 1, \pm 2, \ldots$, it follows that for any m' $\varphi_{m'}(x_n) = 0$ for all x_n, $|x_n| \leqslant \pi$. Let $m'' = (m_1, \ldots, m_{n-2})$, $x'' = (x_1, \ldots, x_{n-2})$, $K'' = K' \cap \{x_{n-1} = 0\}$. For any fixed m'', any x_n, $|x_n| \leqslant \pi$, and all $m_{n-1} = 0, \pm 1, \ldots$, we have

$$0 = \int_{K'} \varphi(x', x_n)\, e^{i(m',x')}\, dx'$$

$$= \int_{-\pi}^{\pi} e^{ix_{n-1}m_{n-1}}\, dx_{n-1} \int_{K''} \varphi(x'', x_{n-1}, x_n)\, e^{i(x'',\,m'')}\, dx'',$$

which implies

$$\int_{K''} \varphi(x'', x_{n-1}, x_n)\, e^{i(m'',\,x'')}\, dx'' = 0$$

for any x_{n-1}, x_n, $|x_{n-1}| \leqslant \pi$, $|x_n| \leqslant \pi$ and all m''. Continuing in this manner, we find that $\varphi(x) = 0$ in $\bar{K}$.

Thus it has been established that any function $f(x) \in \dot{C}^\infty(\bar{K})$ has series expansion (18) that converges uniformly together with derivatives of any order in $\bar{K}$. Evidently, this holds for any cube $K_a = \{|x_i| < a,\ i = 1, \ldots, n\}$.

We now turn to the proof of the theorem. Take a number $a > 0$ so large that $Q \Subset K_a$. By Theorem 1, Subsec. 2, any function $f(x) \in \in H^k(Q)$ has extension $F(x) \in H^k(K_a)$ with compact support in K_a. Any such function $F(x)$ can be approximated, according to Property 3, Subsec. 1, in the norm of $H^k(K_a)$ by averaging functions

$F_h(x)$, which are infinitely differentiable and for sufficiently small h have compact supports in K_a.

As shown above, every function $F_h(x)$ (for sufficiently small h) can be approximated uniformly in $\bar{K}_a$ together with all the derivatives (and therefore also in the norm of $H^k(K_a)$) by partial sums of its Fourier series. Consequently, any function $F_h(x)$ can be approximated in the norm of $H^k(Q)$ by a linear combination of the system $e^{i\frac{\pi}{a}(m,x)}$ with coefficients whose real and imaginary parts are rational numbers. Thus we have constructed a countable set which is everywhere dense in $H^k(Q)$. ■

§ 5. PROPERTIES OF FUNCTIONS BELONGING TO $H^1(Q)$ AND $\overset{\circ}{H}{}^1(Q)$

1. Trace of Functions. Let Q be a region in R_n and S a smooth $(n-1)$-dimensional surface lying in $\bar{Q}$. If in Q there is given a function $f(x)$ defined at every point (that is, if the equality of functions is understood as the equality of their values at every point), then we can consider the value of this function on S. That is, we can consider the function $f|_{x \in S}$ defined at every point of S whose values for all $x \in S$ coincide with the value of $f(x)$. If we consider a function defined a.e. in Q (that is, functions are considered equal if they coincide a.e.), then the value of f on a fixed surface S is determined not uniquely: since $\operatorname{mes} S = 0$, the function can assume any value. Nevertheless, one can speak, in a definite sense, of values on $(n-1)$-dimensional surfaces of an almost everywhere defined function as well.

For the sake of simplicity, assume that the surface $S = S(x_n)$ is the intersection of a region Q with the plane $x_n = \text{const}$. Then, according to Fubini's theorem*, for almost all x_n, the function f has the value $f|_{x \in S(x_n)}$ on $S(x_n)$ which is defined almost everywhere on S (naturally, the equality of functions of $(n-1)$ variables is understood as equality of their values a.e. in the sense of $(n-1)$-dimensional measure). Moreover, it is apparent that for almost all x_n the value on $S(x_n)$ of a function continuous in $\bar{Q}$ is a continuous function on $S(x_n)$, whereas for almost all x_n the value on $S(x_n)$ of a function belonging to $L_2(Q)$ belongs to $L_2(S(x_n))$.

In the investigation of solutions of differential equations, conditions are often prescribed which must be satisfied by the solution on some fixed $(n-1)$-dimensional surface, for instance, on ∂Q (the boundary conditions). Therefore we must generalize the meaning of the value on an $(n-1)$-dimensional surface S of an a.e. defined function—the idea of the trace of a function on S. For an a.e. defined function satisfying some smoothness conditions this idea can be introduced uniquely. In particular, this is easily presented for a continuous function in $\bar{Q}$.

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* More precisely, according to Lemma 4, Sec. 1.11, Chap. II.

By the trace $f|_S$ *of a function* $f \in C(\bar{Q})$ *on an* $(n-1)$*-dimensional surface* S we mean the value on this surface of a function defined at every point and continuous in $\bar{Q}$ that coincides with f almost everywhere (that is, by the trace on S of a continuous function is meant its value extended uniquely with respect to continuity on S). As usual, here the equality of functions defined on S is understood as a.e. equality in the sense of $(n-1)$-dimensional measure.

The notion of the trace of a function on S can also be introduced for functions belonging to certain spaces with integral norms; in particular, for functions in spaces $H^k(Q)$ with $k \geqslant 1$. Since for $k \geqslant 1$ all $H^k(Q)$ are contained in $H^1(Q)$, it is enough to introduce this notion for functions belonging to $H^1(Q)$.

Let S be a surface of class C^1 (see Chap. I, Introduction) lying in $\bar{Q}$, and let S_1 be its simple piece that is projected uniquely onto a region D in the plane $\{x_n = 0\}$ and having the equation

$$x_n = \varphi(x'), \text{ where } x' = (x_1, \ldots, x_{n-1}), \quad \varphi(x') \in C^1(\bar{D}).$$

The region Q is bounded, therefore it can be assumed enclosed in a cube $\{0 < x_i < a,\ i = 1, \ldots, n\}$ for some $a > 0$. Suppose first that $f(x)$ belongs to $\dot{C}^1(\bar{Q})$, and equate it to zero outside $\bar{Q}$. According to Newton-Leibnitz formula

$$f(x)|_{S_1} = f(x', \varphi(x')) = \int_0^{\varphi(x')} \frac{\partial f(x', \xi_n)}{\partial \xi_n}\, d\xi_n,$$

which, on applying Bunyakovskii's inequality, yields

$$|f|_{S_1}|^2 \leqslant \varphi(x') \int_0^{\varphi(x')} \left| \frac{\partial f(x', \xi_n)}{\partial \xi_n} \right|^2 d\xi_n \leqslant a \int_0^a \left| \frac{\partial f(x', \xi_n)}{\partial \xi_n} \right|^2 d\xi_n.$$

Multiplying this inequality by $\sqrt{1 + \varphi_{x_1}^2 + \ldots + \varphi_{x_{n-1}}^2}$ and integrating over D, we obtain

$$\|f\|^2_{L_2(S_1)} = \int_{S_1} |f|_{S_1}|^2\, dS_1 \leqslant C^2 \|f\|^2_{H^1(Q)}, \tag{1}$$

where the constant $C > 0$ does not depend on the function f.

Since the surface S can be covered by a finite number of simple pieces, pieces of type S_1 (that possibly project onto other coordinate planes), we find, summing the respective inequalities (1), that

$$\| f \|_{L_2(S)} \leqslant C \| f \|_{H^1(Q)}, \tag{2}$$

where the constant $C > 0$ does not depend on the function f.

The inequality (2) also holds for any function $f(x) \in C^1(\overline{Q})$. To show this, it suffices to use Theorem 1, Sec. 4.2, on extension (assuming, of course, $\partial Q \in C^1$) and inequality (2) for a function belonging to C^1 and having compact support.

Suppose now that $f \in H^1(Q)$. From Theorem 3, Sec. 4.3, it follows that there is a sequence of functions $f_p(x)$, $p = 1, 2, \ldots$, in $C^1(\overline{Q})$ which converges to f in the norm of $H^1(Q)$. For the function $f_p - f_q$ inequality (2) assumes the form

$$\| f_p - f_q \|_{L_2(S)} \leqslant C \| f_p - f_q \|_{H^1(Q)}. \tag{3}$$

Since $\| f_p - f_q \|_{H^1(Q)} \to 0$ as p, $q \to \infty$, it follows that also $\| f_p - f_q \|_{L_2(S)} \to 0$ as p, $q \to \infty$. This means that the sequence of traces $f_p|_S$ of functions f_p on S is fundamental in $L_2(S)$. Since $L_2(S)$ is complete, there is a function $f_S(x) \in L_2(S)$ to which the sequence of traces $f_p|_S$ converges as $p \to \infty$. Passing to the limit, as $p \to \infty$, in (3), we obtain

$$\| f_q - f_S \|_{L_2(S)} \leqslant C \| f_q - f \|_{H^1(Q)}. \tag{4}$$

Let us show that the function $f_S(x)$ does not depend on the choice of the sequence $f_k(x)$, $k = 1, 2, \ldots$, which approximates $f(x)$ in the norm of $H^1(Q)$. Indeed, let $\tilde{f}_k(x)$, $k = 1, 2, \ldots$, be another sequence of functions in $C^1(\overline{Q})$ for which $\| f - \tilde{f}_k \|_{H^1(Q)} \to 0$ as $k \to \infty$, and let $\tilde{f}_S(x)$ be the limit in the norm of $L_2(S)$ of the sequence $\tilde{f}_k|_S$, $k = 1, 2, \ldots$. Then

$$\| f_S - \tilde{f}_S \|_{L_2(S)} \leqslant \| f_S - f_q \|_{L_2(S)} + \| f_q - \tilde{f}_q \|_{L_2(S)} + \| \tilde{f}_q - \tilde{f}_S \|_{L_2(S)}$$
$$\leqslant C \left(\| f - f_q \|_{H^1(Q)} + \| f_q - \tilde{f}_q \|_{H^1(Q)} + \| \tilde{f}_q - \tilde{f}_S \|_{H^1(Q)} \right),$$

by the inequalities (3) and (4). Since, when $q \to \infty$, the right-hand side of the last inequality tends to zero, we have $f_S = \tilde{f}_S$.

The function $f_S(x)$ (as an element of $L_2(S)$) will be called *the trace of the function* $f(x) \in H^1(Q)$ *on the surface* S and will be denoted by $f|_S$ ($\| f|_S \|_{L_2(S)}$ will be denoted by $\| f \|_{L_2(S)}$).

Thus the trace of a function is defined for any element $f \in H^1(Q)$.

We now show that the notion of the trace is, in fact, a generalization of the notion of the value of a function on an $(n-1)$-dimensional surface. Assume for the sake of simplicity that $S = S(x_n)$ is the intersection of the region Q with the plane $x_n = \text{const}$, and that the function $f \in H^1(Q)$. Consider a sequence of functions $f_m(x)$, $m = 1, 2, \ldots$, in $C^1(\tilde{Q})$ which converges to f in the norm of $H^1(Q)$. By definition, the trace $f|_{S(x_n)}$ for each x_n is the limit in $L_2(S(x_n))$

of the sequence of functions $f_m|_{S(x_n)}$. Since the sequence f_m, $m = 1, 2, \ldots$, converges in $L_2(Q)$ to f, a subsequence f_{m_k}, $k = 1, \ldots$, can be chosen, in view of Remark to Theorem 1, Sec. 2.1, that converges to f a.e. in Q. This means that for almost all x_n the sequence $f_{m_k}|_{S(x_n)}$, $k = 1, 2, \ldots$, converges to the value of f on $S(x_n)$ almost everywhere in the sense of $(n-1)$-dimensional measure. Consequently, the trace and value of f on $S(x_n)$ coincide for almost all x_n.

Thus we have the notions of the trace on S of a function continuous in $\overline{Q}$ and that of a function belonging to $H^1(Q)$. It is claimed that if a function f belongs to $C(\overline{Q})$ and to $H^1(Q)$, then its trace as the trace of a function in $C(\overline{Q})$ (denoted by $f|'_S$) and that of a function in $H^1(Q)$ (denoted by $f|''_S$) coincide. In fact, the function f can be extended, by Theorem 1, Sec. 4.2, into Q', $Q \Subset Q'$, in such a way that its extension F will belong to $C(\overline{Q'})$ and to $H^1(Q')$. Consider the averaging functions $F_h(x)$ for the function F. Since $F_h \to F$ as $h \to 0$ in both the norms of $C(\overline{Q})$ (see Sec. 1.1) and $H^1(Q)$ (see Sec. 4.1, Property 3), we find that, as $h \to 0$, $F_h(x)|_S \to f|'_S$ in the norm of $C(\overline{S})$ and $F_h(x)|_S \to f|''_S$ in the norm of $L_2(S)$; accordingly, $f|'_S = f|''_S$.

The trace $f|_S$ of a function $f(x) \in \overset{\circ}{H}{}^1_S(Q)$ (the definition of this space is given in Sec. 4.3) is zero, since the function $f|_S$ is the limit in the norm of $L_2(S)$ of functions vanishing on S (of traces on S of functions in $\overset{\bullet}{C}{}^1_S(Q)$). In particular, the trace $f|_{\partial Q}$ of a function $f(x) \in \overset{\circ}{H}{}^1(Q)$ is zero. By the way, this establishes the assertion of Subsec. 3 of the preceding section which states that $\overset{\circ}{H}{}^k(Q) \neq H^k(Q)$ for $k \geqslant 1$: the function equal to 1 belonging to any $H^k(Q)$, $k \geqslant 1$, is continuous in $\overline{Q}$, therefore its trace on ∂Q is 1; hence this function does not belong to $\overset{\circ}{H}{}^k(Q)$ for any $k \geqslant 1$.

The trace $f|_S$ of a function $f \in H^1(Q)$ satisfies the inequality (2). To establish this, it is enough to pass to the limit, as $p \to \infty$, in the inequality (2) written for the functions $f_p(x)$ ($f_p(x) \in C^1(\overline{Q})$, $\| f_p - f \|_{H^1(Q)} \to 0$ as $p \to \infty$).

It was assumed so far that the boundary $\partial Q \in C^1$. However, when $S \Subset Q$, for the definition of the trace on S of a function and the proof of inequality (2) this restriction can be done away with. Indeed, in this case there is a region $Q' \Subset Q$ such that $\partial Q' \in C^1$ and $S \in Q'$.

Thus we have proved the following theorem.

Theorem 1. *Suppose that an $(n-1)$-dimensional surface S of class C^1 either belongs to Q', $Q' \Subset Q$, or instead $S \subset \overline{Q}$ and, in addition,*

$\partial Q \in C^1$. Then any function $f(x) \in H^1(Q)$ has on this surface the trace $f|_S$ belonging to $L_2(S)$, and the inequality (2) *holds.*

Let $f(x) \in H^k(Q)$, $k > 1$. Since any generalized derivative $D^\alpha f$ of order $|\alpha| < k$ belongs to $H^1(Q)$, this derivative has, by Theorem 1, the trace $D^\alpha f|_S$ belonging to $L_2(S)$ on any $(n-1)$-dimensional surface S of class C^1. Moreover, the inequalities

$$\| D^\alpha f \|_{L_2(S)} \leqslant C \| f \|_{H^{|\alpha|+1}(Q)} \leqslant C \| f \|_{H^k(Q)} \tag{5}$$

hold with constant $C > 0$ independent of the function f.

2. The Formula of Integration by Parts. Let the functions $f(x)$ and $g(x)$ belong to $H^1(Q)$ and $\partial Q \in C^1$. Then for any $i = 1, \ldots, n$ the formula of integration by parts holds:

$$\int_Q f_{x_i} g\, dx = \int_{\partial Q} f g n_i\, dS - \int_Q f g_{x_i}\, dx, \tag{6}$$

where $n_i = \cos(n, x_i)$ is cosine of the angle between outward normal n to the surface ∂Q and the x_i-axis, and the functions f and g present under the integral sign over ∂Q are traces of functions f and g on ∂Q. Thus, so far as the applicability of formula (6) is concerned, functions belonging to $H^1(Q)$ behave just like functions in $C^1(\overline{Q})$.

To prove (6), consider (Theorem 3, Sec. 4.3) the sequences $f_p(x)$ and $g_p(x)$, $p = 1, 2, \ldots$, of functions in $C^1(\overline{Q})$ which converge, respectively, to the functions $f(x)$ and $g(x)$ in the norm of $H^1(Q)$. Formula (6) holds for functions f_p and g_p:

$$\int_Q f_{px_i} g_q\, dx = \int_{\partial Q} f_p g_q n_i\, dS - \int_Q f_p g_{qx_i}\, dx.$$

Letting here $p \to \infty$ and $q \to \infty$ (and noting that $\| f_p - f \|_{L_2(\partial Q)} \to 0$, $\| g_q - g \|_{L_2(\partial Q)} \to 0$), the relation (6) follows.

It readily follows from (6) that if $g \in H^1(Q)$ and the components $f_i(x)$, $i = 1, \ldots, n$, of a vector $f(x)$, $f(x) = (f_1(x), \ldots, f_n(x))$, belong to $H^1(Q)$, then the relation

$$\int_Q g \operatorname{div} f\, dx = \int_{\partial Q} g(f \cdot n)\, dS - \int_Q f \cdot \nabla g\, dx \tag{7}$$

holds.

3. Properties of Traces of Functions Belonging to $H^1(Q)$. A Criterion for Membership of the Subspace $\overset{\circ}{H}{}^1(Q)$. Let Γ_0 be a sufficiently small (that is, contained in a ball of sufficiently small radius r_0) simple piece of a surface of class C^1 lying in $\overline{Q}$, and let Γ_0 be uniquely projected into a region D in the coordinate plane $\{x_n = 0\}$, $x_n = \varphi(x')$, $x' \in D$, $\varphi(x') \in C^1(\overline{D})$, is the equation of Γ_0.

Let Γ_δ denote the surface $\{x' \in D,\ x_n = \varphi(x') + \delta\}$ and Ω_δ the region $\{x' \in D,\ \varphi(x') < x_n < \varphi(x') + \delta\}$ when $\delta > 0$ or the region $\{x' \in D,\ \varphi(x') + \delta < x_n < \varphi(x')\}$ when $\delta < 0$. Note that for sufficiently small $|\delta|$ (and sufficiently small r_0) at least one of the regions $\Omega_{+|\delta|}$ or $\Omega_{-|\delta|}$ is contained in Q.

Let $x \in \Omega_\delta \subset Q$. Then for any function $f \in C^1(\overline{Q})$ we have

$$f(x', \varphi(x') + \delta) - f(x', \varphi(x')) = \int\limits_{\varphi(x')}^{\varphi(x')+\delta} \frac{\partial f(x', x_n)}{\partial x_n}\, dx_n,$$

whence it follows that

$$|f(x', \varphi(x') + \delta) - f(x', \varphi(x'))|^2 \leqslant \left| \delta \int\limits_{\varphi(x')}^{\varphi(x')+\delta} \left| \frac{\partial f(x', x_n)}{\partial x_n} \right|^2 dx_n \right|.$$

Multiplication of this inequality by $\sqrt{1 + \varphi_{x_1}^2 + \ldots + \varphi_{x_{n-1}}^2}$ and integration over D yield

$$\| f(x^\delta) - f(x^0) \|_{L_2(\Gamma_0)} \leqslant C \sqrt{|\delta|}\, \| f \|_{H^1(\Omega_\delta)}, \tag{8}$$

where $x^0 = (x', \varphi(x')) \in \Gamma_0$, $x^\delta = x^\delta(x^0) = (x', \varphi(x') + \delta) \in \Gamma_\delta$, and $C^2 = \max\limits_{x' \in \overline{D}} \sqrt{1 + \varphi_{x_1}^2 + \ldots + \varphi_{x_{n-1}}^2}$.

It is obvious that, apart from inequality (8), there also holds the inequality

$$\| f(x^\delta) - f(x^0) \|_{L_2(\Gamma_\delta)} \leqslant C \sqrt{|\delta|}\, \| f \|_{H^1(\Omega_\delta)}. \tag{9}$$

Approximating the function $f \in H^1(Q)$ by functions of class $C^1(\overline{Q})$ and using the definition of the trace of a function belonging to $H^1(Q)$, we find that inequalities (8) and (9) hold for all the functions belonging to $H^1(Q)$.

These inequalities express a definite continuity of traces on surfaces Γ_δ of functions belonging to $H^1(Q)$ depending on the displacements of these surfaces.

If the trace on Γ_0 of a function f is zero, $f|_{\Gamma_0} = 0$, then (9) implies the inequality

$$\| f \|_{L_2(\Gamma_\delta)}^2 \leqslant C^2 \delta \| f \|_{H^1(\Omega_\delta)}^2 \leqslant C^2 \rho \| f \|_{H^1(\Omega_\delta)}^2$$

for any ρ and δ, $0 < \delta \leqslant \rho \leqslant \rho_0$, where ρ_0 is such that $\Omega_{\rho_0} \subset Q$ (for the sake of definiteness, we take $\rho_0 > 0$). Integrating the last inequality with respect to $\delta \in (0, \rho)$ and using the absolute continuity of the integral, we find that

$$\| f \|_{L_2(\Omega_\rho)} = o(\rho) \text{ as } \rho \to 0. \tag{10}$$

Thus we have proved that if $f \in H^1(Q)$, $f|_{\Gamma_0} = 0$ and $\Omega_\rho \subset Q$ (in particular, Γ_0 may be a piece of the boundary ∂Q), then (10) holds.

Lemma 1. *If $f \in H^1(Q)$ and its trace on the boundary $f\,|_{\partial Q} = 0$, then*

$$\| f \|_{L_2(Q \setminus Q_\delta)} = o(\delta) \quad as \quad \delta \to 0. \tag{11}$$

Proof. Since the boundary $\partial Q \in C^1$, for every point $y \in \partial Q$ there is a ball $S_{2r}(y)$ of radius $2r$, $r = r(y) > 0$, with centre at this point such that a piece of the boundary $\partial Q \cap S_{2r}(y)$ is projected uniquely onto an $(n - 1)$-dimensional region $D_{2r}(y)$ lying in one of the coordinate planes, say the plane $\{x_n = 0\}$. The equation of the piece $\partial Q \cap S_{2r}(y)$ is of the form $x_n = \varphi(x')$, $x' \in D_{2r}(y)$, $\varphi(x') \in C^1(\overline{D}_{2r}(y))$. Denote by $\Gamma_0 = \Gamma_0(y)$ the surface $\partial Q \cap S_r(y)$, and by $\Gamma_\delta = \Gamma_\delta(y)$ and $\Omega_\delta = \Omega_\delta(y)$ the "parallel" surface and the corresponding region constructed with respect to Γ_0 in the manner described above. Choose $\delta_0 = \delta_0(y)$ so small in absolute value that the region $\Omega_{\delta_0} = \Omega_{\delta_0}(y) \subset Q \cap S_{2r}(y)$.

Since the distance between $\partial Q \setminus S_{2r}(y)$ and $\overline{\Omega}_{\delta_0}(y)$ is positive and the distance between $\partial Q \cap \overline{S}_{2r}(y)$ and $\Gamma_\delta(y)$, where $\delta \in (0, \delta_0)$ if $\delta_0 > 0$ and $\delta \in (\delta_0, 0)$ if $\delta_0 < 0$, is obviously greater than $\gamma\,|\,\delta\,|$ with some constant $\gamma = \gamma(y)$, $0 < \gamma < 1$, a $\gamma_0 = \gamma_0(y)$, $0 < \gamma_0 < 1$, can be found such that for all such δ

$$\inf_{\substack{x \in \partial Q \\ \xi \in \Gamma_\delta}} | x - \xi | > \gamma_0 | \delta |. \tag{12}$$

From the cover of ∂Q by the balls $S_r(y)$, $y \in \partial Q$, we choose a finite subcover $S_{r_1}(x^1), \ldots, S_{r_N}(x^N)$. Then there exists a number $\delta_1 > 0$, $\delta_1 < \min\limits_{1 \leqslant m \leqslant N} | \delta_0(x^m) |$, such that

$$Q \setminus Q_{\delta_1} \subset \bigcup_{m=1}^{N} \Omega_{\delta_0(x^m)}(x^m). \tag{13}$$

Furthermore, by (12), for all δ, $0 < \delta < \delta_1$, and $m = 1, \ldots, N$

$$(Q \setminus Q_{\gamma_1 \delta}) \cap \Omega_{\delta_0(x^m)}(x^m) \subset \Omega_{\delta \cdot \operatorname{sign} \delta_0(x^m)}(x^m), \tag{14}$$

where $\gamma_1 = \min\limits_{1 \leqslant m \leqslant N} \gamma_0(x^m)$.

The inclusions (13) and (14) imply that for any $f \in H^1(Q)$ the inequalities

$$\| f \|^2_{L_2(Q \setminus Q_{\gamma_1 \delta})} \leqslant \sum_{m=1}^{N} \| f \|^2_{L_2((Q \setminus Q_{\gamma_1 \delta}) \cap \Omega_{\delta_0(x^m)}(x^m))}$$

$$\leqslant \sum_{m=1}^{N} \| f \|^2_{L_2(\Omega_{\delta \cdot \operatorname{sign} \delta_0(x^m)}(x^m))}$$

hold for $0 < \delta < \delta_1$. Since $f\,|_{\partial Q} = 0$, (11) now follows from the last inequality and the relation (10). ■

Theorem 2. *In order that a function belonging to* $H^1(Q)$ *may belong to the subspace* $\overset{\circ}{H}{}^1(Q)$ *it is necessary and sufficient that its trace on the boundary of the region be zero.*

Proof. That the condition is necessary is obvious, so we confine ourselves to the proof of sufficiency. Let $f \in H^1(Q)$ and $f|_{\partial Q} = 0$. Take an arbitrary $\varepsilon > 0$. Lemma 1 and Theorem 9, Sec. 1.10, Chap. II, on the absolute continuity of an integral imply the existence of a small $\delta = \delta(\varepsilon)$ such that

$$\|f\|_{L_2(Q \setminus Q_\delta)} < \varepsilon\delta, \qquad \|f\|_{H^1(Q \setminus Q_\delta)} < \varepsilon.$$

Since for a function $f \in H^1(Q)$ there is (Theorem 3, Sec. 4.3; note that $\partial Q \in C^1$) a sequence of functions $f_p(x)$, $p = 1, 2, \ldots$, in $C^1(\overline{Q})$ converging to f in the norm of $H^1(Q)$ (and more so in the norm of $H^1(Q \setminus Q_\delta)$), a number $N = N(\delta) = N(\delta(\varepsilon))$ can be found such that

$$\begin{aligned} &\|f - f_N\|_{H^1(Q)} < \varepsilon, \\ &\|f_N\|_{L_2(Q \setminus Q_\delta)} < 2\varepsilon\delta, \\ &\|f_N\|_{H^1(Q \setminus Q_\delta)} < 2\varepsilon. \end{aligned} \tag{15}$$

Consider the function

$$\zeta_\delta(x) = \int\limits_{Q_{\delta/2}} \omega_{\delta/3}(|x - y|)\, dy,$$

where $\omega_\rho(|x - y|)$ is an averaging kernel. The properties of the averaging kernel imply that $\zeta_\delta(x) \in C^\infty(R_n)$, $\zeta_\delta(x) = 1$ for $x \in Q_{5\delta/6}$, and more so for $x \in Q_\delta$, $\zeta_\delta(x) = 0$ outside $Q_{\delta/6}$, that is, $\zeta_\delta(x) \in \dot{C}^\infty(\overline{Q})$. What is more, for all $x \in R_n$ $0 \leqslant \zeta_\delta(x) \leqslant 1$, $|\nabla \zeta_\delta| \leqslant C/\delta$ where the constant $C > 0$ does not depend on δ.

By (15) we have

$$\begin{aligned} | f_N - f_N\zeta_\delta \|_{H^1(Q)} &= \| f_N - f_N\zeta_\delta \|_{H^1(Q \setminus Q_\delta)} \\ &\leqslant \left(\| f_N(1 - \zeta_\delta) \|^2_{L_2(Q \setminus Q_\delta)} + \| \, |\nabla f_N| (1 - \zeta_\delta) + |f_N| \, |\nabla \zeta_\delta| \, \|^2_{L_2(Q \setminus Q_\delta)} \right)^{1/2} \\ &\leqslant \left(\| f_N \|^2_{L_2(Q \setminus Q_\delta)} + 2 \| \, |\nabla f_N| \, \|^2_{L_2(Q \setminus Q_\delta)} + 2 \| f_N |\nabla \zeta_\delta| \, \|^2_{L_2(Q \setminus Q_\delta)} \right)^{1/2} \\ &\leqslant \left(8\varepsilon^2 + \frac{2C^2}{\delta^2} \| f_N \|^2_{L_2(Q \setminus Q_\delta)} \right)^{1/2} \leqslant \varepsilon (8 + 8C^2)^{1/2} = C_1 \varepsilon. \end{aligned}$$

The functions $f_{N(\delta(\varepsilon))}(x)\, \zeta_{\delta(\varepsilon)}(x)$ belong to $\dot{C}^1(\overline{Q})$ and

$$\begin{aligned} \| f_{N(\delta(\varepsilon))}(x)\, \zeta_{\delta(\varepsilon)}(x) - f(x) \|_{H^1(Q)} &\leqslant \| f - f_{N(\delta(\varepsilon))} \|_{H^1(Q)} \\ &+ \| f_{N(\delta(\varepsilon))} - f_{N(\delta(\varepsilon))}\zeta_{\delta(\varepsilon)} \|_{H^1(Q)} < (1 + C_1)\, \varepsilon. \end{aligned}$$

Accordingly, $f(x) \in \overset{\circ}{H}{}^1(Q)$. ∎

4. On Compactness of Sets in $L_2(Q)$.

Theorem 3. *A set bounded in $H^1(Q)$ is compact in $L_2(Q)$.*

Proof. Let a set $\mathcal{M}$ be bounded in $H^1(Q)$, that is, for all $f \in \mathcal{M}$

$$\|f\|_{H^1(Q)} \leqslant C. \tag{16}$$

Suppose first that $\mathcal{M} \subset \overset{\circ}{H}{}^1(Q)$. We extend all the functions belonging to $\mathcal{M}$ outside Q by putting them equal to zero. In the case under consideration the extended functions belong to $\overset{\circ}{H}{}^1(Q')$ for any region $Q' \supset Q$.

If $f_h(x)$ is an averaging function for $f(x) \in \mathcal{M}$, then inequality (6), Sec. 2.3, holds:

$$\|f_h - f\|^2_{L_2(Q)} \leqslant \frac{C_0}{h^n} \int\limits_{|z|<h} dz \int\limits_Q |f(x+z) - f(x)|^2\, dx. \tag{17}$$

The function $f(x) \in \dot{C}^1(\overline{Q})$, also extended outside Q by assigning to it the value zero, satisfies for any vector z the identity $f(x+z) - f(x) = \int\limits_0^1 \frac{df(x+tz)}{dt}\, dt = \int\limits_0^1 (\nabla f(x+tz)\cdot z)\, dt$, thereby yielding

$$|f(x+z) - f(x)|^2 \leqslant |z|^2 \int\limits_0^1 |\nabla f(x+tz)|^2\, dt$$

and hence

$$\int\limits_Q |f(x+z) - f(x)|^2\, dx \leqslant |z|^2 \|f\|^2_{H^1(Q)}. \tag{18}$$

The inequality (18) also holds for any $f \in \mathcal{M}$; this can be proved by the usual limiting process.

It follows from (17) and (18) that

$$\|f_h - f\|^2_{L_2(Q)} \leqslant C_0 \|f\|^2_{H^1(Q)} \frac{h^2}{h^n} \int\limits_{|z|<h} dz \leqslant C_1^2 h^2,$$

where the constant C_1 is independent, in view of (16), of both h and f.

If it is now shown that for any fixed $h > 0$ the set $\mathcal{M}_h$ consisting of averaging functions $f_h(x)$ for all $f(x) \in \mathcal{M}$ is compact in $C(\overline{Q})$ (and therefore in $L_2(Q)$), the assertion of the theorem will follow from Corollary of Theorem 2, Sec. 3.7, Chap. II.

According to Property (d) of the averaging kernel (see Chap. I, Introduction), we have

$$|f_h(x)| \leqslant \frac{C_0}{h^n} \int\limits_Q |f(x)|\, dx \leqslant C_0' \|f\|_{L_2(Q)} \leqslant C_0' \|f\|_{H^1(Q)} \leqslant \text{const}$$

and

$$\left|\frac{\partial f_h}{\partial x_i}\right| \leqslant \frac{C_1}{h^{n+1}} \int\limits_Q |f(x)|\,dx \leqslant \text{const}, \qquad i=1, \ldots, n,$$

where, by (16), the constant does not depend on f. Now an application of Arzela's theorem shows that the set $\{f_h(x)\} = \mathscr{M}_h$[*] is compact in $C(\bar{Q})$.

Suppose now that $\mathscr{M} \subset H^1(Q)$. Denote by $\mathscr{M}'$ the set of functions $F(x)$ belonging to $\overset{\circ}{H}{}^1(Q')$ obtained by extension, according to Theorem 1, Sec. 4.2, of functions $f(x)$ belonging to $\mathscr{M}$ into some region Q', $Q \Subset Q'$. Since $\|F\|_{H^1(Q')} \leqslant \text{const}\,\|f\|_{H^1(Q)}$ with constant independent of f, the set $\mathscr{M}'$ is bounded in $\overset{\circ}{H}{}^1(Q')$. By what has been proved just now, it is compact in $L_2(Q')$. Hence the set $\mathscr{M}$ is compact in $L_2(Q)$. ■

5. On Compactness of the Set of Traces of Functions Belonging to $H^1(Q)$.

Theorem 4. *If a set of functions is bounded in* $H^1(Q)$, *then the set of their traces on the* $(n-1)$*-dimensional surface* $\Gamma \subset \bar{Q}$ *of class* C^1 *is compact in* $L_2(\Gamma)$.

Proof. Let the set $\mathscr{M}$ be bounded in $H^1(Q)$ and let $\mathscr{M}_\Gamma$ be the set of traces on Γ of functions belonging to $\mathscr{M}$. We denote by $\mathscr{M}'$ the set bounded in $H^1(Q')$ that consists of extensions into $Q' \Supset Q$ of functions belonging to $\mathscr{M}$ (Theorem 1, Sec. 4.2, $\partial Q \in C^1$).

Let Γ_0 be the part of the surface Γ which is uniquely projected into a region D in the plane $\{x_n = 0\}$, and let $x_n = \varphi(x')$, $x' \in D$, be the equation of Γ_0, $\varphi(x') \in C^1(\bar{D})$. There exists a $\delta > 0$ such that the region $\Omega_{2\delta} = \{x' \in D,\ \varphi(x') < x_n < \varphi(x') + 2\delta\}$ lies in Q'.

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[*] Suppose that the set $\mathfrak{M}$ of continuous functions in $\bar{Q}$ is uniformly bounded and equicontinuous: $\|g\|_{C(\bar{Q})} \leqslant \text{const}$ for all $g \in \mathfrak{M}$ and for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that for all $g \in \mathfrak{M}$ $|g(x') - g(x'')| < \varepsilon$ for arbitrary x', x'' in $\bar{Q}$ satisfying $|x' - x''| < \delta$ (in our case $\mathfrak{M} = \mathscr{M}_h$, and equicontinuity of $\mathscr{M}_h$ follows from uniform boundedness of derivatives). Let us show that the set $\mathfrak{M}$ is compact in $C(\bar{Q})$.

Let $\{g_k\}$ be an arbitrary infinite sequence of functions belonging to $\mathfrak{M}$. For every natural m take a finite set of points $\{x_q^m\}$, $q = 1, \ldots, p(m)$, in $\bar{Q}$ so that for every $x \in \bar{Q}$ there is a point in this set that is at a distance less than $\delta(2^{-m})$ from x. From the sequence $\{g_k\}$ we choose a subsequence $\{g_{k_1}\}$ converging at every point of the set $\{x_q^1\}$; then $\|g_{k_1} - g_{l_1}\|_{C(\bar{Q})} < 3 \cdot 2^{-1}$ for $k_1, l_1 \geqslant N_1$. From the sequence $\{g_{k_1}\}$ choose a subsequence $\{g_{k_2}\}$ converging at every point of the set $\{x_q^2\}$, and so forth. Thus for every m there is a sequence $\{g_{k_m}\}$ with the property that $\|g_{k_m} - g_{l_m}\|_{C(\bar{Q})} < 3 \cdot 2^{-m}$ for $k_m, l_m \geqslant N_m$. Evidently, the diagonal sequence $\{g_{m_m}\}$ is fundamental in $C(\bar{Q})$.

For any function $f(x) \in C^1(\bar{Q}')$ and any points $x = (x', x_n) \in \Gamma_0$ and $(x', y_n) \in \Omega_{2\delta}$ we have

$$f(x', y_n) - f(x) = \int_{x_n}^{y_n} \frac{\partial f(x', \xi_n)}{\partial \xi_n} \, d\xi_n,$$

which yields

$$|f(x)|^2 \leqslant 2 |f(x', y_n)|^2 + 4\delta \int_{x_n}^{x_n+2\delta} \left| \frac{\partial f(x', \xi_n)}{\partial \xi_n} \right|^2 d\xi_n.$$

We integrate this inequality with respect to $y_n \in (\delta, 2\delta)$ to obtain

$$\delta |f(x)|^2 \leqslant 2 \int_{\delta}^{2\delta} |f(x', y_n)|^2 \, dy_n + 4\delta^2 \int_{x_n}^{x_n+2\delta} \left| \frac{\partial f(x', \xi_n)}{\partial \xi_n} \right|^2 d\xi_n,$$

and then integrate the resulting inequality over Γ_0 with respect to x (that is, multiply it by $\sqrt{1 + \varphi_{x_1}^2 + \ldots + \varphi_{x_{n-1}}^2}$ and integrate over D) to have

$$\delta \int_{\Gamma_0} |f|^2 \, dS \leqslant \text{const} \left(2 \int_{Q'} |f|^2 \, dx + 4\delta^2 \int_{Q'} |\nabla f|^2 \, dx \right).$$

Since the surface Γ can be divided into a finite number of pieces of Γ_0 type and for each of such pieces the inequality just established holds, summing these inequalities we obtain

$$\|f\|_{L_2(\Gamma)}^2 \leqslant \frac{C_1'}{\delta} \|f\|_{L_2(Q')}^2 + C_2' \delta \|f\|_{H^1(Q')}^2,$$

where the constants C_1' and C_2' are independent of both f and δ. By the usual technique, it is found that this inequality is true not only for any $f \in C^1(\bar{Q}')$ but also for any function belonging to $H^1(Q')$.

By Remark to the theorem on extension (see Sec. 4.2), the last inequality yields the inequality

$$\|f\|_{L_2(\Gamma)}^2 \leqslant \frac{C_1}{\delta} \|f\|_{L_2(Q)}^2 + C_2 \delta \|f\|_{H^1(Q)}^2, \tag{19}$$

true for any $f \in H^1(Q)$.

By Theorem 3 (of the preceding subsection), the set $\mathcal{M}$ is compact in $L_2(Q)$. Therefore from any infinite sequence of elements of the set $\mathcal{M}$ a subsequence f_p, $p = 1, 2, \ldots$, can be chosen which is fundamental in $L_2(Q)$: given $\varepsilon > 0$ an N can be found such that for all $p \geqslant N$ and $q \geqslant N$ $\|f_p - f_q\|_{L_2(Q)} < \varepsilon$. But then the sequence of traces $f_p|_S$, $p = 1, 2, \ldots$, will be fundamental in $L_2(S)$, because the inequality (19) applied to $f_p - f_q$ and the inequality (16) imply, for all p, $q \geqslant N$,

$$\|f_p - f_q\|^2_{L_2(Q)} \leqslant \frac{C_1\varepsilon^2}{\delta} + C_2\delta \|f_p - f_q\|^2_{H^1(Q)} \leqslant \varepsilon(C_1 + 4C_2C^2) = C_3\varepsilon,$$

provided we take $\delta = \varepsilon$. ▮

6. Equivalent Norms in Spaces $H^1(Q)$ and $\mathring{H}^1(Q)$. Suppose that in a region Q, $\partial Q \in C^1$, there is defined a real symmetric matrix $P(x) = (p_{ij}(x))$, $i, j = 1, \ldots, n$, continuous in $\overline{Q}$. This means that the real-valued functions $p_{ij}(x) \in C(\overline{Q})$ and $p_{ij} = p_{ji}$, $i, j = 1, \ldots, n$. Suppose further that a real-valued function $q(x) \in C(\overline{Q})$ is defined in Q, while the real-valued function $r(x) \in C(\partial Q)$ is defined on ∂Q.

On $H^1(Q)$ we define the Hermitian bilinear form (see Sec. 2.4, Chap. II)

$$W(f, g) = \int_Q \sum_{i,j=1}^n p_{ij} f_{x_i} \overline{g}_{x_j}\, dx + \int_Q q f \overline{g}\, dx + \int_{\partial Q} r f \overline{g}\, dS \tag{20}$$

(in the rightmost integral, of course, $f = f|_{\partial Q}$, $g = g|_{\partial Q}$).

Theorem 5. *If the matrix $P(x)$ is positive-definite, that is, for any complex vector $\xi = (\xi_1, \ldots, \xi_n)$ and for all $x \in \overline{Q}$*

$$\sum_{i,j=1}^n p_{ij}(x)\xi_i\overline{\xi}_j \geqslant \gamma \sum_{i=1}^n |\xi_i|^2 \tag{21}$$

with constant $\gamma > 0$, the functions $q(x) \geqslant 0$ on $\overline{Q}$, $r(x) \geqslant 0$ on ∂Q and either $q(x) \not\equiv 0$ or $r(x) \not\equiv 0$, then the bilinear form (20) defines on $H^1(Q)$ a scalar product equivalent to the scalar product

$$(f, g)_{H^1(Q)} = \int_Q (\nabla f \nabla \overline{g} + f\overline{g})\, dx. \tag{22}$$

Proof. According to the definition (see Sec. 2.4, Chap. II), for the proof of this theorem it is enough to establish the existence of two constants $C_1 > 0$ and $C_2 > 0$ such that the inequalities

$$W(f, f) \leqslant C_1^2 \|f\|^2_{H^1(Q)}, \qquad \|f\|^2_{H^1(Q)} \leqslant C_2^2 W(f, f) \tag{23}$$

hold for all $f \in H^1(Q)$.

First note that by the hypothesis all the three terms in the expression for $W(f, f)$ (in (20) $g = f$) are nonnegative.

Since

$$\int_Q \sum_{i,j=1}^n p_{ij} f_{x_i} \overline{f}_{x_j}\, dx \leqslant A \int_Q \sum_{i,j=1}^n |f_{x_i}|\,|f_{x_j}|\, dx$$
$$\leqslant An \int_Q |\nabla f|^2\, dx \leqslant An \|f\|^2_{H^1(Q)},$$

where $A = \max\limits_{1 \leqslant i,\, j \leqslant n} \| p_{ij} \|_{C(\overline{Q})}$,

$$\int\limits_Q q\, |f|^2\, dx \leqslant A_1 \| f \|^2_{L_2(Q)} \leqslant A_1 \| f \|^2_{H^1(Q)},$$

where $A_1 = \| q \|_{C(\overline{Q})}$, and according to inequality (2) of Subsec. 1,

$$\int\limits_{\partial Q} r\, |f|^2\, dS \leqslant A_2 \| f \|^2_{L_2(\partial Q)} \leqslant C^2 A_2 \| f \|^2_{H^1(Q)},$$

where $A_2 = \| r \|_{C(\partial Q)}$, the first of inequalities (23) holds with the constant $C_1^2 = An + A_1 + A_2 C^2$.

We shall establish the second inequality in (23). Supposing, on the contrary, that there is no such constant C_2^2, a function $f_m(x) \in H^1(Q)$ can be found for any integer $m \geqslant 1$ such that $\| f_m \|^2_{H^1(Q)} > mW(f_m, f_m)$ or, equivalently, a function $g_m(x) \in H^1(Q)$ $(g_m = f_m / \| f_m \|_{H^1(Q)})$ can be found such that

$$\| g_m \|_{H^1(Q)} = 1 \tag{24}$$

and

$$W(g_m, g_m) = \int\limits_Q \sum_{i,\, j=1}^{n} p_{ij} g_{mx_i} \bar{g}_{mx_j}\, dx + \int\limits_Q q\, |g_m|^2\, dx + \int\limits_{\partial Q} r\, |g_m|^2\, dS < 1/m.$$

This inequality implies that each of the three terms in $W(g_m, g_m)$ is less than $1/m$, therefore (using the inequality (21)) the following inequalities hold:

$$\int\limits_Q |\nabla g_m|^2\, dx < \frac{1}{m\gamma}, \quad \int\limits_Q q\, |g_m|^2\, dx < \frac{1}{m}, \quad \int\limits_{\partial Q} r\, |g_m|^2\, dS < \frac{1}{m}. \tag{25}$$

By virtue of (24), the sequence g_m, $m = 1, 2, \ldots$, is bounded in $H^1(Q)$; accordingly (Theorem 3, Subsec. 4), from it a subsequence can be chosen which is fundamental in $L_2(Q)$. With no loss of generality, it can be assumed that the sequence g_m, $m = 1, 2, \ldots$, itself is fundamental in $L_2(Q)$, that is, $\| g_m - g_p \|_{L_2(Q)} \to 0$ as $m, p \to \infty$. Since, by the first inequality in (25),

$$\begin{aligned} \| g_m - g_p \|^2_{H^1(Q)} &= \| g_m - g_p \|^2_{L_2(Q)} + \| \, |\nabla (g_m - g_p)| \, \|^2_{L_2(Q)} \\ &\leqslant \| g_m - g_p \|^2_{L_2(Q)} + 2 \| \, |\nabla g_m| \, \|^2_{L_2(Q)} + 2 \| \, |\nabla g_p| \, \|^2_{L_2(Q)} \\ &\leqslant \| g_m - g_p \|^2_{L_2(Q)} + \frac{2}{m\gamma} + \frac{2}{p\gamma}, \end{aligned}$$

it follows that $\| g_m - g_p \|_{H^1(Q)} \to 0$ as $m, p \to \infty$, that is, the sequence g_m, $m = 1, 2, \ldots$, is fundamental in $H^1(Q)$ as well. Thus this sequence converges in the norm of $H^1(Q)$ to an element

$g \in H^1(Q)$. Letting $m \to \infty$ in (24) and (25), we obtain the following relations:

(a) $\| g \|_{H^1(Q)} = 1$,

(b) $\int\limits_Q |\nabla g|^2\, dx = 0$,

(c) $\int\limits_Q q\, |g|^2\, dx = 0$,

(d) $\int\limits_{\partial Q} r\, |g|^2\, dS = 0$.

The relations (b) and (a) imply that $g = \text{const} = 1/\sqrt{|Q|}$ in Q and $g|_{\partial Q} = 1/\sqrt{|Q|}$ on ∂Q, which contradicts (c) if $q(x) \not\equiv 0$ or (d) if $r(x) \not\equiv 0$. ∎

Let $P(x) = p(x)\, E$, where E is the identity matrix. Theorem 5 has the following corollary.

Corollary. *The bilinear form*

$$W(f, g) = \int\limits_Q (p \nabla f \nabla \bar{g} + q f \bar{g})\, dx + \int\limits_{\partial Q} r(x) f \bar{g}\, dS,$$

where $p(x) \in C(\bar{Q})$, $q(x) \in C(\bar{Q})$, $r(x) \in C(\partial Q)$, $p(x) \geqslant \text{const} > 0$, $q(x) \geqslant 0$ *in* $\bar{Q}$, $r(x) \geqslant 0$ *on* ∂Q *and either* $q(x) \not\equiv 0$ *in* Q *or* $r(x) \not\equiv 0$ *on* ∂Q, *defines in* $H^1(Q)$ *a scalar product equivalent to the scalar product* (22).

Theorem 6. *If the matrix* $P(x)$ *is positive-definite and the function* $q(x) \geqslant 0$ *in* $\bar{Q}$, *then the Hermitian bilinear form*

$$W_1(f, g) = \int\limits_Q \sum_{i, j=1}^{n} p_{ij} f_{x_i} \bar{g}_{x_j}\, dx + \int\limits_Q q f \bar{g}\, dx$$

defines a scalar product in $\mathring{H}^1(Q)$ *which is equivalent to the scalar product* (22).

Proof. Since $\mathring{H}^1(Q) \subset H^1(Q)$, it follows from Theorem 5 that a scalar product equivalent to the scalar product (22) can be defined in $\mathring{H}^1(Q)$ by means of the bilinear form (20) with $r(x) \equiv 1$ on ∂Q and $q(x) \geqslant 0$ in $\bar{Q}$. But for the functions $f(x)$ and $g(x)$ belonging to $\mathring{H}^1(Q)$ the values of bilinear forms W and W_1 coincide. ∎

Let $P(x) = p(x)\, E$. Theorem 6 implies the following result.

Corollary. *The bilinear form*

$$W(f, g) = \int\limits_Q (p \nabla f \nabla \bar{g} + q f \bar{g})\, dx,$$

where $p(x) \in C(\overline{Q})$, $q(x) \in C(\overline{Q})$, $p(x) \geqslant \text{const} > 0$, $q(x) \geqslant 0$ *in* $\overline{Q}$, *defines a scalar product in* $\mathring{H}^1(Q)$ *equivalent to the scalar product* (22).

In particular, the scalar product

$$(f, g)'_{\mathring{H}^1(Q)} = \int_Q \nabla f \nabla \bar{g}\, dx$$

is equivalent to the scalar product (22).

The last assertion readily yields *Steklov's inequality*

$$\| f \|^2_{L_2(Q)} \leqslant \text{const} \int_Q | \nabla f |^2\, dx,$$

which is true for any $f \in \mathring{H}^1(Q)$.

§ 6. PROPERTIES OF FUNCTIONS BELONGING TO $H^k(Q)$

In this section we shall examine mutual relationship between the spaces $H^k(Q)$ and $C^l(\overline{Q})$. It will be demonstrated that if a function belongs to $H^k(Q)$ for sufficiently large k, then it will also belong to $C^l(\overline{Q})$ (that is, the function can be so modified on a set of measure zero that it becomes continuous in $\overline{Q}$ together with all the derivatives up to order l).

To obtain this result, it is necessary to represent a sufficiently smooth function in Q in terms of the integral over Q of a combination of its derivatives.

1. Representation of Functions by Means of Integrals.

Theorem 1. *Let the function* $f(x) \in C^2(\overline{Q})$ *and let the space have dimension* $n \geqslant 2$. *Then for any point* $x \in Q$ *the following identity holds*:

$$f(x) = \int_Q U(x-\xi)\, \Delta f(\xi)\, d\xi + \int_{\partial Q} \left(f(\xi) \frac{\partial U(x-\xi)}{\partial n_\xi} - \frac{\partial f(\xi)}{\partial n_\xi} U(x-\xi) \right) dS_\xi, \tag{1}$$

where

$$U(x) = \begin{cases} -\dfrac{1}{(n-2)\,\sigma_n\, |x|^{n-2}} & \textit{when} \quad n > 2, \\ -\dfrac{1}{2\pi} \ln \dfrac{1}{|x|} & \textit{when} \quad n = 2, \end{cases} \tag{2}$$

and $\sigma_n = 2\pi^{n/2}/\Gamma(n/2)$ *is the surface area of the* $(n-1)$*-dimensional unit sphere**.

Proof. The function $U(x-\xi)$ (called the fundamental solution for the Laplace operator) as a function of ξ satisfies for $\xi \neq x$ the relation $\Delta_\xi U(\xi - x) = \left(\frac{\partial^2}{\partial \xi_1^2} + \ldots + \frac{\partial^2}{\partial \xi_n^2}\right) U(\xi - x) = 0$, as can be checked by direct differentiation.

Fix a point $x \in Q$ and take $\varepsilon > 0$ so small that the ball $\{|\xi - x| \leqslant \varepsilon\} \subset Q$. In the region $Q_\varepsilon = Q \setminus \{|x-\xi| \leqslant \varepsilon\}$ Green's formula (see Sec. 1.2)

$$\int_{Q_\varepsilon} \Delta f(\xi)\, U(x-\xi)\, d\xi = \int_{\partial Q} \left(U(\xi - x) \frac{\partial f(\xi)}{\partial n} - f(\xi) \frac{\partial U(\xi - x)}{\partial n_\xi} \right) dS_\xi$$
$$+ \int_{|\xi - x| = \varepsilon} \left(U(\xi - x) \frac{\partial f(\xi)}{\partial n} - f(\xi) \frac{\partial U(\xi - x)}{\partial n_\xi} \right) dS_\xi \qquad (3)$$

holds for the function $U(\xi - x)$ (regarded as a function of ξ) and any function $f(\xi) \in C^2(\bar{Q})$.

As $\frac{\partial}{\partial n_\xi} = -\frac{\partial}{\partial |\xi - x|}$ on the sphere $|\xi - x| = \varepsilon$, the second term on the right side of (3) has the form (when $n > 2$)

$$-\frac{1}{(n-2)\sigma_n \varepsilon^{n-2}} \int_{|\xi - x| = \varepsilon} \frac{\partial f(\xi)}{\partial n} dS_\xi + \frac{1}{\sigma_n \varepsilon^{n-1}} \int_{|\xi - x| = \varepsilon} f(\xi)\, dS_\xi$$
$$= f(x) + \frac{1}{\sigma_n \varepsilon^{n-1}} \int_{|\xi - x| = \varepsilon} (f(\xi) - f(x))\, dS_\xi$$
$$- \frac{1}{(n-2)\sigma_n \varepsilon^{n-2}} \int_{|\xi - x| = \varepsilon} \frac{\partial f(\xi)}{\partial n} dS_\xi = f(x) + O(\varepsilon), \qquad (4)$$

since the surface area of the sphere $|\xi - x| = \varepsilon$ is $\sigma_n \varepsilon^{n-1}$, and for $|\xi - x| = \varepsilon$ $f(\xi) - f(x) = O(\varepsilon)$ and $\left|\frac{\partial f(\xi)}{\partial n}\right| \leqslant \text{const}$.

The function $U(\xi - x)$ is integrable over Q, therefore the limit, as $\varepsilon \to 0$, of the left-hand side of (3) is equal to the integral over Q of the function $U(\xi - x)\, \Delta f(\xi)$. Letting $\varepsilon \to 0$ in (3) and using (4), we obtain (1) when $n > 2$. When $n = 2$, the above proof remains valid, the only difference being that the second term on the right side of (3), in contrast to (4), is now $f(x)+O(\varepsilon \ln \varepsilon)$. ▮

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* The representation (1) holds in one-dimensional case also ($Q = (a, b)$). The identity $f(x) \frac{1}{2} \int_a^b |x - \xi| f''(\xi)\, d\xi + \frac{1}{2}(f(a) + f(b)) - \frac{1}{2}((a - x) f'(a) + (b - x) f'(b))$, which is easily verified, can be put in form (1) if we introduce the function $U(x - \xi) = \frac{1}{2}|x - \xi|$. However, we won't have occasions to use the identity (1) when $n = 1$.

2. Continuity and Continuous Differentiability of Functions Belonging to $H^k(Q)$. In Theorem 1, any function $f \in \dot{C}^2(\overline{Q})$ was expressed in terms of integral over Q of its second derivatives. If the function is still more smooth, $f \in \dot{C}^k(\overline{Q})$, $k > 2$, then, along with representation (1), there are representations in terms of derivatives of order k. To obtain such representations, we require the following simple result.

Lemma 1. *Let $n \geqslant 3$. Then for any (real) μ the function*

$$u_\mu(x) = \begin{cases} \dfrac{|x|^{\mu+2}}{(\mu+2)(\mu+n)} & \textit{when } \mu \neq -2,\ \mu \neq -n, \\ (\ln|x|)/(n-2) & \textit{when } \mu = -2, \\ -\dfrac{\ln|x|}{|x|^{n-2}(n-2)} & \textit{when } \mu = -n \end{cases}$$

satisfies the equation $\Delta u_\mu = |x|^\mu$ for all $x \in R_n$, $x \neq 0$.

That the lemma is true can be seen by direct calculation.

Let the function $f \in \dot{C}^2(\overline{Q})$. By formula (1), we have for all $x \in Q$

$$f(x) = \int_Q U(x-\xi)\,\Delta f(\xi)\,d\xi.$$

In particular, when $n=2$

$$f(x) = \frac{1}{2\pi}\int_Q \Delta f(\xi) \ln|x-\xi|\,d\xi, \tag{5}$$

when $n=3$

$$f(x) = -\frac{1}{4\pi}\int_Q \frac{\Delta f(\xi)}{|x-\xi|}\,d\xi, \tag{6}$$

when $n>3$

$$f(x) = -\frac{1}{(n-2)\sigma_n}\int_Q \frac{\Delta f(\xi)}{|x-\xi|^{n-2}}\,d\xi. \tag{7}$$

Suppose that $n=4$, and the function $f(x) \in \dot{C}^3(\overline{Q})$. Integration of (7) by parts yields, on using the relation $\frac{1}{|x-\xi|^2} = \frac{1}{2}\Delta_\xi \ln|x-\xi|$ (Lemma 1)

$$f(x) = \frac{1}{4\sigma_4}\int_Q \Delta f \cdot \Delta_\xi \ln|x-\xi|\,d\xi = \frac{1}{4\sigma_4}\int_Q \nabla(\Delta f(\xi))\,\nabla_\xi \ln|x-\xi|\,d\xi. \tag{8}$$

When $n = 5$, the relation (7) and the identity $|x-\xi|^{-3} = -\frac{1}{2}\nabla_\xi \frac{1}{|x-\xi|}$ (Lemma 1) yield the representation

$$f(x) = \frac{1}{2\cdot 3\sigma_5}\int_Q \Delta f(\xi)\,\Delta_\xi \frac{1}{|x-\xi|}\,d\xi$$

$$= -\frac{1}{2\cdot 3\sigma_5}\int_Q \nabla(\Delta f(\xi))\,\nabla_\xi \frac{1}{|x-\xi|}\,d\xi \qquad (9)$$

for the function $f(x) \in \dot{C}^3(\bar{Q})$, and so on. Suppose that $f(x) \in \dot{C}^{2p}(\bar{Q})$, $p \geqslant 2$. Then for $n > 4p-3$ the identities

$$\frac{1}{|x-\xi|^{4p-4}} = C'_{4p-2}\Delta_\xi^{p-1}\frac{1}{|x-\xi|^{2p-2}}, \quad \frac{1}{|x-\xi|^{4p-3}} = C'_{4p-1}\Delta_\xi^{p-1}\frac{1}{|x-\xi|^{2p-1}},$$

which follow from Lemma 1, and (7) yield

$$f(x) = C''_{4p-2}\int_Q \frac{\Delta^p f(\xi)}{|x-\xi|^{2p-2}}\,d\xi \text{ when } n = 4p-2 \qquad (10_{4p-2})$$

and

$$f(x) = C''_{4p-1}\int_Q \frac{\Delta^p f(\xi)}{|x-\xi|^{2p-1}}\,d\xi \text{ when } n = 4p-1, \qquad (10_{4p-1})$$

where C'_i and C''_i are absolute constants. Since

$$\frac{1}{|x-\xi|^{4p-2}} = C'_{4p}\Delta_\xi^{p}\left(\frac{1}{|x-\xi|^{2p-2}}\right), \quad \frac{1}{|x-\xi|^{4p-1}} = C'_{4p+1}\Delta_\xi^{p}\left(\frac{1}{|x-\xi|^{2p-1}}\right)$$

for $n > 4p-1$, $p \geqslant 2$, we have, by (7),

$$f(x) = C''_{4p}\int_Q \nabla(\Delta^p f(\xi))\,\nabla_\xi\left(\frac{1}{|x-\xi|^{2p-2}}\right)d\xi \text{ when } n = 4p, \qquad (10_{4p})$$

and

$$f(x) = C''_{4p+1}\int_Q \nabla(\Delta^p f(\xi))\,\nabla_\xi\left(\frac{1}{|x-\xi|^{2p-1}}\right)d\xi \text{ when } n = 4p+1 \qquad (10_{4p+1})$$

for $f \in \dot{C}^{2p+1}(\bar{Q})$, $p \geqslant 2$, where C'_i, C''_i are absolute constants.

Since $\left|\nabla_\xi \frac{1}{|x-\xi|^s}\right| = \frac{s}{|x-\xi|^{s+1}}$, $s \geqslant 1$, the relations (6), (8), (9), (10_{4p-2})-(10_{4p+1}) yield the inequalities

$$|f(x)| \leqslant C_{4p-2}\int_Q \frac{|\Delta^p f(\xi)|}{|x-\xi|^{2p-2}}\,d\xi \text{ for } n = 4p-2,\ p > 1, \qquad (11_{4p-2})$$

$$|f(x)| \leqslant C_{4p-1}\int_Q \frac{|\Delta^p f(\xi)|}{|x-\xi|^{2p-1}}\,d\xi \text{ for } n = 4p-1,\ p \geqslant 1, \qquad (11_{4p-1})$$

for all $f \in \dot{C}^{2p}(\bar{Q})$ and the inequalities

$$|f(x)| \leqslant C_{4p} \int\limits_Q \frac{|\nabla\Delta^p f(\xi)|}{|x-\xi|^{2p-1}}\, d\xi \text{ for } n=4p,\ p \geqslant 1, \qquad (11_{4p})$$

$$|f(x)| \leqslant C_{4p+1} \int\limits_Q \frac{|\nabla\Delta^p f(\xi)|}{|x-\xi|^{2p}}\, d\xi \text{ for } n=4p+1,\ p \geqslant 1, \quad (11_{4p+1})$$

for all $f \in \dot{C}^{2p+1}(\bar{Q})$, where C_i are absolute constants.

From (5), by means of Bunyakovskii's inequality, we obtain

$$|f(x)| \leqslant \frac{1}{2\pi}\Big(\int\limits_Q |\Delta f|^2\, d\xi\Big)^{1/2}\Big(\int\limits_Q |\ln|x-\xi||^2\, d\xi\Big)^{1/2} \leqslant C\|f\|_{H^2(Q)},\ n=2,$$

from (11_{4p-2})

$$|f(x)| \leqslant C_{4p-2}\Big(\int\limits_Q |\Delta^p f|^2\, d\xi\Big)^{1/2}\Big(\int\limits_Q \frac{d\xi}{|x-\xi|^{4p-4}}\Big)^{1/2}$$

$$\leqslant C\|f\|_{H^{2p}(Q)},\quad n=4p-2>2,$$

and similarly from (11_{4p-1})-(11_{4p+1})

$$|f(x)| \leqslant C\|f\|_{H^{2p}(Q)}, \qquad n=4p-1 \geqslant 3,$$

$$|f(x)| \leqslant C\|f\|_{H^{2p+1}(Q)}, \qquad n=4p \geqslant 4,$$

$$|f(x)| \leqslant C\|f\|_{H^{2p+1}(Q)}, \qquad n=4p+1 \geqslant 5,$$

where the constant C depends on n and the region Q.

Thus the inequality

$$\|f\|_{C(\bar{Q})} \leqslant C\|f\|_{H^{[\frac{n}{2}]+1}(Q)} \qquad (12)$$

holds for all $f \in \dot{C}^{[\frac{n}{2}]+1}(\bar{Q})$, $n \geqslant 1$, where the constant does not depend on f. For $n=1$, this inequality readily follows from the representation

$$f(x) = \frac{1}{2}\int\limits_a^b \operatorname{sign}(x-\xi)\cdot f'(\xi)\, d\xi$$

for any function $f(x) \in \dot{C}^1([a, b])$.

If the function $f \in \dot{C}^{l+1+[\frac{n}{2}]}(\bar{Q})$ for some $l>0$, then, apart from (12), it also satisfies the inequality

$$\|f\|_{C^l(\bar{Q})} \leqslant C_l\|f\|_{H^{l+1+[\frac{n}{2}]}(Q)}, \qquad (13)$$

where the constant $C_l>0$ does not depend on f.

Indeed, for any vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonzero integer components, $|\alpha| \leqslant l$, by (12) we have

$$\|D^\alpha f\|_{C(\bar{Q})} \leqslant C\|D^\alpha f\|_{H^{1+[\frac{n}{2}]}(Q)} \leqslant C\|f\|_{H^{1+|\alpha|+[\frac{n}{2}]}(Q)} \leqslant C\|f\|_{H^{1+l+[\frac{n}{2}]}(Q)}.$$

Summing these inequalities over all α, $|\alpha| \leqslant l$, we obtain the inequality (13).

Suppose now $f \in \mathring{H}^{l+1+[\frac{n}{2}]}(Q)$ and $f_m(x)$, $m = 1, 2, \ldots$, is a sequence of functions in $\dot{C}^{l+1+[\frac{n}{2}]}(\bar{Q})$ that converges to f in the norm of $H^{l+1+[\frac{n}{2}]}(Q)$. By (13)

$$\|f_m - f_s\|_{C^l(\bar{Q})} \leqslant C\|f_m - f_s\|_{H^{l+1+[\frac{n}{2}]}(Q)} \to 0$$

as $m, s \to \infty$, that is, f_m, $m = 1, 2, \ldots$, is a fundamental sequence in the norm of $C^l(\bar{Q})$. This means that the limit function f belongs to $C^l(\bar{Q})$. Passing to the limit, as $m \to \infty$, in the inequality $\|f_m\|_{C^l(\bar{Q})} \leqslant C\|f_m\|_{H^{l+1+[\frac{n}{2}]}(Q)}$, we see that the inequality (13) holds for any $f \in \mathring{H}^{l+1+[\frac{n}{2}]}(Q)$.

Let $f \in H_{\text{loc}}^{l+1+[\frac{n}{2}]}(Q)$. Take any subregion $Q' \Subset Q$ and construct the function $\zeta(x) \in \dot{C}^\infty(\bar{Q})$ which is equal to 1 in Q'. The function $f \cdot \zeta \in \mathring{H}^{l+1+[\frac{n}{2}]}(Q)$, so it belongs to $\dot{C}^l(\bar{Q})$, which means that f belongs to $C^l(\bar{Q}')$. Since Q' is arbitrary, f belongs to $C^l(Q)$. Thus we have proved the following assertion.

Theorem 2. *A function belonging to the space* $H_{\text{loc}}^{l+1+[\frac{n}{2}]}(Q)$ *belongs to the space* $C^l(Q)$, *that is* $H_{\text{loc}}^{l+1+[\frac{n}{2}]}(Q) \subset C^l(Q)$.

Suppose now that $f \in H^{l+1+[\frac{n}{2}]}(Q)$ and let $\partial Q \in C^{l+1+[\frac{n}{2}]}$. Then by the theorem on extension for (any) region Q', $Q' \Supset Q$, there exists a function F belonging to $\mathring{H}^{l+1+[\frac{n}{2}]}(Q')$ that coincides in Q with f; moreover, $\|F\|_{H^{l+1+[\frac{n}{2}]}(Q')} \leqslant C'\|f\|_{H^{l+1+[\frac{n}{2}]}(Q)}$, where the constant

C' does not depend on f. The function $F \in C^l(\bar{Q}')$ and satisfies the inequality $\| F \|_{C^l(\bar{Q}')} \leqslant C'' \| F \|_{H^{l+1+[\frac{n}{2}]}(Q')}$ (inequality (13) for the function F in Q'). Accordingly, $f \in C^l(\bar{Q})$ and the inequality

$$\| f \|_{C^l(\bar{Q}')} \leqslant C'C'' \| f \|_{H^{l+1+[\frac{n}{2}]}(Q)}$$

holds. Thus we have shown the following theorem.

Theorem 3. *If* $\partial Q \in C^{l+1+[\frac{n}{2}]}$, *then* $H^{l+1+[\frac{n}{2}]}(Q) \subset C^l(\bar{Q})$. *Moreover, the inequality* (13), *where* $C > 0$ *does not depend on* f, *holds for any function* $f \in H^{l+1+[\frac{n}{2}]}(Q)$.

§ 7. SPACES $C^{r,0}$ AND $C^{2s,s}$. SPACES $H^{r,0}$ AND $H^{2s,s}$

We have so far examined function spaces (C^k, H^k, $k = 0, 1, \ldots$) composed of functions whose differentiability properties were the same with respect to all the independent variables: for example, the space H^k consists of all the functions belonging to L_2 whose all the generalized derivatives up to order k belong to L_2. In the theory of differential equations one has to use also sets composed of functions whose differentiability properties differ with respect to different variables. In particular, the spaces of functions introduced below will be used in Chap. VI on parabolic equations.

Let D be a bounded region in the space R_n ($x = (x_1, \ldots, x_n)$ is a point in R_n) and $Q_T = \{x \in D,\ 0 < t < T\}$ a cylinder of height $T > 0$ in the space $R_{n+1} = R_n \times \{-\infty < t < \infty\}$.

1. Banach Spaces $C^{r,0}(\bar{Q}_T)$ and $C^{2s,s}(\bar{Q}_T)$. We denote by $C^{r,0}(\bar{Q}_T)$, where the integer $r \geqslant 1$, the set of all functions $f(x, t)$ that are continuous in $\bar{Q}_T$ together with the derivatives $\dfrac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ for all (nonnegative integers) $\alpha_1, \ldots, \alpha_n$, $\alpha_1 + \ldots + \alpha_n \leqslant r$.

We let $C^{2s,s}(\bar{Q}_T)$, where the integer $s \geqslant 1$, denote the set of functions $f(x, t)$ that are continuous in $\bar{Q}_T$ together with the derivatives $\dfrac{\partial^{\alpha_1+\ldots\alpha_n+\beta} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \partial t^{\beta}}$ for all (nonnegative integers) $\alpha_1, \ldots, \alpha_n$, β, $\alpha_1 + \ldots + \alpha_n + 2\beta \leqslant 2s$.

When $r = 0$ and $s = 0$, the spaces $C^{r,0}(\bar{Q}_T)$ and $C^{2s,s}(\bar{Q}_T)$ will mean the space $C^{0,0}(\bar{Q}_T) = C(\bar{Q}_T)$.

It is clear that the set $C^{r,\,0}(\bar{Q}_T)$, $r \geqslant 0$, is a Banach space with the norm

$$\| f \|_{C^{r,\,0}(\bar{Q}_T)} = \sum_{\alpha_1+\ldots+\alpha_n=r} \max_{\bar{Q}_T} \left| \frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \right|,$$

while the set $C^{2s,\,s}(\bar{Q}_T)$, $s \geqslant 0$, is a Banach space with the norm

$$\| f \|_{C^{2s,\,s}(\bar{Q}_T)} = \sum_{\alpha_1+\ldots+\alpha_n+2\beta\leqslant 2s} \max_{\bar{Q}_T} \left| \frac{\partial^{\alpha_1+\ldots+\alpha_n+\beta} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \partial t^{\beta}} \right|.$$

2. Hilbert Spaces $H^{r,\,0}(Q_T)$ and $H^{2s,\,s}(Q_T)$. We denote by $H^{r,\,0}(Q_T)$, where the integer $r \geqslant 1$, the set of all functions $f(x, t)$ that together with the generalized derivatives $\frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$ for all (nonnegative integers) $\alpha_1, \ldots, \alpha_n$, $\alpha_1 + \ldots + \alpha_n \leqslant r$, belong to $L_2(Q_T)$. Similarly, $H^{2s,\,s}(Q_T)$, where the integer $s \geqslant 1$, denotes the set of all functions $f(x, t)$ that together with the generalized derivatives $\frac{\partial^{\alpha_1+\ldots+\alpha_n+\beta} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \partial t^{\beta}}$ for all (nonnegative integers) $\alpha_1, \ldots, \alpha_n, \beta$, $\alpha_1 + \ldots + \alpha_n + 2\beta \leqslant 2s$, belong to $L_2(Q_T)$.

The space $H^{r,\,0}(Q_T)$ with $r = 0$ and the space $H^{2s,\,s}(Q_T)$ with $s = 0$ will mean the space $H^{0,\,0}(Q_T) = L_2(Q_T)$.

The following properties of the sets $H^{r,\,0}(Q_T)$ and $H^{2s,\,s}(Q_T)$ are an immediate consequence of their definitions.

1. The set $H^{r,\,0}(Q_T)$, $r \geqslant 0$, is a Hilbert space with the scalar product

$$(f, g)_{H^{r,\,0}(Q_T)} = \int\limits_{Q_T} \sum_{\alpha_1+\ldots+\alpha_n\leqslant r} \frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \cdot \overline{\frac{\partial^{\alpha_1+\ldots+\alpha_n} g}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}}\, dx\, dt,$$

while the set $H^{2s,\,s}(Q_T)$, $s \geqslant 0$, is a Hilbert space with the scalar product

$$(f, g)_{H^{2s,\,s}(Q_T)} = \int\limits_{Q_T} \sum_{2\beta+\alpha_1+\ldots+\alpha_n\leqslant 2s} \frac{\partial^{\alpha_1+\ldots+\alpha_n+\beta} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \partial t^{\beta}} \cdot \overline{\frac{\partial^{\alpha_1+\ldots+\alpha_n+\beta}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \partial t^{\beta}}}\, dx\, dt.$$

2. For any r and s, $0 \leqslant r \leqslant 2s$, $H^{2s,\,s}(Q_T) \subset H^{2s,\,0}(Q_T) \subset H^{r,\,0}(Q_T)$.

3. If $f(x, t) \in H^{r,\,0}(Q_T)$, then $\frac{\partial^{\alpha_1+\ldots+\alpha_n} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}} \in H^{r-r',\,0}(Q_T)$ for $\alpha_1 + \ldots + \alpha_n = r' \leqslant r$.

4. If $f(x, t) \in H^{2s,s}(Q_T)$, then $\dfrac{\partial^{\alpha_1+\ldots+\alpha_n+\beta} f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}\,\partial t^{\beta}} \in H^{2(s-s'),\,s-s'}(Q_T)$ for $\alpha_1+\ldots+\alpha_n+2\beta \leqslant 2s'$, where $s' \leqslant s$.

We shall now show that if the boundary ∂D of the region D is sufficiently smooth, then the functions belonging to $H^{r,0}(Q_T)$ and $H^{2s,s}(Q_T)$ can be extended into a wider (than Q_T) region while preserving the smoothness. Namely, we shall prove the following.

5. Suppose that D' is an arbitrary region in R_n such that $D \Subset D'$, and t^0, t^1 are any numbers satisfying the inequalities $t^0 < 0$, $t^1 > T$. Let $Q'_{t^0,\,t^1}$ denote the cylinder $\{x \in D',\ t^0 < t < t^1\}$. If $\partial D \in C^r$, $r \geqslant 1$, then for any function $f \in H^{r,0}(Q_T)$ there is an extension $F \in H^{r,0}(Q'_{t^0,\,t^1})$ whose support is compact in $Q'_{t^0,\,t^1}$. Moreover, the inequality

$$\| F \|_{H^{r,0}(Q'_{t^0,\,t^1})} \leqslant C \| f \|_{H^{r,0}(Q_T)}, \tag{1}$$

where the positive constant C does not depend on f, holds. If $\partial D \in C^{2s}$, $s \geqslant 1$, then for any function $f \in H^{2s,s}(Q_T)$ there is an extension $F \in H^{2s,s}(Q'_{t^0,\,t^1})$ with compact support in $Q'_{t^0,\,t^1}$; moreover the inequality

$$\| F \|_{H^{2s,s}(Q'_{t^0,\,t^1})} \leqslant C \| f \|_{H^{2s,s}(Q_T)}, \tag{2}$$

where the constant $C > 0$ does not depend on f, holds.

The desired extension F of a function f belonging to $H^{r,0}(Q_T)$ or to $H^{2s,s}(Q_T)$ is obtained in two stages: first, f is extended to F_1 through the curved surface of the cylinder Q_T into a wider cylinder $Q'_T = \{x \in D',\ 0 < t < T\}$ of the same height T, and then the function F_1 is extended through the top $\{x \in D',\ t = T\}$ and base $\{x \in D',\ t = 0\}$ of the cylinder Q'_T.

To construct the function F_1, we employ the same technique as the one used in extending into D' a function belonging to $H^k(D)$ (see Sec. 4.2). We use the extension constructed there of functions belonging to a rectangular parallelepiped.

Let $\Pi_{a,T}$, $a > 0$, denote the rectangular parallelepiped $\{|x_i| < a,\ i = 1, \ldots, n,\ 0 < t < T\}$ and $\Pi^+_{a,T}$, $\Pi^-_{a,T}$ the rectangular parallelepipeds $\{|x_i| < a,\ i = 1, \ldots, n-1,\ 0 < x_n < a,\ 0 < t < T\}$ and $\{|x_i| < a,\ i = 1, \ldots, n-1,\ -a < x_n < 0,\ 0 < t < T\}$, respectively. Suppose that the function $z(x, t) \in C^k(\overline{\Pi}^+_{a,T})$ for some $k \geqslant 1$. The extension $Z(x, t)$ of the function $z(x, t)$ into the parallelepiped $\Pi_{a,T}$ is defined in the parallelepiped $\Pi^-_{a,T}$ in the following manner:

$$Z(x, t) = \sum_{i=1}^{k+1} A_i z\left(x', -\frac{x_n}{i}, t\right), \tag{3}$$

where $x' = (x_1, \ldots, x_{n-1})$ and $A_1, \ldots, A_{k+1}$ is the solution of the linear algebraic system

$$\sum_{i=1}^{k+1} A_i\left(-\frac{1}{i}\right)^s = 1, \quad s = 0, 1, \ldots, k.$$

While proving Lemma 1 in Sec. 4.2, it was established that $Z(x, t) \in C^k(\overline{\Pi}_{a,T})$, and for any $\alpha_1, \ldots, \alpha_n, \beta$, $\alpha_1 \geqslant 0, \ldots, \beta \geqslant 0$, $\alpha_1 + \ldots + \alpha_n + \beta \leqslant k$,

$$\left\| \frac{\partial^{\alpha_1+\ldots+\alpha_n+\beta} Z}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \partial t^{\beta}} \right\|_{L_2(\Pi_{a,T})} \leqslant C \left\| \frac{\partial^{\alpha_1+\ldots+\beta} z}{\partial x_1^{\alpha_1} \ldots \partial t^{\beta}} \right\|_{L_2(\Pi_{a,T}^+)},$$

where the constant $C > 0$ does not depend on z. Therefore for any $r \leqslant k$

$$\| Z \|_{H^{r,0}(\Pi_{a,T})} \leqslant C \| z \|_{H^{r,0}(\Pi_{a,T}^+)}, \tag{4}$$

and for any s, $2s \leqslant k$,

$$\| Z \|_{H^{2s,s}(\Pi_{a,T})} \leqslant C \| z \|_{H^{2s,s}(\Pi_{a,T}^+)}, \tag{5}$$

where the constant $C > 0$ does not depend on z.

Since the set $C^\infty(\overline{\Pi}_{a,T}^+)$ and hence also the set $C^k(\overline{\Pi}_{a,T}^+)$ is everywhere dense in the space $H^{r,0}(\Pi_{a,T}^+)$ and in $H^{2s,s}(\Pi_{a,T}^+)$ for any $k \geqslant r$ or $k \geqslant 2s$, respectively (this is proved in exactly the same way as analogous assertions regarding the space $H^k(D)$, see Property 6, Sec. 4.1), and since these spaces are complete, it follows from what has been said above that any function z belonging to $H^{r,0}(\Pi_{a,T}^+)$ or to $H^{2s,s}(\Pi_{a,T}^+)$ has an extension Z into $\Pi_{a,T}$ that belongs to $H^{r,0}(\Pi_{a,T})$ or to $H^{2s,s}(\Pi_{a,T})$, respectively; moreover, the function Z is defined in $\Pi_{a,T}^-$ by (3), and the inequality (4) or inequality (5) respectively holds. Arguing exactly in the same way as in the proof of Theorem 1 (Sec. 4.2), we obtain the function $F_1(x, t)$ which is the extension of $f(x, t)$ into the cylinder Q'_T; furthermore, if $f \in H^{r,0}(Q_T)$, then $F_1 \in H^{r,0}(Q'_T)$ and the inequality

$$\| F_1 \|_{H^{r,0}(Q'_T)} \leqslant C_1 \| f \|_{H^{r,0}(Q_T)}$$

holds; if $f \in H^{2s,s}(Q_T)$, then $F_1 \in H^{2s,s}(Q'_T)$ and the inequality

$$\| F_1 \|_{H^{2s,s}(Q'_T)} \leqslant C_2 \| f \|_{H^{2s,s}(Q_T)}$$

holds (the constants $C_1, C_2 > 0$ do not depend on f). What is more, $F_1 = 0$ in $Q'_T \setminus Q''_T$, where $Q''_T = \{x \in D'', 0 < t < T\}$ and D'' is a region in R_n such that $D \Subset D'' \Subset D'$.

We shall now construct the required extension F of f into the cylinder Q'_{t_0, t_1}.

When $f \in H^{r,0}(Q_T)$, for F we take the function equal to F_1 in Q'_T and to zero in $Q'_{t^0,t^1} \setminus Q'_T$. Evidently, F belongs to $H^{r,0}(Q'_{t^0,t^1})$, its support is compact in Q'_{t^0,t^1}, and it satisfies the inequality (1).

When $f \in H^{2s,s}(Q_T)$, then we define its extension F into the cylinder Q'_{t^0,t^1} by putting $F = \zeta(t) F_2(x, t)$, where $\zeta(t) \in C^\infty \times \times (-\infty, +\infty)$, $\zeta(t) = 1$ for $t \in (0, T)$, $\zeta(t) = 0$ for $t > \frac{T+t^1}{2}$ and for $t < \frac{t^0}{2}$, while the function $F_2(x, t)$ is equal to $F_1(x, t)$ in Q'_T, to $\sum_{i=1}^{s+1} A'_i F_1\left(x, \frac{tT}{it^0}\right)$ in $\{x \in D', \ t^0 < t < 0\}$ and to $\sum_{i=1}^{s+1} A''_i F_1\left(x, T - \frac{t-T}{i} \cdot \frac{T}{t^1 - T}\right)$ in $\{x \in D', \ T < t < t^1\}$, where $A'_1, \ldots, A'_{s+1}$ is the solution of the linear algebraic system of equations $\sum_{i=1}^{s+1} A'_i \left(\frac{T}{it^0}\right)^p = 1$, $p = 0, \ldots, s$ while $A''_1, \ldots, A''_{s+1}$ is the solution of the system $\sum_{i=1}^{s+1} A''_i \left(-\frac{T}{i(t^1 - T)}\right)^p = 1$, $p = 0, \ldots, s$. Evidently, the function $F \in H^{2s,s}(Q'_{t^0,t^1})$, has compact support in Q'_{t^0,t^1}, and satisfies the inequality (2).

Property 5 coupled with Lemma 1, Sec. 3.2, readily implies the following assertion (the corresponding assertion regarding the space H^k was obtained in Sec. 4.3).

6. If the boundary $\partial D \in C^r$, $r \geqslant 1$, then the set $C^\infty(\overline{Q}_T)$ is everywhere dense in $H^{r,0}(Q_T)$. If $\partial D \in C^{2s}$, $s \geqslant 1$, then the set $C^\infty(\overline{Q}_T)$ is everywhere dense in $H^{2s,s}(Q_T)$.

7. Let $f(x, t) \in H^{1,0}(Q_T)$ and S be an $(n-1)$-dimensional surface of class C^1 lying in $\overline{D}$; in particular, S may coincide with the boundary ∂D of D.

By $\Gamma_{S,T}$ we denote the cylindrical surface $\{x \in S, \ 0 < t < T\}$; the curved surface $\Gamma_{\partial D, T} = \{x \in \partial D, \ 0 < t < T\}$ of the cylinder Q_T will be denoted by Γ_T.

By Property 6 ($\partial D \in C^1$), there is a sequence of functions f_k, $k = 1, 2, \ldots$, in $C^1(\overline{Q}_T)$ such that $\lim_{k \to \infty} \| f_k - f \|_{H^{1,0}(Q_T)} = 0$. Since the functions $f_k(x, t)$, $k = 1, 2, \ldots$, regarded as functions of x, belong to $C^1(\overline{D})$ for any $t \in [0, T]$, the inequalities

$$\| f_k - f_s \|^2_{L_2(S)} \leqslant C^2 \| f_k - f_s \|^2_{H^1(D)}, \qquad k, s = 1, 2, \ldots, \tag{6}$$

hold for any $t \in [0, T]$ in which the positive constant C depends only on the region D and the surface S (see inequality (3), Sec. 5.1). Integration of (6) with respect to $t \in (0, T)$ yields the inequalities

$$\| f_k - f_s \|_{L_2(\Gamma_{S,T})} \leqslant C \| f_k - f_s \|_{H^{1,0}(Q_T)}, \qquad k, s = 1, 2, \ldots.$$

Since f_k, $k = 1, 2, \ldots$, is a fundamental sequence in $H^{1,0}(Q_T)$, the last inequalities imply that the sequence of values $f_k|_{(x,t)\in\Gamma_{S,T}}$, $k = 1, 2, \ldots$, of these functions on $\Gamma_{S,T}$ is fundamental in $L_2(\Gamma_{S,T})$. Accordingly, there exists a function $f_{\Gamma_{S,T}} \in L_2(\Gamma_{S,T})$ to which the sequence $f_k|_{(x,t)\in\Gamma_{S,T}}$, $k = 1, \ldots$, converges in $L_2(\Gamma_{S,T})$, and by repeating the arguments of Sec. 5.1, it can be easily shown that the function $f_{\Gamma_{S,T}}$ does not depend on the choice of the sequence f_k, $k = 1, 2, \ldots$, which approximates the function f.

It is natural to call the function $f_{\Gamma_{S,T}}$ the *trace of function* f (belonging to $H^{1,0}(Q_T)$) on the cylindrical surface $\Gamma_{S,T}$ and denote it by $f|_{\Gamma_{S,T}}$.

As in Sec. 5.1, it is easily shown that

$$\| f \|_{L_2(\Gamma_{S,T})} \leqslant C \| f \|_{H^{1,0}(Q_T)}$$

(here $\| f \|_{L_2(\Gamma_{S,T})} = \| f|_{\Gamma_{S,T}} \|_{L_2(\Gamma_{S,T})}$), where the constant $C > 0$ does not depend on f.

Note that if $\mathcal{M}$ is a bounded set of functions in $H^{1,0}(Q_T)$, then, by the last inequality, the set $\mathcal{M}'$ of traces of these functions on $\Gamma_{S,T}$ is bounded in $L_2(\Gamma_{S,T})$ but, in contrast to the case of the space $H^1(Q_T)$, is not compact.

The above notion of trace enables us to extend the formula of integration by parts to functions belonging to $H^{1,0}(Q_T)$. Namely, for any two functions f and g in $H^{1,0}(Q_T)$ there holds the formula of integration by parts (Ostrogradskii's formula)

$$\int_{Q_T} f_{x_i} g \, dx \, dt = \int_{\Gamma_T} f g n_i \, dS \, dt - \int_{Q_T} f g_{x_i} \, dx \, dt,$$

where n_i is the ith component of the n-dimensional (unit) vector which is outward normal to the surface ∂D, $i = 1, 2, \ldots, n$, and the functions f and g present under the integral sign over the curved surface Γ_T of the cylinder Q_T are traces of functions f and g on Γ_T. This formula can be easily proved (compare Sec. 5.2) by approximating in $H^{1,0}(Q_T)$ the functions f and g by functions belonging to $C^1(\bar{Q}_T)$.

If $f \in H^{r,0}(Q_T)$, $r \geqslant 1$, then any derivative of this function with respect to $x_1, \ldots, x_n$ of order less than r has the trace on the curved surface Γ_T of the cylinder Q_T. If $f \in H^{2s,\,s}(Q_T)$, $s \geqslant 1$, then any derivative $\frac{\partial^{\alpha_1+\ldots+\alpha_n+\beta} f}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}\,\partial t^{\beta}}$, $\alpha_1 + \ldots + \alpha_n + 2\beta < 2s$, has the trace on the curved surface Γ_T of the cylinder Q_T.

§ 8. EXAMPLES OF OPERATORS IN FUNCTION SPACES

1. Integral Operators. Fredholm Integral Equation. Let Q be a bounded region in the n-dimensional space R_n. On $Q \times Q$ we consider the measurable function $K(x, y)$. Assume that the function $f(y)$ is such that $K(x, y) f(y) \in L_1(Q)$ for almost all $x \in Q$ (for instance, $f = 0$). With every such function $f(y)$ we associate a function

$$g(x) = \int_Q K(x, y) f(y)\, dy. \tag{1}$$

This mapping can be regarded as an operator (linear, obviously) from $L_1(Q)$ into $L_1(Q)$, from $L_2(Q)$ into $L_2(Q)$, from $C(\overline{Q})$ into $C(\overline{Q})$ and so on. The function $K(x, y)$ is referred to as *kernel* of this operator. Of course, this operator may not be defined on the whole space, for example, for the operator from $C(\overline{Q})$ into $C(\overline{Q})$ the domain of definition is the set of all those functions in $C(\overline{Q})$ for which $g(x) \in C(\overline{Q})$. However, if the kernel $K(x, y) \in C(\overline{Q \times Q})$, then, as can be easily seen, this operator is defined everywhere (in $L_1(Q)$, $L_2(Q)$, $C(\overline{Q})$) and is bounded.

We shall examine the operator defined by formula (1) with the kernel $K(x, y) = K_0(x, y)\,|x - y|^{-\alpha}$, where $K_0(x, y) \in C(\overline{Q \times Q})$ and $0 \leqslant \alpha < n$, regarded as an operator from $C(\overline{Q})$ into $C(\overline{Q})$ and as an operator from $L_2(Q)$ into $L_2(Q)$; in both cases it will be denoted by K:

$$g = Kf. \tag{2}$$

The operator K is called the *Fredholm integral operator*. According to the results of Sec. 1.12, Chap. II, for any function $f \in C(\overline{Q})$ the function $g \in C(\overline{Q})$. This means that the operator K from $C(\overline{Q})$ into $C(\overline{Q})$ is defined on the whole of $C(\overline{Q})$.

Since the functions $\int_Q |K(x, y)|\, dy$ and $\int_Q |K(x, y)|\, dx$ are continuous in $\overline{Q}$, they are bounded, that is,

$$A = \max\left\{\max_{y\in\overline{Q}} \int_Q |K(x, y)|\, dx,\ \max_{x\in\overline{Q}} \int_Q |K(x, y)|\, dy\right\} < \infty. \tag{3}$$

Because for any point $x \in Q$

$$|g(x)| \leqslant \|f\|_{C(\bar{Q})} \int_Q |K(x, y)|\,dy \leqslant A \|f\|_{C(\bar{Q})},$$

so $\|g\|_{C(\bar{Q})} \leqslant A \|f\|_{C(\bar{Q})}$, which implies that the operator K from $C(\bar{Q})$ into $C(\bar{Q})$ is bounded and $\|K\| \leqslant A$.

Let $f(x) \in L_2(Q)$. The functions $|f(y)|^2 \int_Q |K(x, y)|\,dx$ and $\int_Q |K(x, y)|\,dx$ belong to $L_1(Q)$ (the latter even to $C(\bar{Q})$), therefore, by Corollary to Fubini's theorem, the functions $K(x, y)|f(y)|^2$ and $K(x, y)$ belong to $L_1(Q \times Q)$. This means that the function $K(x, y) f(y)$ also belongs to $L_1(Q \times Q)$, since $|K(x, y) f(y)| \leqslant \leqslant \frac{|K(x, y)|}{2} + \frac{|K(x, y)|\,|f(y)|^2}{2}$. Then by Fubini's theorem the functions $g(x) = \int_Q K(x, y) f(y)\,dy$, $\int_Q |K(x, y)|\,dy$ and $\int_Q |K(x, y)| \times \times |f(y)|^2\,dy$ belong to $L_1(Q)$. For almost all $x \in Q$ we have the inequality

$$|g(x)|^2 \leqslant \int_Q |K(x, y)|\,dy \cdot \int_Q |K(x, y)|\,|f(y)|^2\,dy$$

$$\leqslant A \int_Q |K(x, y)|\,|f(y)|^2\,dy,$$

implying that $g(x) \in L_2(Q)$. Integrate this inequality over Q and apply Fubini's theorem to obtain

$$\|g\|^2_{L_2(Q)} \leqslant A \int_Q dx \int_Q |K(x, y)|\,|f(y)|^2\,dy$$

$$= A \int_Q |f(y)|^2 \left(\int_Q |K(x, y)|\,dx \right) dy \leqslant A^2 \|f\|^2_{L_2(Q)}.$$

Thus the operator K from $L_2(Q)$ into $L_2(Q)$ is defined on the whole of $L_2(Q)$, is bounded and $\|K\| \leqslant A$.

Lemma 1. *The operator K acting from $L_2(Q)$ into $L_2(Q)$ is completely continuous. The operator K acting from $C(\bar{Q})$ into $C(\bar{Q})$ is completely continuous.*

Proof. 1. We first consider the operator K acting from $L_2(Q)$ into $L_2(Q)$. The function $K_N(x, y)$ defined for any $N > 0$ as follows

$$K_N(x, y) = \begin{cases} K(x, y) & \text{when } |x - y| \geqslant N^{-1} \\ K_0(x, y) N^{\alpha} & \text{when } |x - y| < N^{-1} \end{cases}$$

belongs to $C(\overline{Q \times Q})$. Since for any point $x \in \overline{Q}$ we have the inequality

$$\int_Q |K(x,\ y) - K_N(x,\ y)|\, dy$$

$$= \int_{Q \cap \{|x-y|<N^{-1}\}} |K_0(x,\ y)| \left(\frac{1}{|x-y|^{\alpha}} - N^{\alpha} \right) dy$$

$$\leqslant B \int_{|x-y|<N^{-1}} \frac{dy}{|x-y|^{\alpha}} = B \int_{|\xi|<N^{-1}} \frac{d\xi}{|\xi|^{\alpha}}$$

$$= B\sigma_n \int_0^{N^{-1}} \frac{d\rho}{\rho^{\alpha+1-n}} = \frac{B\sigma_n}{(n-\alpha)\, N^{n-\alpha}},$$

where $B = \| K_0 \|_{C(\overline{Q \times Q})}$ and σ_n is the surface area of the $(n-1)$-dimensional unit sphere, given an $\varepsilon > 0$ we can find an N such that

$$\max_{x \in \overline{Q}} \int_Q |K(x,\ y) - K_N(x,\ y)|\, dy < \frac{\varepsilon}{2}.$$

Because $K_N(x,\ y) \in C(\overline{Q \times Q})$, there is a polynomial $P(x,\ y)$ such that $|P(x,\ y) - K_N(x,\ y)| < \frac{\varepsilon}{2|Q|}$ for all $(x,\ y) \in \overline{Q \times Q}$. This implies that

$$\max_{x \in \overline{Q}} \int_Q |K(x,\ y) - P(x,\ y)|\, dy \leqslant \max_{x \in \overline{Q}} \int_Q |K(x,\ y) - K_N(x,\ y)\, dy$$

$$+ \max_{x \in \overline{Q}} \int_Q |K_N(x,\ y) - P(x,\ y)|\, dy < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad (4)$$

Similarly, it can be shown that

$$\max_{y \in \overline{Q}} \int_Q |K(x,\ y) - P(x,\ y)|\, dx < \varepsilon. \qquad (4')$$

The polynomial $P(x,\ y)$ and the function $G(x,\ y) = K(x,\ y) - P(x,\ y) = \frac{K_0(x,\ y) - P(x,\ y)\,|x-y|^{\alpha}}{|x-y|^{\alpha}}$ can be regarded as kernels of integral operators of the type (2); we denote them by P and G, respectively. Moreover, we have the relation

$$K = P + G,$$

and, by (4) and (4′), the estimate

$$\| G \| < \varepsilon.$$

Thus the operator K has been expressed as the sum of an operator G with arbitrarily small norm and the finite-dimensional operator P (this operator transforms $L_2(Q)$ into the set of polynomials whose degrees do not exceed that of $P(x, y)$). Therefore, by Theorem 4, Sec. 3.9, Chap. II, the operator K is completely continuous.

2. We now examine the operator K acting from $C(\overline{Q})$ into $C(\overline{Q})$. Because K is bounded, it maps a bounded set $\mathcal{M}$ in $C(\overline{Q})$ into a bounded set $\mathcal{M}'$. According to the results of Sec. 1.12, Chap. II, given an $\varepsilon > 0$ a $\delta > 0$ can be found such that $\int_Q |K(x', y) - K(x'', y)|\, dy < \varepsilon$ whenever $|x' - x''| < \delta$. Therefore for $|x' - x''| < \delta$

$$|g(x') - g(x'')| \leqslant \int_Q |K(x', y) - K(x'', y)|\,|f(y)|\, dy \leqslant \varepsilon \|f\|_{C(\overline{Q})}.$$

Thus the set $\mathcal{M}'$ of functions continuous in $\overline{Q}$ is uniformly bounded and equi-continuous. Hence, by Arzela's theorem, this set is compact. ■

The equation $\varphi = \mu K \varphi + f$, where μ is a complex parameter and K the Fredholm integral operator, that is, the equation

$$\varphi(x) = \mu \int_Q K(x, y)\, \varphi(y)\, dy + f(x), \tag{5}$$

is known as the *Fredholm integral equation* (of the second kind).

We shall examine Eq. (5) in $L_2(Q)$ ($f \in L_2(Q)$ and the desired solution φ will be sought in $L_2(Q)$).

In view of Lemma 1, Fredholm's theorems (Secs. 4.3-4.7, Chap. II) are applicable to Eq. (5). In particular, if μ is not a characteristic value of the operator K (such numbers are at most countable), there exists a bounded operator $(I - \mu K)^{-1}$, that is Eq. (5) has a unique solution $\varphi \in L_2(Q)$ with any free term $f \in L_2(Q)$.

If the kernel $K(x, y)$ is such that $K(x, y) = \overline{K(y, x)}$, the operator K from $L_2(Q)$ into $L_2(Q)$ is *selfadjoint*.

In fact, by Fubini's theorem,

$$\begin{aligned}(K\varphi, \psi)_{L_2(Q)} &= \int_Q \int_Q K(x, y)\, \varphi(y)\, dy\, \overline{\psi(x)}\, dx \\ &= \int_Q \varphi(y) \left(\int_Q K(x, y)\, \overline{\psi(x)}\, dx \right) dy \\ &= \int_Q \varphi(y) \left(\overline{\int_Q K(y, x)\, \psi(x)\, dx} \right) dy = (\varphi, K\psi)_{L_2(Q)}\end{aligned}$$

for any $\varphi, \psi \in L_2(Q)$.

Therefore all the results obtained in Sec. 5 of Chap. II for a general completely continuous selfadjoint operator are also true for the operator K. In particular, all the eigenvalues and characteristic values of K are real and there exists an orthonormal basis composed of eigenfunctions of this operator for the space $L_2(Q)$ (Corollary 2 to Theorem 2, Sec. 5.2, Chap. II).

2. Differential Operators. Suppose that in an n-dimensional region Q there is defined a bounded measurable function $a_\alpha(x)$ for every vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, with the integers $\alpha_i \geqslant 0$, $i = 1, \ldots, n$, $|\alpha| \leqslant k$, where $k \geqslant 1$. A linear operator acting from $L_2(Q)$ into $L_2(Q)$ that associates with the function f the function

$$(\mathcal{L}f)(x) = \sum_{|\alpha| \leqslant k} a_\alpha(x) D^\alpha f(x) \tag{6}$$

is called the *linear differential operator* (from $L_2(Q)$ into $L_2(Q)$). It will be assumed that the operator $\mathcal{L}$ is of order k, that is, at least one of the coefficients $a_\alpha(x)$ is different from zero for $|\alpha| = k$ (the set where $a_\alpha(x) \neq 0$ is not a set of measure zero).

The operator $\mathcal{L}$ is, of course, not defined on the whole of $L_2(Q)$. Nevertheless, the set of functions f for which expression (6) makes sense ($D^\alpha f$ is the generalized derivative) contains $H^k(Q)$. Accordingly, $H^k(Q)$ can be taken as the domain of definition of $\mathcal{L}$.

If all the functions $a_\alpha(x)$, $|\alpha| \leqslant k$, are continuous in $\overline{Q}$, formula (6) also defines a linear operator from $C(\overline{Q})$ into $C(\overline{Q})$ (the linear differential operator from $C(\overline{Q})$ into $C(\overline{Q})$). In this case as the domain of definition of $\mathcal{L}$ one can take $C^k(\overline{Q})$.

A particular case of $\mathcal{L}$ acting from $L_2(Q)$ into $L_2(Q)$ (from $C(\overline{Q})$ into $C(\overline{Q})$) is the operator D^α, $|\alpha| = k$, that associates with f in $H^k(Q)$ ($C^k(\overline{Q})$) its generalized (classical) derivative. The operator D^α from $L_2(Q)$ into $L_2(Q)$ is unbounded, because the sequence $f_m(x) = e^{im(x_1+\ldots+x_n)}$, $m = 1, 2, \ldots$, of functions in $H^k(Q)$ which is bounded in $L_2(Q)$ $\left(\| f_m \|_{L_2(Q)} = \sqrt{|Q|},\ m = 1, 2, \ldots\right)$ is mapped into the sequence $g_m(x) = (im)^{|\alpha|} e^{im(x_1+\ldots+x_n)}$, $m = 1, 2, \ldots$, unbounded in $L_2(Q)$ $\left(\| g_m \|_{L_2(Q)} = m^{|\alpha|} \sqrt{|Q|} \to \infty \text{ as } m \to \infty\right)$.

It can be similarly shown that the operator $\mathcal{L}$, $k \geqslant 1$, from $L_2(Q)$ into $L_2(Q)$ is also unbounded, and so are the operators D^α and $\mathcal{L}$ from $C(\overline{Q})$ into $C(\overline{Q})$.

If $\mathcal{L}$ is regarded as an operator from $H^k(Q)$ into $L_2(Q)$ or from $C^k(\overline{Q})$ into $C(\overline{Q})$, then it is bounded, because for any $f \in H^k(Q)$ ($C^k(\overline{Q})$)

$$\| \mathcal{L}f \|_{L_2(Q)} \leqslant \text{const} \| f \|_{H^k(Q)} \quad \left(\| \mathcal{L}f \|_{C(\overline{Q})} \leqslant \text{const} \| f \|_{C^k(\overline{Q})} \right).$$

PROBLEMS ON CHAPTER III

1. A ball $S = \{\| x \| < 1\}$ in a Banach space is said to be *strictly convex* if for any points x and y, $x \neq y$, on the unit sphere $\| x \| = \| y \| = 1$ and any $\alpha \in (0, 1)$ the point $\alpha x + (1 - \alpha) y \in S$, that is, $\| \alpha x + (1 - \alpha) y \| < 1$. Is the unit ball strictly convex in spaces $C(\overline{Q})$, $L_1(Q)$, $L_2(Q)$?

2. Let x be a point on the unit sphere in $C(\overline{Q})$ $(L_1(Q))$. Find the set of all points y on the unit sphere such that all the points of the segment $\alpha x + (1 - \alpha)y$, $0 \leqslant \alpha \leqslant 1$, lie on this sphere.

3. The set $\dot{C}^k(\overline{Q})$ is a linear manifold in $C^k(\overline{Q})$. Denote by $\mathring{C}^k(\overline{Q})$ the closure of this set in the norm $\max\limits_{x \in \overline{Q}} \sum\limits_{|\alpha| \leqslant k} |D^\alpha f(x)|$: $\mathring{C}^k(\overline{Q}) = \overline{\dot{C}^k(\overline{Q})}$. What functions is $\mathring{C}^k(\overline{Q})$ composed of?

4. Show that if $\partial Q \in C^k$, then $C^\infty(\overline{Q})$ is everywhere dense in $C^k(\overline{Q})$.

Let B be a Banach space, and $\mathcal{M}$, $\mathcal{N}$ its subspaces. We say that B is the direct sum of $\mathcal{M}$ and $\mathcal{N}$: $B = \mathcal{M} \oplus \mathcal{N}$ if any element f of B is uniquely expressed as the sum $f_1 + f_2$, where $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{N}$. If the Hilbert space $H = \mathcal{M} \oplus \mathcal{N}$ and $\mathcal{M} \perp \mathcal{N}$, then $\mathcal{M}$ $(\mathcal{N})$ is called orthogonal complement of $\mathcal{N}$ $(\mathcal{M})$ in H.

5. Express $C^k([a, b])$ as a direct sum of the subspace $\mathring{C}^k([a, b])$ and a subspace $\mathcal{N}$. Find the dimension of $\mathcal{N}$.

6. The set of functions belonging to $L_2(Q)$ that vanish (a.e.) in Q', $Q' \subset Q$, is a subspace of $L_2(Q)$. Find its orthogonal complement.

7. Consider the function $f(x) = r^\alpha \varphi$ in the plane $x = (x_1, x_2) = (r \cos \varphi, r \sin \varphi)$, $0 \leqslant \varphi < 2\pi$. For what α does the function f belong to $H^1(Q)$, where Q is (a) the disc $\{r < 1\}$, (b) $\{r < 1, \varphi \neq 0\}$?

8. Suppose that the sequence of functions $f_m(x)$, $m = 1, 2, \ldots$, in $C^k(\overline{Q})$ converges weakly in $L_2(Q)$ to a function f and the sequence $D^\alpha f_m$, $m = 1, 2, \ldots$, for some $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = k$, is bounded in $L_2(Q)$. Show that f has generalized derivative $D^\alpha f$.

9. Suppose that the sequence of functions $f_m(x)$, $m = 1, 2, \ldots$, in $C^1(\overline{Q})$ converges weakly in $L_2(Q)$, and the sequences $\frac{\partial f_m}{\partial x_i}$, $i = 1, \ldots, n$, $m = 1, 2, \ldots$, are bounded in $L_2(Q)$. Show that the sequence f_m, $m = 1, 2, \ldots$, converges strongly in $L_2(Q)$. Give an example of a sequence that satisfies the formulated conditions but is not compact in $H^1(Q)$.

10. Show that if the sequence of functions $f_m(x)$, $m = 1, 2, \ldots$, in $\dot{C}^k(\overline{Q})$, $k \geqslant 1$, converges weakly in $L_2(Q)$ to a function f and for all $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = k$, $\| D^\alpha f_m \|_{L_2(Q)} \leqslant \text{const}$, $m = 1, 2, \ldots$, then (a) $f \in \mathring{H}^k(Q)$, (b) the sequence f_m $m = 1, 2, \ldots$, converges to f strongly in $H^{k-1}(Q)$.

11. Prove that for any function $f(x) \in H^k(K)$ $(f(x) \in C^k(\overline{K}))$, where K is an n-dimensional cube, there is an extension $F(x)$, with compact support, into a wider region Q, $Q \Supset K$, that belongs to $H^k(Q)$ $(C^k(\overline{Q}))$, and satisfies the inequality $\| F \|_{H^k(Q)} \leqslant C \| f \|_{H^k(K)}$, where the constant $C > 0$ does not depend on f.

12. Let x^0 be a point in a region Q of the n-dimensional space R_n, $n > 1$. Show that the closure of the linear manifold of functions that are continuously differentiable in $\overline{Q}$ and vanish in some neighbourhood (different for different functions) of x^0 coincides with $H^1(Q)$.

13. Show that the set $\widetilde{H}^1(a, b)$ of all functions $f \in H^1(a, b)$ for which $f(a) = f(b)$ is a subspace of the space $H^1(a, b)$. Show that $\overset{\circ}{H}{}^1(a, b) \subset \widetilde{H}^1(a, b) \subset H^1(a, b)$. Find the orthogonal complement of $\overset{\circ}{H}{}^1(a, b)$ in $\widetilde{H}^1(a, b)$ and that of $\widetilde{H}^1(a, b)$ in $H^1(a, b)$, and construct orthonormal bases for the spaces $\overset{\circ}{H}{}^1(a, b)$, $\widetilde{H}^1(a, b)$ and $H^1(a, b)$.

Let a function $f \in L_2(K)$, where K is the cube $\{|x_i| < a,\ i = 1, \ldots, n\}$. According to Fubini's theorem, for almost all $x_n = \xi \in (-a, a)$ the function $f(x', \xi)$ is defined and belongs to $L_2(K')$, where K' is the $(n-1)$-dimensional cube $\{|x_i| < a,\ i = 1, \ldots, n-1\}$. This function will be referred to as the value of f on the section $K \cap \{x_n = \xi\}$. Similarly, for almost all $x' = \xi' \in K'$ a function $f(\xi', x_n)$ is defined and belongs to $L_2(-a, a)$. This function will be referred to as the value of f on the section $K \cap \{x' = \xi'\}$.

For a function $f \in H^1(K)$ there exists the trace $f|_{x_n=\xi}$, for all $x_n = \xi \in [-a, a]$, belonging to $L_2(K')$.

14. Prove that if $f \in H^1(K)$, then for almost all $\xi' \in K'$ its value $f(\xi', x_n)$ on the section $K \cap \{x' = \xi'\}$ belongs to the space $H^1(-a, a)$, for almost all $\xi \in (-a, a)$ its trace $f|_{n_n=\xi}$ and its value $f(x', \xi)$ on the section $K \cap \{x_n = \xi\}$ belong to $H^1(K')$.

15. Prove that the set of traces of all functions in $H^1(Q)$ on an $(n-1)$-dimensional surface $S \subset \overline{Q}$ does not coincide with $L_2(S)$.

16. Prove the following assertions:

(a) If $f \in H^1(Q)$, then $|f|$ also belongs to $H^1(Q)$,

(b) If the functions $f_1, \ldots, f_N$ belong to $H^1(Q)$, then the functions $\max(f_1, \ldots, f_N)$ and $\min(f_1, \ldots, f_N)$ also belong to $H^1(Q)$.

17. We shall say that a function $f(x)$ belongs to the class $C^{\alpha}(Q)$ for some α, $0 < \alpha < 1$, if for any strictly interior subregion Q', $Q' \Subset Q$, there is a constant $C = C(Q')$ such that for all points x', x'' in Q' the inequality $|f(x') - f(x'')| \leqslant C|x' - x''|^{\alpha}$ holds. If this inequality holds with some constant C for all x', x'' in $\overline{Q}$, then we say that the function $f(x)$ belongs to the class $C^{\alpha}(\overline{Q})$.

Show that if $f \in H_{\text{loc}}^{\left[\frac{n}{2}\right]+1}(Q)$ (Q is an n-dimensional region), then $f \in C^{\alpha}(Q)$ for any $\alpha < [n/2] + 1 - n/2$, and if $f \in \overset{\circ}{H}{}^{\left[\frac{n}{2}\right]+1}(Q)$ or $f \in H^{\left[\frac{n}{2}\right]+1}(Q)$ and $\partial Q \in C^{\left[\frac{n}{2}\right]+1}$, then $f \in C^{\alpha}(\overline{Q})$ for any $\alpha < [n/2] + 1 - n/2$.

18. Prove that any bounded set in $H^{k+1+\left[\frac{n}{2}\right]}(Q)$ $\left(Q \text{ is an } n\text{-dimensional region, } \partial Q \in C^{k+1+\left[\frac{n}{2}\right]}\right)$ is compact in $C^k(\overline{Q})$.

19. Suppose that the functions $k(x)$, $a(x)$, $\rho(x)$ belong to $C(\overline{Q})$, $\sigma(x) \in C(\partial Q)$, $k(x) > 0$, $a(x) \geqslant 0$, $\rho(x) \geqslant 0$ in $\overline{Q}$, $\sigma(x) \geqslant 0$ on ∂Q. Show that the bilinear form

$$W_1(f, g) = \int_Q (k\nabla f \nabla \bar{g} + af\bar{g})\,dx + \left(\int_Q \rho f\,dx\right)\left(\int_Q \rho \bar{g}\,dx\right)$$

with $a + \rho \not\equiv 0$ and the bilinear form

$$W_2(f, g) = \int_Q (k\nabla f \nabla \bar{g} + af\bar{g})\,dx + \left(\int_{\partial Q} \sigma f\,dS\right)\left(\int_{\partial Q} \sigma \bar{g}\,dS\right)$$

when either $a \not\equiv 0$ or $\sigma \not\equiv 0$ defined on $H^1(Q)$ determine scalar products in $H^1(Q)$ which are equivalent to the scalar product

$$(f,\ g)_{H^1(Q)} = \int\limits_Q (\nabla f \nabla \bar{g} + f\bar{g})\, dx.$$

20. Suppose that the function $k(x) \in C^2([0,\ 1])$ and $k(x) > 0$ for $x > 0$. By $H_k(0,\ 1)$ denote the completion of the set of functions in $C^1([0,\ 1])$ vanishing for $x = 1$ in the norm generated by the scalar product $\int\limits_0^1 k(x)\, f'(x)\, \bar{g}'(x)\, dx$.

Prove that $H_k(0,\ 1) \subset L_2(0,\ 1)$ if and only if $\overline{\lim\limits_{x \to +0}}\, k(x) \cdot x^{-2} > 0$.

21. Show that the scalar products $(f,\ g)' = \int\limits_Q \sum\limits_{|\alpha| \leqslant k} D^\alpha f D^\alpha \bar{g}\, dx$ and $(f,\ g)'' = \int\limits_Q \sum\limits_{|\alpha| = k} D^\alpha f D^\alpha \bar{g}\, dx$ are equivalent in the space $\mathring{H}^k(Q)$.

22. Let $f \in L_2(0,\ 1)$. The linear functional $l_f(u) = (f,\ u)_{L_2(Q)}$ is bounded in $\mathring{H}^k(0,\ 1)$ for any $k \geqslant 0$. By Riesz's theorem, there exists (a unique) element $F \in \mathring{H}^k(0,\ 1)$ such that $l_f(u) = (F,\ u)_{\mathring{H}^k(0,\ 1)}$ for all $u \in \mathring{H}^k(0,\ 1)$. Find F and show that $F \in \mathring{H}^k(0,\ 1) \cap H^{2k}(0,\ 1)$. $\Big($For scalar product take (a) $(f,\ g) = \int\limits_0^1 f^{(k)} \bar{g}^{(k)} dx$, (b) $(f,\ g) = \int\limits_0^1 (f^{(k)} \bar{g}^{(k)} + f\bar{g})\, dx$, where $f^{(k)} = \dfrac{d^k f}{dx^k}$. $\Big)$

SUGGESTED READING ON CHAPTER III

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CHAPTER IV

ELLIPTIC EQUATIONS

§ 1. GENERALIZED SOLUTIONS OF BOUNDARY-VALUE PROBLEMS. EIGENVALUE PROBLEMS

1. Classical and Generalized Solutions of Boundary-Value Problems. Suppose that in an n-dimensional region Q there is given the elliptic equation

$$\mathcal{L}u \equiv \operatorname{div}\,(k(x)\,\nabla u) - a(x)\,u = f(x), \tag{1}$$

whose coefficients are real-valued and satisfy the conditions

$$a(x) \in C(\overline{Q}), \quad k(x) \in C^1(\overline{Q}), \quad k(x) \geqslant k_0 > 0 \text{ for all } x \in Q.$$

The function $u(x)$ and the free term $f(x)$ of the equation may be, in general, complex-valued.

A function $u(x)$ belonging to $C^2(Q) \cap C(\overline{Q})$ is called a (*classical*) *solution of the first boundary-value problem* or the *Dirichlet problem* for Eq. (1) if it satisfies Eq. (1) in Q and the condition

$$u|_{\partial Q} = \varphi(x), \tag{2}$$

where $\varphi(x)$ is a given function, on the boundary ∂Q.

The function $u(x) \in C^2(Q) \cap C^1(\overline{Q})$ is called a (*classical*) *solution of the third boundary-value problem* for Eq. (1) if it satisfies Eq. (1) in Q and on the boundary ∂Q the condition

$$\left(\frac{\partial u}{\partial n} + \sigma(x)\,u\right)\Big|_{\partial Q} = \varphi(x), \tag{3}$$

where $\sigma(x) \in C(\partial Q)$ and $\varphi(x)$ are given functions. It will be assumed that $\sigma(x) \geqslant 0$.

If the function $\sigma(x)$ in (3) is identically zero, then the third boundary-value problem is termed the *second boundary-value problem* or the *Neumann problem.*

When $n = 1$, Eq. (1) becomes an ordinary differential equation

$$\mathcal{L}u \equiv (k(x)\,u')' - a(x)\,u = f(x). \tag{1_1}$$

The region Q in this case is an interval (α, β), while the boundary conditions of the first and third boundary-value problems are, respectively, of the form

$$u\,|_{x=\alpha} = \varphi_0, \quad u\,|_{x=\beta} = \varphi_1 \tag{2_1}$$

and

$$(-u' + \sigma_0 u)\,|_{x=\alpha} = \varphi_0, \quad (u' + \sigma_1 u)\,|_{x=\beta} = \varphi_1, \tag{3_1}$$

where φ_0, φ_1, $\sigma_0 \geqslant 0$, $\sigma_1 \geqslant 0$ are some given constants.

Suppose that the function $u(x)$ is a classical solution in Q of the first boundary-value problem (1), (2). We multiply (1) by an arbitrary function $\overline{v(x)} \in \mathring{C}^1(Q)$ and integrate the resulting identity over Q. By means of Ostrogradskii's formula, we obtain

$$\int_Q (k\nabla u\, \nabla\bar{v} + au\bar{v})\,dx = -\int_Q f\bar{v}\,dx \tag{4}$$

(the integral over the boundary ∂Q vanishes, because v has compact support).

If we additionally assume that the partial derivatives of the solution $u_{x_i} \in L_2(Q)$, $i = 1, \ldots, n$, that is, that $u(x) \in H^1(Q)$ and $f(x) \in L_2(Q)$, then the integral identity (4) holds not only for all $v(x) \in \mathring{C}^1(\overline{Q})$ but also for all $v \in \mathring{H}^1(Q)$. To see this, we consider any function $v \in \mathring{H}^1(Q)$ and a sequence of functions $v_k(x)$, $k = 1, 2, \ldots$, in $\mathring{C}^1(\overline{Q})$ that converges to v in the norm of $H^1(Q)$. The identity (4) holds for each of $v_k\,(x)$. Letting in this $k \to \infty$, we find that the identity (4) holds for v also.

Thus, if $f \in L_2(Q)$, the classical solution u of the problem (1), (2) belonging to the space $H^1(Q)$ satisfies the integral identity (4) for all $v \in \mathring{H}^1(Q)$.

We introduce the following definition.

A function $u \in H^1(Q)$ is called the *generalized solution of the problem* (1), (2) with $f \in L_2(Q)$ if it satisfies the identity (4) for all $v \in \mathring{H}^1(Q)$ and the boundary condition (2). In the boundary condition (2), the equality is understood as the equality of elements of $L_2(\partial Q)$, and $u|_{\partial Q}$ is the trace of u.

Note that the above definition of a generalized solution is not a complete generalization of the corresponding classical notion, because in order that a classical solution $u(x)$ be a generalized solution it should be subject to additional conditions of "integral character", namely, it must be such that $u \in H^1(Q)$ and $\mathscr{L}u \in L_2(Q)$, where $\mathscr{L}$ is the operator in (1).

In an analogous manner one can introduce the notion of generalized solution of third (second) boundary-value problem for Eq. (1).

Suppose that the function $u(x)$ is a classical solution of the third boundary-value problem (1), (3). Assume that the right-hand side $f(x)$ of Eq. (1) belongs to $L_2(Q)$ and the function $\varphi(x)$ present in the boundary condition (3) belongs to $L_2(\partial Q)$. We multiply (1) by an arbitrary function $\overline{v(x)} \in H^1(Q)$ and integrate the resulting identity over Q. Then Ostrogradskii's formula yields the integral identity

$$\int_Q (k\nabla u\nabla\bar{v} + au\bar{v})\,dx + \int_{\partial Q} k\sigma u\bar{v}\,dS = -\int_Q f\bar{v}\,dx + \int_{\partial Q} k\varphi\bar{v}\,dS, \tag{5}$$

which is satisfied by the classical solution $u(x)$ for all $v(x) \in H^1(Q)$.

We now introduce the following definition.

The function $u \in H^1(Q)$ is termed a *generalized solution of the third* (*second, if* $\sigma(x) \equiv 0$) *boundary-value problem for Eq.* (1) with $f \in L_2(Q)$, $\varphi \in L_2(\partial Q)$, if it satisfies (5) for all $v \in H^1(Q)$.

In defining the generalized solutions the functions v in identities (4) and (5) were assumed to be complex-valued, but they may as well be assumed real-valued. Indeed, if the function $u \in H^1(Q)$ satisfies, for example, the identity (4) for all complex-valued functions $v \in \overset{\circ}{H}{}^1(Q)$, then it obviously satisfies the same identity for all real-valued $v \in \overset{\circ}{H}{}^1(Q)$. Conversely, if the function $u \in H^1(Q)$ satisfies (4) for all real-valued $v \in \overset{\circ}{H}{}^1(Q)$, then the same identity holds also for any complex-valued $v = \operatorname{Re} v + i \operatorname{Im} v$ belonging to $\overset{\circ}{H}{}^1(Q)$, because it holds for functions $\operatorname{Re} v$ and $\operatorname{Im} v$ belonging to $\overset{\circ}{H}{}^1(Q)$.

Note that we have, in fact, already encountered (Sec. 3.1, Chap. I) generalized solutions of boundary-value problems for Eq. (1) (in the two-dimensional case) in deriving the equilibrium conditions of a membrane: the integral identities (4) and (5) appearing in the definition of generalized solutions coincide with the identities (4) and (1) of Sec. 3.1, Chap. I.

The definitions of generalized solutions of boundary-value problems for Eq. (1) apply equally well to one-dimensional case. A function $u \in H^1(\alpha, \beta)$ satisfying the boundary conditions (2_1) (from Theorem 3, Sec. 6.2, Chap. III, it follows that $u \in C([\alpha, \beta])$) is a generalized solution of the first boundary-value problem for Eq. (1_1) if for any $v \in \overset{\circ}{H}{}^1(\alpha, \beta)$

$$\int_\alpha^\beta (ku'\bar{v}' + au\bar{v})\,dx = -\int_\alpha^\beta f\bar{v}\,dx. \tag{4_1}$$

A function $u \in H^1(\alpha, \beta)$ is a generalized solution of the third (second) boundary-value problem for Eq. (1_1) if for any $v \in H^1(\alpha, \beta)$

$$\int_\alpha^\beta (ku'\bar{v}' + au\bar{v})\,dx + k(\beta)\,\sigma_1 u(\beta)\,\bar{v}(\beta) + k(\alpha)\,\sigma_0 u(\alpha)\,\bar{v}(\alpha)$$
$$= -\int_\alpha^\beta f\bar{v}\,dx + k(\beta)\,\varphi_1\bar{v}(\beta) + k(\alpha)\,\varphi_0\bar{v}(\alpha). \qquad (5_1)$$

The present section is devoted to the study of generalized solutions of boundary-value problems. Since the generalized solutions are elements of the Hilbert space $H^1(Q)$, the general results of Chap. II will be widely used.

The investigation of classical solutions of the boundary-value problems is a considerably more difficult problem and it is natural to divide this into two simpler problems: first, the generalized solution is constructed, and then by establishing (under appropriate conditions) its smoothness it is shown to be a classical solution. The smoothness of generalized solutions will be proved in the next section.

2. Existence and Uniqueness of Generalized Solution in the Simplest Case. To examine the questions of existence and uniqueness of generalized solutions of boundary-value problems, it is convenient to start with the case when the boundary conditions are homogeneous (that is, $\varphi = 0$). By definition, a generalized solution of the boundary-value problem (1), (2) with $\varphi = 0$ is the function $u \in \mathring{H}^1(Q)$ satisfying for all $v \in \mathring{H}^1(Q)$ the integral identity (4):

$$\int_Q (k\nabla u\,\nabla\bar{v} + au\bar{v})\,dx = -\int_Q f\bar{v}\,dx$$

The generalized solution of the third (second) boundary-value problem (1), (3) with $\varphi = 0$ is the function $u \in H^1(Q)$ satisfying for all $v \in H^1(Q)$ the integral identity

$$\int_Q (k\nabla u\,\nabla\bar{v} + au\bar{v})\,dx + \int_{\partial Q} k\sigma u\bar{v}\,dS = -\int_Q f\bar{v}\,dx. \qquad (6)$$

Suppose that $a(x) \geqslant 0$ in Q. Then, by Theorem 6, Sec. 5.6, Chap. III, a scalar product

$$(u, v)_{\mathring{H}^1(Q)} = \int_Q (k\nabla u\,\nabla\bar{v} + au\bar{v})\,dx, \qquad (7)$$

equivalent to the usual scalar product $\left((u, v) = \int_Q (\nabla u\,\nabla\bar{v} + u\bar{v})\,dx\right)$ can be introduced in the space $\mathring{H}^1(Q)$. Using this, the identity (4) may be put in the form

$$(u, v)_{\mathring{H}^1(Q)} = -(f, v)_{L_2(Q)}. \tag{8}$$

For a fixed $f \in L_2(Q)$ $(f, v)_{L_2(Q)}$ is a linear functional on $\mathring{H}^1(Q)$, $v \in \mathring{H}^1(Q)$. Since

$$|(f, v)_{L_2(Q)}| \leqslant \| f \|_{L_2(Q)} \| v \|_{L_2(Q)} \leqslant C \| f \|_{L_2(Q)} \| v \|_{\mathring{H}^1(Q)}$$

in which the positive constant C does not depend on f and v, this functional is bounded and its norm does not exceed $C \| f \|_{L_2(Q)}$.

By Riesz's theorem (Theorem 1, Sec. 3.2, Chap. II), there is a function F_1 in $\mathring{H}^1(Q)$ such that $(f, v)_{L_2(Q)} = (F_1, v)_{\mathring{H}^1(Q)}$ for all $v \in \mathring{H}^1(Q)$. Such a function is unique and satisfies the inequality $\| F_1 \|_{\mathring{H}^1(Q)} \leqslant C \| f \|_{L_2(Q)}$. Accordingly, in $\mathring{H}^1(Q)$ there is a unique function $u = F_1$ satisfying the identity (8).

Thus we have proved the following theorem.

Theorem 1. *If* $a(x) \geqslant 0$ *in* Q, *then for any* $f \in L_2(Q)$ *there exists a unique generalized solution* u *of the problem* (1), (2) (*with* $\varphi = 0$). *Moreover,*

$$\| u \|_{\mathring{H}^1(Q)} \leqslant C \| f \|_{L_2(Q)}, \tag{9}$$

where the positive constant C *does not depend on* f.

If $a(x) \geqslant 0$ and at least one of the functions $a(x)$ or $\sigma(x)$ does not vanish identically, then, by Corollary to Theorem 5, Sec. 5.6, Chap. III, the scalar product

$$(u, v)_{H^1(Q)} = \int_Q (k \nabla u \nabla \bar{v} + a u \bar{v})\, dx + \int_{\partial Q} k \sigma u \bar{v}\, dS, \tag{10}$$

equivalent to the usual scalar product, can be introduced in $H^1(Q)$. Therefore identity (6) can be rewritten as

$$(u, v)_{H^1(Q)} = -(f, v)_{L_2(Q)}. \tag{11}$$

Since for a fixed $f \in L_2(Q)$ the functional $(f, v)_{L_2(Q)}$, which is linear in $v \in H^1(Q)$, is bounded: $|(f, v)_{L_2(Q)}| \leqslant \| f \|_{L_2(Q)} \| v \|_{L_2(Q)} \leqslant C \| f \|_{L_2(Q)} \| v \|_{H^1(Q)}$, where the constant $C > 0$ does not depend on f or v, by Riesz's theorem there is a unique function F_2 in $H^1(Q)$ such that $(f, v)_{L_2(Q)} = -(F_2, v)_{H^1(Q)}$ for any $v \in H^1(Q)$, and $\| F_2 \|_{H^1(Q)} \leqslant C \| f \|_{L_2(Q)}$. Therefore in $H^1(Q)$ there exists a unique function $u = F_2$ satisfying (11).

Thus we have proved the following theorem.

Theorem 2. *If* $a(x) \geqslant 0$ *in* Q *and at least one of the functions* $a(x)$ *or* $\sigma(x)$ *does not vanish identically, then for any* $f \in L_2(Q)$ *there* *exists a unique generalized solution u of the problem* (1), (3) (*with* $\varphi = 0$). *Moreover,*

$$\|u\|_{H^1(Q)} \leqslant C\|f\|_{L_2(Q)}, \tag{12}$$

where the positive constant C does not depend on f.

Remark. If f is a real-valued function, then the solutions of the boundary-value problems obtained in Theorems 1 and 2 are also real-valued. Indeed, let $u = \operatorname{Re} u + i \operatorname{Im} u$ be the generalized solution of one of these boundary-value problems. Since the coefficients of the equation and the function f are real-valued, from (4) (or (6)) it follows that the function $\operatorname{Re} u$ is also a generalized solution of the same problem (the function v in (4) and (6) may be considered real-valued). The uniqueness of the solution implies that $u = \operatorname{Re} u$.

3. Eigenfunctions and Eigenvalues. A nonzero function $u(x)$ is called an *eigenfunction of the first boundary-value problem for the operator* $\mathscr{L} = \operatorname{div}(k(x)\nabla) - a(x)$ if there exists a number λ such that the function $u(x)$ is a classical solution of the following problem:

$$\mathscr{L}u = \lambda u, \quad x \in Q, \tag{13}$$

$$u|_{\partial Q} = 0. \tag{14}$$

The number λ is called the *eigenvalue* (corresponding to the eigenfunction $u(x)$).

It is obvious that to every eigenfunction there corresponds only one eigenvalue but not vice-versa. In particular, if $u(x)$ is an eigenfunction, then so is the function $cu(x)$ for any constant $c \neq 0$ corresponding to the same eigenvalue. Accordingly, we may consider eigenfunctions normalized, for instance, by the condition $\|u\|_{L_2(Q)} = 1$.

Let λ be an eigenvalue and $u(x)$ an eigenfunction of the first boundary-value problem, and let $u(x) \in \mathring{H}^1(Q)$. Multiplying (13) by an arbitrary $\bar{v} \in \mathring{H}^1(Q)$ and integrating the resulting equation over Q, we obtain the integral identity

$$\int_Q (k\nabla u\nabla\bar{v} + au\bar{v})\,dx = -\lambda\int_Q u\bar{v}\,dx \tag{15}$$

which is satisfied by u for all $v \in \mathring{H}^1(Q)$.

A nonzero function $u \in \mathring{H}^1(Q)$ is called the *generalized eigenfunction of the first boundary-value problem for the operator $\mathscr{L}$* if there is a number λ such that the function u satisfies the integral identity (15) for all $v \in \mathring{H}^1(Q)$; the number λ is called the *eigenvalue* (corresponding to the generalized eigenfunction u).

It will be assumed that $\| u \|_{L_2(Q)} = 1$.

A nonzero function $u(x)$ is called the *eigenfunction of the third (second) boundary-value problem for the operator* $\mathscr{L} = \operatorname{div}\,(k(x)\,\nabla) - a(x)$ if there is a number λ (the *eigenvalue* corresponding to $u(x)$) such that $u(x)$ is a classical solution of the following problem

$$\mathscr{L}u = \lambda u, \quad x \in Q,$$
$$\left(\frac{\partial u}{\partial n} + \sigma(x)\,u\right)\Big|_{\partial Q} = 0.$$

As is easy to see, the eigenfunction of the third (second) boundary-value problem satisfies for all $v \in H^1(Q)$ the integral identity

$$\int_Q (k\nabla u\nabla \bar{v} + au\bar{v})\,dx + \int_{\partial Q} k\sigma u\bar{v}\,dS = -\lambda \int_Q u\bar{v}\,dx. \qquad (16)$$

A nonzero function $u \in H^1(Q)$ is called the *generalized eigenfunction of the third (second) boundary-value problem for the operator* $\mathscr{L}$ if there is a number λ (the *eigenvalue* corresponding to u) such that the function u satisfies the identity (16) for all $v \in H^1(Q)$.

It will be assumed that $\| u \|_{L_2(Q)} = 1$.

The further consideration in this section will be confined to generalized eigenfunctions and their corresponding eigenvalues. It will be convenient to regard the identities (15) and (16) defining the generalized eigenfunctions as identities in the scalar products in spaces $L_2(Q)$ and $\mathring{H}^1(Q)$ or $H^1(Q)$, respectively.

Put $m = \min\limits_{x \in \bar{Q}} a(x)$ (here it is not assumed that $a(x) \geqslant 0$). Then the function

$$\tilde{a}(x) = a(x) - m + 1 \geqslant 1 \text{ in } Q.$$

Therefore a scalar product (equivalent to the usual scalar product) in $\mathring{H}^1(Q)$ can be defined by the formula

$$(u, v)_{\mathring{H}^1(Q)} = \int_Q (k\nabla u\nabla \bar{v} + \tilde{a}u\bar{v})\,dx, \qquad (17)$$

while in $H^1(Q)$ by

$$(u, v)_{H^1(Q)} = \int_Q (k\nabla u\nabla \bar{v} + \tilde{a}u\bar{v})\,dx + \int_{\partial Q} k\sigma u\bar{v}\,dS. \qquad (18)$$

Then (15) and (16) can be rewritten as

$$(u, v)_{\mathring{H}^1(Q)} = (-\lambda - m + 1)\,(u, v)_{L_2(Q)} \qquad (19)$$

and

$$(u, v)_{H^1(Q)} = (-\lambda - m + 1)\,(u, v)_{L_2(Q)}. \qquad (20)$$

We shall first prove the following assertions.

Lemma 1. *There is a bounded linear operator A acting from $L_2(Q)$ into $\overset{\circ}{H}{}^1(Q)$ with the domain of definition $L_2(Q)$ such that for all $v \in \overset{\circ}{H}{}^1(Q)$ the following relation holds:*

$$(u, v)_{L_2(Q)} = (Au, v)_{\overset{\circ}{H}{}^1(Q)}. \tag{21}$$

The operator A has an inverse A^{-1}. If A is regarded as an operator from $\overset{\circ}{H}{}^1(Q)$ into $\overset{\circ}{H}{}^1(Q)$, it is selfadjoint, positive and completely continuous.

Lemma 1′. *There is a bounded linear operator A' acting from $L_2(Q)$ into $H^1(Q)$ with the domain of definition $L_2(Q)$ such that for all $v \in H^1(Q)$ the following relation holds:*

$$(u, v)_{L_2(Q)} = (A'u, v)_{H^1(Q)}. \tag{21′}$$

*The operator A' has an inverse A'^{-1}. If A' is regarded as an operator from $H^1(Q)$ into $H^1(Q)$, it is selfadjoint, positive, and completely continuous**.

We shall prove Lemma 1; Lemma 1′ is proved in the same way.

Proof of Lemma 1. For any (fixed) function $u \in L_2(Q)$ the functional $l(v) = (u, v)_{L_2(Q)}$, which is linear in v, $v \in \overset{\circ}{H}{}^1(Q)$, is bounded, because

$$| l(v) | = | (u, v)_{L_2(Q)} | \leqslant \| u \|_{L_2(Q)} \| v \|_{L_2(Q)} \leqslant C \| u \|_{L_2(Q)} \| v \|_{\overset{\circ}{H}{}^1(Q)}.$$

Therefore, by Riesz's theorem, there exists a unique function $U \in \overset{\circ}{H}{}^1(Q)$, $\| U \|_{\overset{\circ}{H}{}^1(Q)} = \| l \| \leqslant C \| u \|_{L_2(Q)}$, such that $l(v) = (U, v)_{\overset{\circ}{H}{}^1(Q)}$ for all $v \in \overset{\circ}{H}{}^1(Q)$. This means that on $L_2(Q)$ an operator A is defined (which is obviously linear): $Au = U$, for which (21) holds. Since $\| Au \|_{\overset{\circ}{H}{}^1(Q)} \leqslant C \| u \|_{L_2(Q)}$, the operator A from $L_2(Q)$ into $\overset{\circ}{H}{}^1(Q)$ is bounded. If for some $u \in L_2(Q)$ $Au = 0$, then, by (21), $(u, v)_{L_2(Q)} = 0$ for all $v \in \overset{\circ}{H}{}^1(Q)$, that is, $u = 0$. This implies that the operator A^{-1} exists.

The operator A from $\overset{\circ}{H}{}^1(Q)$ into $\overset{\circ}{H}{}^1(Q)$ is selfadjoint, as can be seen from (21): $(Au, v)_{\overset{\circ}{H}{}^1(Q)} = (u, v)_{L_2(Q)} = \overline{(v, u)}_{L_2(Q)} = \overline{(Av, u)}_{\overset{\circ}{H}{}^1(Q)} = (u, Av)_{\overset{\circ}{H}{}^1(Q)}$; (21) also implies that the operator A is positive.

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* The form of operators A and A' depends, of course, on the scalar product defined in $\overset{\circ}{H}{}^1(Q)$ and $H^1(Q)$, respectively. Here scalar products (17) and (18) are used.

Let us show that A as an operator from $\overset{\circ}{H}^1(Q)$ into $\overset{\circ}{H}^1(Q)$ is completely continuous. Consider an arbitrary set of functions bounded in $\overset{\circ}{H}^1(Q)$. By Theorem 3, Sec. 5.4, Chap. III, this set is compact in $L_2(Q)$. This means that from any of its infinite subsets we can choose a sequence u_s, $s = 1, 2, \ldots$, which is fundamental in $L_2(Q)$. Since the operator A from $L_2(Q)$ into $\overset{\circ}{H}^1(Q)$ is bounded, and hence continuous, the sequence Au_s, $s = 1, 2, \ldots$, is fundamental in $\overset{\circ}{H}^1(Q)$. ∎

By Lemma 1, the identity (19) can be written in the form of an operator equation in the space $\overset{\circ}{H}^1(Q)$:

$$-(\lambda + m - 1)\,Au = u,\ u \in \overset{\circ}{H}^1(Q). \tag{22}$$

Similarly, (20) can be written, using Lemma 1′, as an operator equation in the space $H^1(Q)$:

$$-(\lambda + m - 1)\,A'u = u,\quad u \in H^1(Q). \tag{22'}$$

Thus the number λ is an eigenvalue of the first (third) boundary-value problem for the operator $\mathscr{L}$ and u is the corresponding generalized eigenfunction if and only if $-(\lambda + m - 1)$ is the characteristic value of the completely continuous selfadjoint operator A from $\overset{\circ}{H}^1(Q)$ into $\overset{\circ}{H}^1(Q)$ (A' from $H^1(Q)$ into $H^1(Q)$) and u is the corresponding eigenelement.

Therefore from the results of Sec. 5, Chap. II it follows that there is at most a countable set of eigenvalues of the first (third) boundary-value problem; this set does not have finite limit points; all the eigenvalues are real; to every eigenvalue there corresponds a finite number (the multiplicity of the eigenvalue) of mutually orthogonal eigenfunctions in $\overset{\circ}{H}^1(Q)$ (in $H^1(Q)$); the eigenfunctions corresponding to different eigenvalues are orthogonal in $\overset{\circ}{H}^1(Q)$ (in $H^1(Q)$).

Note that corresponding to each eigenvalue λ of the first (third) boundary-value problem one can choose exactly k, k being the multiplicity of λ, mutually orthogonal real eigenfunctions in $\overset{\circ}{H}^1(Q)$ (in $H^1(Q)$). Let $u = \operatorname{Re} u + i \operatorname{Im} u$ be the eigenfunction corresponding to the eigenvalue λ. Since λ and the coefficients $k(x)$ and $a(x)$ are real, the functions $\operatorname{Re} u$ and $\operatorname{Im} u$, as follows from (15) or (16), are also eigenfunctions corresponding to the same λ (the function v in (15) or (16) may be considered real-valued). It is not difficult to check that the maximum number of mutually orthogonal real eigenfunctions is k.

Let

$$\lambda_1, \lambda_2, \ldots, \lambda_s, \ldots \tag{23}$$

be the sequence of all the eigenvalues of the first (third) boundary-value problem for the operator $\mathscr{L}$ in which each eigenvalue is repeated according to its multiplicity. Let

$$u_1,\ u_2,\ \ldots,\ u_s,\ \ldots \tag{24}$$

be a system of mutually orthogonal generalized eigenfunctions ($\| u_s \|_{L_2(Q)} = 1$) in $\overset{\circ}{H}{}^1(Q)$ ($H^1(Q)$); each u_s corresponds to the eigenvalue λ_s:

$$-(\lambda_s + m - 1)\, A u_s = u_s, \qquad s = 1,\ \ldots, \tag{25}$$

for the first boundary-value problem and

$$-(\lambda_s + m - 1)\, A' u_s = u_s, \quad s = 1,\ \ldots, \tag{25'}$$

for the third boundary-value problem.

Scalar multiplication in $\overset{\circ}{H}{}^1(Q)$ ($H^1(Q)$) of (25) ((25′)) by u_s gives, in view of (21) ((21′)),

$$\| u_s \|^2_{\overset{\circ}{H}{}^1(Q)} = -(\lambda_s + m - 1)\, \| u_s \|^2_{L_2(Q)} = -(\lambda_s + m - 1), \tag{26}$$

$$\| u_s \|^2_{H^1(Q)} = -(\lambda_s + m - 1)\, \| u_s \|^2_{L_2(Q)} = -(\lambda_s + m - 1), \tag{26'}$$

which may be written (the scalar products in $\overset{\circ}{H}{}^1(Q)$ and $H^1(Q)$ are defined by formulas (17) and (18)) in the form

$$\int_Q k\, |\nabla u_s|^2\, dx + \int_Q (a + \lambda_s)\, |u_s|^2\, dx = 0 \tag{27}$$

for the first boundary-value problem and

$$\int_Q k\, |\nabla u_s|^2\, dx + \int_Q (a + \lambda_s)\, |u_s|^2\, dx + \int_{\partial Q} k\sigma\, |u_s|^2\, dS = 0 \tag{27'}$$

for the third boundary-value problem.

It follows from equality (27) that for all $s = 1, 2, \ldots$

$$\lambda_s < -m = -\min_{x \in \overline{Q}} a(x). \tag{28}$$

Similarly, from (27′) it follows that for all $s = 1, 2, \ldots$

$$\lambda_s \leqslant -m = -\min_{x \in \overline{Q}} a(x), \tag{28'}$$

and for all $s = 1, \ldots$ strict inequality holds if either $a(x) \not\equiv \text{const}$ or $\sigma(x) \not\equiv 0$. If, however, $\sigma(x) \equiv 0$ (the second boundary-value problem) and $a(x) \equiv \text{const}$, $a(x) \equiv m$, then among the eigenvalues of the second boundary-value problem there is one that equals $-m$ and the corresponding eigenfunction is equal to const $= 1/\sqrt{|Q|}$. This eigenvalue has multiplicity 1, because, by (27′), all the eigenfunctions corresponding to it satisfy the relation $\int\limits_Q k\,|\nabla u|^2\,dx =$ $= 0$, that is, are constants.

It follows from (26) ((26′)) that the system

$$\frac{u_1}{\sqrt{1-m-\lambda_1}}, \ \ldots, \ \frac{u_s}{\sqrt{1-m-\lambda_s}}, \ \ldots \tag{$\tilde{24}$}$$

is orthonormal in $\mathring{H}^1(Q)$ (in $H^1(Q)$). By Corollary 1 to Theorem 2, Sec. 5.2, Chap. II, this system is an orthonormal basis for $\mathring{H}^1(Q)$ (for $H^1(Q)$). And since the space $\mathring{H}^1(Q)$ ($H^1(Q)$) is infinite-dimensional, it follows that the set ($\tilde{24}$), and therefore also (23), is infinite. Hence $\lambda_s \to -\infty$ as $s \to \infty$.

Scalar multiplication in $\mathring{H}^1(Q)$ ($H^1(Q)$) of (25) ((25′)) by u_j, $j \neq s$, and the use of (21) ((21′)), gives the identity $-(\lambda_s + m - 1) \times$ $\times (u_s, u_j)_{L_2(Q)} = 0$, that is, the system (24) is orthonormal in $L_2(Q)$. As the linear manifold spanned by the system ($\tilde{24}$) (and therefore by the system (24)) is everywhere dense in $\mathring{H}^1(Q)$ ($\mathring{H}^1(Q)$), it is also everywhere dense in $L_2(Q)$. Accordingly, the system (24) is an orthonormal basis for $L_2(Q)$, that is, any element $f \in L_2(Q)$ can be expanded in a convergent Fourier series in $L_2(Q)$:

$$f = \sum_{s=1}^{\infty} f_s u_s, \qquad f_s = (f, u_s)_{L_2(Q)} \tag{29}$$

and the Parseval-Steklov equality holds:

$$\|f\|^2_{L_2(Q)} = \sum_{s=1}^{\infty} |f_s|^2.$$

Suppose that the function $f \in \mathring{H}^1(Q)$ ($H^1(Q)$). It can be expanded in a Fourier series with respect to the orthonormal basis ($\tilde{24}$) that converges in $\mathring{H}^1(Q)$ ($H^1(Q)$):

$$= \sum_{s=1}^{\infty} \left(f, \frac{u_s}{\sqrt{1-m-\lambda_s}}\right)_{\mathring{H}^1(Q)} \frac{u_s}{\sqrt{1-m-\lambda_s}} \tag{30}$$

for the first boundary-value problem ($f \in \mathring{H}^1(Q)$) and

$$f = \sum_{s=1}^{\infty} \left(f, \frac{u_s}{\sqrt{1-m-\lambda_s}}\right)_{H^1(Q)} \frac{u_s}{\sqrt{1-m-\lambda_s}} \tag{30′}$$

for the third boundary-value problem ($f \in H^1(Q)$). Moreover, the following Parseval-Steklov equalities hold:

$$\sum_{s=1}^{\infty} \left| \left(f, \frac{u_s}{\sqrt{1-m-\lambda_s}} \right)_{\overset{\circ}{H}{}^1(Q)} \right|^2 = \| f \|^2_{\overset{\circ}{H}{}^1(Q)}$$

and

$$\sum_{s=1}^{\infty} \left| \left(f, \frac{u_s}{\sqrt{1-m-\lambda_s}} \right)_{H^1(Q)} \right|^2 = \| f \|^2_{H^1(Q)}.$$

The series (30) ((30′)), of course, converges to f in the norm of $L_2(Q)$ also. A comparison of this series with the series (29) shows that $f_s = (f, u_s)_{L_2(Q)} = \left(f, \frac{u_s}{\sqrt{1-m-\lambda_s}} \right)_{\overset{\circ}{H}{}^1(Q)} \frac{1}{\sqrt{1-m-\lambda_s}}$ $\left(f_s = \left(f, \frac{u_s}{\sqrt{1-m-\lambda_s}} \right)_{H^1(Q)} \frac{1}{\sqrt{1-m-\lambda_s}} \right)$. Therefore

$$\| f \|^2_{\overset{\circ}{H}{}^1(Q)} = \sum_{s=1}^{\infty} \left| \left(f, \frac{u_s}{\sqrt{1-m-\lambda_s}} \right)_{\overset{\circ}{H}{}^1(Q)} \right|^2 = \sum_{s=1}^{\infty} (1-m-\lambda_s) | f_s |^2$$

$$= (1-m) \| f \|^2_{L_2(Q)} - \sum_{s=1}^{\infty} \lambda_s | f_s |^2$$

$$\left(\| f \|^2_{H^1(Q)} = (1-m) \| f \|^2_{L_2(Q)} - \sum_{s=1}^{\infty} \lambda_s | f_s |^2 \right),$$

whence it follows, in view of (28), that

$$\sum_{s=1}^{\infty} | \lambda_s | \, | f_s |^2 \leqslant - \sum_{s=1}^{\infty} \lambda_s | f_s |^2 + 2 | m | \sum_{s=1}^{\infty} | f_s |^2$$

$$\leqslant \| f \|^2_{\overset{\circ}{H}{}^1(Q)} + (2 | m | + | m-1 |) \| f \|^2_{L_2(Q)}$$

and similarly (in view of (28′)) that

$$\sum_{s=1}^{\infty} | \lambda_s | \, | f_s |^2 \leqslant \| f \|^2_{H^1(Q)} + (2 | m | + | m-1 |) \| f \|^2_{L_2(Q)}.$$

Thus we obtain the inequality

$$\sum_{s=1}^{\infty} | \lambda_s | \, | f_s |^2 \leqslant C \| f \|^2_{\overset{\circ}{H}{}^1(Q)}, \tag{31}$$

where λ_s, $s = 1, 2, \ldots$, are eigenvalues of the first boundary-value problem with $f \in \mathring{H}^1(Q)$, and the inequality

$$\sum_{s=1}^{\infty} |\lambda_s| \, |f_s|^2 \leqslant C \|f\|^2_{H^1(Q)}, \tag{32}$$

where λ_s, $s = 1, 2, \ldots$, are eigenvalues of the third boundary-value problem with $f \in H^1(Q)$. The constant C in (31) and (32) does not depend on f. Thus we have proved the following theorem.

Theorem 3. *The eigenvalues* $\lambda_1, \lambda_2, \ldots$ *of the first or third (second) boundary-value problem for the operator* $\mathcal{L} = \operatorname{div}(k(x)\nabla) - a(x)$ *are real and* $\lambda_s \to -\infty$ *as* $s \to \infty$. *When* $a(x) \not\equiv \text{const}$, *the eigenvalues of the first and third, with* $\sigma \not\equiv 0$, *boundary-value problems as well as those of the second boundary-value problem* ($\sigma \equiv 0$) *satisfy the inequality* $\lambda_s < -\min\limits_{x \in \bar{Q}} a(x)$ *for all* $s = 1, 2, \ldots$, *When* $a(x)$ *is a constant*, $a(x) \equiv m$, *the eigenvalues of the second boundary-value problem satisfy the inequality* $\lambda_s \leqslant -m$, $s = 1, 2, \ldots$, *and there is an eigenvalue that is equal to* $-m$ *whose multiplicity is* 1 *and to which there corresponds the generalized eigenfunction* $1/\sqrt{|Q|}$. *The generalized eigenfunctions* $u_1(x), u_2(x), \ldots$ *of the boundary-value problems under consideration constitute an orthonormal basis for* $L_2(Q)$, *that is, any function* $f \in L_2(Q)$ *can be expanded in a Fourier series* (29) *which converges in* $L_2(Q)$. *When* $f \in \mathring{H}^1(Q)$, *the series* (29) *in terms of the generalized eigenfunctions of the first boundary-value problem converges in* $\mathring{H}^1(Q)$ *and the inequality* (31) *holds. When* $f \in H^1(Q)$, *the series* (29) *in terms of the generalized eigenfunctions of the third (second) boundary-value problem converges in* $H^1(Q)$ *and the inequality* (32) *holds.*

4. Variational Properties of Eigenvalues and Eigenfunctions. The operator A defined by the relation (21) and acting from $\mathring{H}^1(Q)$ into $\mathring{H}^1(Q)$ is selfadjoint, completely continuous and positive (Lemma 1), therefore, by Theorem 1, Sec. 5.1, Chap II, its first characteristic value, obviously positive, is

$$\mu_1 = \inf_{f \in \mathring{H}^1(Q)} \frac{\|f\|^2_{\mathring{H}^1(Q)}}{(Af, f)_{\mathring{H}^1(Q)}} = \inf_{f \in \mathring{H}^1(Q)} \frac{\|f\|^2_{\mathring{H}^1(Q)}}{\|f\|^2_{L_2(Q)}}$$

(the norm of f in $\mathring{H}^1(Q)$ is defined corresponding to the scalar product (17)). The functional $\|f\|^2_{\mathring{H}^1(Q)} / \|f\|^2_{L_2(Q)}$ assumes the value μ_1 when $f = u_1$, where u_1 is the first eigenelement of the operator A. Therefore the first eigenvalue of the first boundary-value problem for the operator $\mathscr{L}$ is given by

$$\lambda_1 = -m+1-\inf_{f\in \mathring{H}^1(Q)} \frac{\|f\|^2_{\mathring{H}^1(Q)}}{\|f\|^2_{L_2(Q)}} = -\inf_{f\in \mathring{H}^1(Q)} \frac{\int\limits_Q (k\,|\nabla f|^2+a\,|f|^2)\,dx}{\int\limits_Q |f|^2\,dx} \tag{33}$$

and the exact lower bound of the functional

$$\int\limits_Q (k\,|\nabla f|^2+a\,|f|^2)\,dx \Big/ \int\limits_Q |f|^2\,dx$$

on the space $\mathring{H}^1(Q)$ is attained through the first eigenfunction u_1.

The results of Sec. 5.1, Chap. II, imply that $(k+1)$th characteristic value μ_{k+1} of the operator A is equal to $\inf\limits_{\substack{f\in \mathring{H}^1(Q)\\ (f,\,u_i)_{\mathring{H}^1(Q)}=0\\ i=1,\,\ldots,\,k}} \dfrac{\|f\|^2_{\mathring{H}^1(Q)}}{\|f\|^2_{L_2(Q)}}$.

Since, according to (21), $(f,\,u_i)_{\mathring{H}^1(Q)} = \mu_i\,(f,\,Au_i)_{\mathring{H}^1(Q)} = \mu_i\,(f,\,u_i)_{L_2(Q)}$, $i=1,\,2,\,\ldots$, it follows that

$$\mu_{k+1} = \inf_{\substack{f\in \mathring{H}^1(Q)\\ (f,\,u_i)_{L_2(Q)}=0\\ i=1,\,\ldots,\,k}} \frac{\|f\|^2_{\mathring{H}^1(Q)}}{\|f\|^2_{L_2(Q)}}.$$

Thus the $(k+1)$th eigenvalue of the first boundary-value problem for the operator $\mathscr{L}$ is given by

$$\lambda_{k+1} = -m+1-\inf_{\substack{f\in \mathring{H}^1(Q)\\ (f,\,u_i)_{L_2(Q)}=0\\ i=1,\,\ldots,\,k}} \frac{\|f\|^2_{\mathring{H}^1(Q)}}{\|f\|^2_{L_2(Q)}}$$

$$= -\inf_{\substack{f\in \mathring{H}^1(Q)\\ (f,\,u_i)_{L_2(Q)}=0\\ i=1,\,\ldots,\,k}} \frac{\int\limits_Q (k\,|\nabla f|^2+a\,|f|^2)\,dx}{\int\limits_Q |f|^2\,dx} \tag{34}$$

The functional

$$\int\limits_Q (k\,|\nabla f|^2+a\,|f|^2)\,dx \Big/ \int\limits_Q |f|^2\,dx$$

attains its exact lower bound through the eigenfunction u_{k+1} on the subspace of the space $\mathring{H}^1(Q)$ that is composed of all the functions orthogonal in the space with the scalar product, $L_2(Q)$, to the eigenfunctions $u_1,\,\ldots,\,u_k$ of this boundary-value problem.

In exactly the same manner, for the third (second) boundary-value problem for the operator $\mathscr{L}$

$$\lambda_1 = -m + 1 - \inf_{f \in H^1(Q)} \frac{\|f\|^2_{H^1(Q)}}{\|f\|^2_{L_2(Q)}}$$

$$= -\inf_{f \in H^1(Q)} \frac{\int\limits_Q (k|\nabla f|^2 + a|f|^2)\,dx + \int\limits_{\partial Q} k\sigma|f|^2\,dS}{\int\limits_Q |f|^2\,dx}, \tag{33'}$$

$$\lambda_{k+1} = -m + 1 - \inf_{\substack{f \in H^1(Q) \\ (f,\, u_i)_{L_2(Q)} = 0 \\ i = 1, \ldots, k}} \frac{\|f\|^2_{H^1(Q)}}{\|f\|^2_{L_2(Q)}}$$

$$= -\inf_{\substack{f \in H^1(Q) \\ (f,\, u_i)_{L_2(Q)} = 0 \\ i = 1, \ldots, k}} \frac{\int\limits_Q (k|\nabla f|^2 + a|f|^2)\,dx + \int\limits_{\partial Q} k\sigma|f|^2\,dS}{\int\limits_Q |f|^2\,dx}. \tag{34'}$$

The exact lower bound of the functional

$$\frac{\int\limits_Q (k|\nabla f|^2 + a|f|^2)\,dx + \int\limits_{\partial Q} k\sigma|f|^2\,dS}{\int\limits_Q |f|^2\,dx}$$

on $H^1(Q)$ is attained through the first eigenfunction u_1. The exact lower bound of this functional is attained through the $(k+1)$th eigenfunction u_{k+1} on the subspace of $H^1(Q)$ consisting of all the elements orthogonal to the space with the scalar product, $L_2(Q)$, to the eigenfunctions $u_1, \ldots, u_k$ of the respective boundary-value problem.

Formulas (33) and (33′) can be combined into one:

$$\lambda_1 = -\inf_{f \in G} \frac{\int\limits_Q (k|\nabla f|^2 + a|f|^2)\,dx + \int\limits_{\partial Q} k\sigma|f|^2\,dS}{\int\limits_Q |f|^2\,dx}, \tag{33''}$$

moreover, λ_1 is the first eigenvalue of the third (second, if $\sigma \equiv 0$) boundary-value problem for the operator $\mathscr{L}$ if $G = H^1(Q)$, and λ_1 is the first eigenvalue of the first boundary-value problem if $G = \mathring{H}^1(Q)$ $\left(\text{when } f \in \mathring{H}^1(Q), \text{ the integral } \int\limits_{\partial Q} k\sigma\,|f|^2\,dS = 0\right)$.

Similarly, formulas (34) and (34′) can be combined into one:

$$\lambda_{k+1} = -\inf_{\substack{f \in G \\ (f,\, u_i)_{L_2(Q)} = 0 \\ i = 1, \ldots, k}} \frac{\int\limits_Q (k|\nabla f|^2 + a|f|^2)\,dx + \int\limits_{\partial Q} k\sigma|f|^2\,dS}{\int\limits_Q |f|^2\,dx}. \tag{34''}$$

Sometimes the application of formulas (34), (34′), (34″) in finding the $(k+1)$th eigenvalue λ_{k+1} becomes difficult as they depend on the knowledge of the preceding eigenfunctions $u_1, \dots, u_k$. A formula for calculating λ_{k+1} will be obtained below which is free from this defect.

We take arbitrary k functions $\varphi_1, \dots, \varphi_k$ belonging to $L_2(Q)$ and denote by $R(\varphi_1, \dots, \varphi_k)$ the subspace of $\overset{\circ}{H}{}^1(Q)$ which consists of functions f that are orthogonal to the functions $\varphi_1, \dots, \varphi_k$ in the space with the scalar product, $L_2(Q)$: $(f, \varphi_s)_{L_2(Q)} = 0$, $s = 1, \dots, k$. Let

$$d(\varphi_1, \dots, \varphi_k) = -m + 1 - \inf_{f \in R(\varphi_1, \dots, \varphi_k)} \frac{\| f \|^2_{\overset{\circ}{H}{}^1(Q)}}{\| f \|^2_{L_2(Q)}},$$

and let d_{k+1} be the exact lower bound of the number set $\{d(\varphi_1, \dots, \varphi_k)\}$ taken over all the possible systems of functions $\varphi_1, \dots, \varphi_k$ belonging to $L_2(Q)$:

$$d_{k+1} = \inf_{\substack{(\varphi_1, \dots, \varphi_k) \\ \varphi_s \in L_2(Q) \\ s=1, \dots, k}} d(\varphi_1, \dots, \varphi_k).$$

We shall show that $d_{k+1} = \lambda_{k+1}$, where λ_{k+1} is the $(k+1)$th eigenvalue of the first boundary-value problem for the operator $\mathscr{L}$.

Since $d(u_1, \dots, u_k) = \lambda_{k+1}$ (formula (34)), it follows that $d_{k+1} \leqslant \lambda_{k+1}$. We shall now establish the reverse inequality. To do this, it is enough to construct for an arbitrary fixed choice of the system $\varphi_1, \dots, \varphi_k$ a function f in $R(\varphi_1, \dots, \varphi_k)$ such that $\| f \|_{L_2(Q)} = 1$ and

$$\| f \|^2_{\overset{\circ}{H}{}^1(Q)} \leqslant -\lambda_{k+1} - m + 1.$$

The function f will be sought in the form

$$f = \sum_{s=1}^{k+1} f_s u_s, \qquad f_s = (f, u_s)_{L_2(Q)}.$$

Then the conditions $f \in R(\varphi_1, \dots, \varphi_k)$ and $\| f \|_{L_2(Q)} = 1$ become

$$(f, \varphi_p)_{L_2(Q)} = \sum_{s=1}^{k+1} f_s (u_s, \varphi_p)_{L_2(Q)} = 0, \qquad p = 1, \dots, k, \tag{35}$$

$$\| f \|^2_{L_2(Q)} = \sum_{s=1}^{k+1} | f_s |^2 = 1. \tag{36}$$

Since the linear system (35) regarding the vector $(f_1, \dots, f_{k+1})$ is a homogeneous system of k equations in $k+1$ unknowns, it has always a nontrivial solution. It is always possible to satisfy the normalizing condition (36). Since, in view of (26) and (36),

$$\|f\|^2_{\mathring{H}^1(Q)}=\sum_{s=1}^{k+1}|f_s|^2\|u_s\|^2_{\mathring{H}^1(Q)}=\sum_{s=1}^{k+1}|f_s|^2(-\lambda_s-m+1)$$
$$\leqslant\sum_{s=1}^{k+1}|f_s|^2(-\lambda_{k+1}-m+1)\equiv-\lambda_{k+1}-m+1$$

(recall that $\lambda_1\geqslant\lambda_2\geqslant\ldots\geqslant\lambda_{k+1}$), it follows that f is the desired function.

Thus the $(k+1)$th eigenvalue of the first boundary-value problem for the operator $\mathscr{L}$ is given by the formula

$$\lambda_{k+1}=\inf_{\substack{(\varphi_1,\ldots,\varphi_k)\\ \varphi_s\in L_2(Q)\\ s=1,\ldots,k}}\left(-m+1-\inf_{\substack{f\in\mathring{H}^1(Q)\\ (f,\varphi_i)_{L_2(Q)}=0\\ i=1,\ldots,k}}\frac{\|f\|^2_{\mathring{H}^1(Q)}}{\|f\|^2_{L_2(Q)}}\right)$$
$$=-m+1-\sup_{\substack{(\varphi_1,\ldots,\varphi_k)\\ \varphi_s\in L_2(Q)\\ s=1,\ldots,k}}\inf_{\substack{f\in\mathring{H}^1(Q)\\ (f,\varphi_i)_{L_2(Q)}=0\\ i=1,\ldots,k}}\frac{\|f\|^2_{\mathring{H}^1(Q)}}{\|f\|^2_{L_2(Q)}}$$
$$=-\sup_{\substack{(\varphi_1,\ldots,\varphi_k)\\ \varphi_s\in L_2(Q)\\ s=1,\ldots,k}}\inf_{\substack{f\in\mathring{H}^1(Q)\\ (f,\varphi_i)_{L_2(Q)}=0\\ i=1,\ldots,k}}\frac{\int\limits_Q(k|\nabla f|^2+a|f|^2)\,dx}{\int\limits_Q|f|^2\,dx},\qquad(37)$$

expressing the so-called *minimax property* of the eigenvalues.

Exactly in the same manner the formula for $(k+1)$th eigenvalue of the third (and second) boundary-value problem for the operator $\mathscr{L}$ is established:

$$\lambda_{k+1}=-m+1-\sup_{\substack{(\varphi_1,\ldots,\varphi_k)\\ \varphi_s\in L_2(Q)\\ s=1,\ldots,k}}\inf_{\substack{f\in H^1(Q)\\ (f,\varphi_i)_{L_2(Q)}=0\\ i=1,\ldots,k}}\frac{\|f\|^2_{H^1(Q)}}{\|f\|^2_{L_2(Q)}}$$
$$=-\sup_{\substack{(\varphi_1,\ldots,\varphi_k)\\ \varphi_i\in L_2(Q)\\ i=1,\ldots,k}}\inf_{\substack{f\in H^1(Q)\\ (f,\varphi_i)_{L_2(Q)}=0\\ i=1,\ldots,k}}\frac{\int\limits_Q(k|\nabla f|^2+a|f|^2)\,dx+\int\limits_{\partial Q}k\sigma|f|^2\,dS}{\int\limits_Q|f|^2\,dx}.\qquad(37')$$

Formulas (37) and (37′) can be combined into one:

$$\lambda_{k+1}=-\sup_{\substack{(\varphi_1,\ldots,\varphi_k)\\ \varphi_i\in L_2(Q)\\ i=1,\ldots,k}}\inf_{\substack{f\in G\\ (f,\varphi_i)_{L_2(Q)}=0\\ i=1,\ldots,k}}\frac{\int\limits_Q(k|\nabla f|^2+a|f|^2)\,dx+\int\limits_{\partial Q}k\sigma|f|^2\,dS}{\int\limits_Q|f|^2\,dx}\qquad(37'')$$

in which λ_{k+1} is the $(k+1)$th eigenvalue of the first boundary-value problem for the operator $\operatorname{div}(k(x)\nabla) - a(x)$ if $G = \mathring{H}^1(Q)$, and λ_{k+1} is the $(k+1)$th eigenvalue of the third (second, if $\sigma \equiv 0$) boundary-value problem if $G = H^1(Q)$.

The minimax property of eigenvalues furnishes a possible way of comparing the eigenvalues of various boundary-value problems.

Theorem 4. 1. *Let* λ_k^{I}, λ_k^{II}, λ_k^{III} *be the kth eigenvalues of the first, second, and third (for some* $\sigma \geqslant 0$*) boundary-value problems for the operator* $\mathcal{L} = \operatorname{div}(k(x)\nabla) - a(x)$. *Then* $\lambda_k^{\mathrm{I}} \leqslant \lambda_k^{\mathrm{III}} \leqslant \lambda_k^{\mathrm{II}}$ *for all* $k = 1, 2, \ldots$.

2. *Let* λ_k' *be the kth eigenvalue of the first, second or third (for some* $\sigma = \sigma' \geqslant 0$*) boundary-value problems for the operator* $\mathcal{L}' = \operatorname{div}(k'(x)\nabla) - a'(x)$, *and let* λ_k'' *be the kth eigenvalue of the first, second or third (for some* $\sigma = \sigma'' \geqslant 0$*) boundary-value problems for the operator* $\mathcal{L}'' = \operatorname{div}(k''(x)\nabla) - a''(x)$. *If* $k' \leqslant k''$, $a' \leqslant a''$ *in* Q *and in the case of the third boundary-value problem* $\sigma' \leqslant \sigma''$ *on* ∂Q, *then* $\lambda_k' \geqslant \lambda_k''$ *for all* $k = 1, 2, \ldots$.

3. *Let* Q' *be a subregion of the region* Q, $Q' \subset Q$, *and* $\lambda_k(Q)$, $\lambda_k(Q')$ *be the kth eigenvalues of the first boundary-value problem for the operator* $\mathcal{L} = \operatorname{div}(k(x)\nabla) - a(x)$ *in* Q *and* Q', *respectively. Then* $\lambda_k(Q) \geqslant \lambda_k(Q')$ *for all* $k = 1, 2, \ldots$.

Proof. Let $k > 1$. Since the value of the functional present in (37″) after the sign inf in the case of the third boundary-value problem ($\sigma \geqslant 0$) is not less than its value for the second boundary-value problem ($\sigma = 0$) and the set G in both cases is the same, $G = H^1(Q)$, it follows that $\lambda_k^{\mathrm{III}} \leqslant \lambda_k^{\mathrm{II}}$. The inequality $\lambda_k^{\mathrm{I}} \leqslant \lambda_k^{\mathrm{III}}$ also follows from (37″), because the set G over which inf is taken in the case of the third boundary-value problem is wider than the set G for the first boundary-value problem: $H^1(Q) \supset \mathring{H}^1(Q)$.

Assertion 1 for $k = 1$ follows from (34)″.

2. Assertion 2 follows from (37″) (for $k > 1$) and from (34″) (when $k = 1$), because the value of the functional appearing after the inf sign for the operator $\mathcal{L}''$ is not less than the corresponding value for the operator $\mathcal{L}'$.

3. Since the set $\mathring{H}^1(Q)$ contains the set $\mathring{H}^1(Q)$ of functions belonging to $\mathring{H}^1(Q)$ and vanishing on $Q \setminus Q'$, we have for $k > 1$

$$\lambda_k(Q) = -\sup_{\substack{(\varphi_1, \ldots, \varphi_{k-1}) \\ \varphi_s \in L_2(Q) \\ s=1, \ldots, k-1}} \inf_{\substack{f \in \mathring{H}^1(Q) \\ (f, \varphi_s)_{L_2(Q)}=0 \\ s=1, \ldots, k-1}} T(f)$$

$$\geqslant -\sup_{\substack{(\varphi_1, \ldots, \varphi_{k-1}) \\ \varphi_s \in L_2(Q) \\ s=1, \ldots, k-1}} \inf_{\substack{f \in \overset{\circ}{\tilde{H}}{}^1(Q) \\ (f, \varphi_s)_{L_2(Q)}=0 \\ s=1, \ldots, k-}} T(f)$$

$$= - \sup_{\substack{(\varphi_1, \ldots, \varphi_{k-1}) \\ \varphi_s \in L_2(Q') \\ s=1, \ldots, k-1}} \inf_{\substack{f \in \mathring{H}^1(Q') \\ (f, \varphi_s)_{L_2(Q')}=0 \\ s=1, \ldots, k-1}} T(f) = \lambda_k(Q'),$$

where

$$T(f) = \frac{\int\limits_Q (k\,|\nabla f|^2 + a\,|f|^2)\,dx}{\int\limits_Q |f|^2\,dx}\,.$$

If $k=1$, then

$$\lambda_1(Q) = - \inf_{f \in \mathring{H}^1(Q)} T(f) \geqslant - \inf_{f \in \tilde{\mathring{H}}^1(Q)} T(f) = \lambda_1(Q'). \quad \blacksquare$$

5. Asymptotic Behaviour of Eigenvalues of the First Boundary-Value Problem. We first consider the eigenvalues of the first boundary-value problem for the Laplace operator Δ (of the operator $\mathscr{L} = \operatorname{div}(k\nabla) - a$ with $k \equiv 1$, $a \equiv 0$) in the cube $K_l = \{0 < x_i < l,\ i = 1, \ldots, n\}$ with side $l > 0$. The generalized eigenfunction $u(x)$, corresponding to the eigenvalue λ, of the first boundary-value problem for the operator Δ in K_l is defined to be a function belonging to $\mathring{H}^1(K_l)$ which satisfies the identity

$$\int\limits_{K_l} \nabla u \nabla \bar{v}\,dx = -\lambda \int\limits_{K_l} u\bar{v}\,dx$$

for all $v \in \mathring{H}^1(K_l)$.

It can be easily verified that the function $u_{m_1 \ldots m_n}(x) = \left(\frac{2}{l}\right)^{n/2} \prod\limits_{i=1}^{n} \sin \frac{\pi m_i x_i}{l}$ with integers $m_1 > 0, \ldots, m_n > 0$ is an eigenfunction of the boundary-value problem under discussion; the corresponding eigenvalue is $-\frac{\pi^2}{l^2}(m_1^2 + \ldots + m_n^2)$. The system of functions $u_{m_1 \ldots m_n}(x)$ for all integers $m_i > 0$, $i = 1, \ldots, n$, is orthonormal in $L_2(K_l)$. Since any function belonging to $L_2(K_l)$ and orthogonal to all $u_{m_1 \ldots m_n}$ is zero (this is proved just as the corresponding assertion in Sec. 4.4, Chap. III, for the system of functions $u_{m_1 \ldots m_n} = \exp\{i\,(m_1 x_1 + \ldots + m_n x_n)\}$ in the cube $\{|x_i| < \pi,\ i = 1, \ldots, n\}$), this system constitutes an orthonormal basis for $L_2(K_l)$, and accordingly contains all the eigenfunctions of the first boundary-value problem for the operator Δ in K_l.

Thus there is a one-to-one correspondence between the set of all the eigenfunctions of the problem in question and the set of all the points $(m_1, \ldots, m_n)$ with positive integer coordinates, and therefore also the set of all the cubes $K_{m_1 \ldots m_n} = \{m_i - 1 \leqslant x_i \leqslant m_i,\ i = 1, \ldots, n\}$. And the eigenvalue corresponding to the function

$u_{m_1 \ldots m_n}(x)$ is equal to the square of the distance from the point $(m_1, \ldots, m_n)$ to the origin multiplied by $-\pi^2/l^2$. Thus the multiplicity of the eigenvalue λ equals the number of points with integer coordinates lying on a sphere of radius $\sqrt{-\lambda}\, l/\pi$. In particular, the number $-\frac{\pi^2}{l^2} n$ is the first eigenvalue; its multiplicity is 1. The corresponding eigenfunction is $u_{1 \ldots 1}(x) = \left(\frac{2}{l}\right)^{n/2} \sin \frac{\pi x_1}{l} \ldots \sin \frac{\pi x_n}{l}$. The next eigenvalue is $-\frac{\pi^2}{l^2}(n+3)$; its multiplicity is n. The corresponding eigenfuctions are $u_{\underbrace{1 \ldots}_{i-1} 1, 2, 1 \ldots 1}(x) = \left(\frac{2}{l}\right)^{n/2} \times$ $\times \sin \frac{\pi x_1}{l} \ldots \sin \frac{\pi x_{i-1}}{l} \sin \frac{2\pi x_i}{l} \sin \frac{\pi x_{i+1}}{l} \ldots \sin \frac{\pi x_n}{l}$, $i = 1, \ldots, n$.

Let $N(\rho)$ denote the number of eigenvalues (having regard to the multiplicity) which do not exceed some $\rho > 0$ in absolute value. $N(\rho)$ is equal to the number of points $(m, \ldots, m_n)$ with positive integer coordinates for which $m_1^2 + \ldots + m_n^2 \leqslant \frac{l^2}{\pi^2} \rho$ or, what is the same, is equal to the volume of a solid $M_{\sqrt{\rho}\, l/\pi}$ composed of all the cubes $K_{m_1 \ldots m_n}$ for which $m_1^2 + \ldots + m_n^2 < \frac{l^2}{\pi^2} \rho$. Since $M_{\sqrt{\rho}\, l/\pi} \subset S_{\sqrt{\rho}\, l/\pi} = \left\{ |x| < \frac{l}{\pi} \sqrt{\rho},\ x_i \geqslant 0,\ i = 1, \ldots, n \right\}$, $N(\rho) \leqslant |S_{\sqrt{\rho}\, l/\pi}| = \frac{\sigma_n}{2^n n} \frac{l^n}{\pi^n} \rho^{n/2}$. On the other hand, for $\rho > n \frac{\pi^2}{l^2}$ $M_{\sqrt{\rho}\, l/\pi} \supset S_{\sqrt{\rho}\, l/\pi - \sqrt{n}}$, therefore for $\rho > \frac{n\pi^2}{l^2}$, $N(\rho) \geqslant \frac{\sigma_n}{2^n n} \times$ $\times \left(\frac{l\sqrt{\rho}}{\pi} - \sqrt{n} \right)^n$.

Let the eigenvalues be numbered, as usual, in increasing order: $0 > \lambda_1 \geqslant \lambda_2 \geqslant \ldots$ (every eigenvalue in this sequence occurs according to its multiplicity). Consider any eigenvalue λ_s of multiplicity p_s, $p_s \geqslant 1$, and assume that $\lambda_{s-p_s'}, \ldots, \lambda_s, \ldots, \lambda_{s+p_s''}$ for some $p_s' \geqslant 0$, $p_s'' \geqslant 0$, $p_s' + p_s'' + 1 = p_s$ are all the eigenvalues equal to λ_s.

The number p_s is equal to the volume of the solid composed of the cubes $K_{m_1 \ldots m_n}$ whose vertices $(m_1, \ldots, m_n)$ lie on a sphere of radius $\frac{l}{\pi} |\lambda_s|^{1/2}$ with the origin as the centre. This solid is contained in $S_{|\lambda_s|^{1/2} l/\pi} \setminus S_{|\lambda_s|^{1/2} l/\pi - \sqrt{n}}$, so $p_s \leqslant \frac{\sigma_n}{2^n n} \left[\left(\frac{l}{\pi} |\lambda_s| \right)^{n/2} - \right.$ $\left. - \left(\frac{l}{\pi} |\lambda_s| - \sqrt{n} \right)^{n/2} \right]$.

In particular, noting that $\lambda_s \to -\infty$ as $s \to \infty$, we find that $\lim\limits_{s \to \infty} \frac{p_s}{|\lambda_s|^{n/2}} = 0$.

From the definition of the function $N(\rho)$ it follows that $s+p_s'' = N(|\lambda_s|)$. Accordingly, $\frac{\sigma_n}{n2^n}\left(\frac{l|\lambda_s|}{\pi}-\sqrt{n}\right)^{n/2} \leqslant s+p_s'' \leqslant \leqslant \frac{\sigma_n}{n2^n}\left(\frac{l|\lambda_s|}{\pi}\right)^{n/2}$. And since $0 \leqslant p_s'' \leqslant p_s$, the ratio $s/|\lambda_s|^{n/2}$ has a limit that equals $\frac{\sigma_n}{2^n n} l^{n/2}\pi^{-n/2}$. Therefore (recalling that $\ldots \leqslant \lambda_2 \leqslant \leqslant \lambda_1 < 0$) there are constants C_0 and C_1, $0 < C_0 \leqslant C_1$, such that

$$-\frac{C_1}{l} s^{2/n} \leqslant \lambda_s \leqslant -\frac{C_0}{l} s^{2/n} \tag{38}$$

for all $s = 1, 2, \ldots$.

Since the eigenvalues do not depend on the coordinate system chosen, the inequalities (38) for the eigenvalues of the first boundary-value problem for the Laplace operator continue to hold if the cube K_l is replaced by any other cube with side l. It is easy to see that the eigenvalues of the first boundary-value problem for the operator $k_0\Delta - a_0$, where $k_0 > 0$ and a_0 are constants, in a cube with side l are $-k_0\frac{\pi^2}{l^2}(m_1^2+\ldots+m_n^2)-a_0$, where $m_1, \ldots, m_n$ are positive integers. Thus the inequalities (38), with some constants C_0 and C_1 (depending on k_0) hold also for them for all s, starting with some s_0 (depending on k_0 and a_0).

We now examine the general case.

Theorem 5. *Let* λ_s, $s = 1, 2, \ldots$, *be the eigenvalues of the first boundary-value problem for the operator* $\mathscr{L} = \operatorname{div}(k(x)\nabla) - a(x)$ *in the region* Q. *There exist constants* C_0 *and* C_1, $0 < C_0 \leqslant C_1$, *and a number* s_0 *such that the inequalities*

$$-C_1 s^{2/n} \leqslant \lambda_s \leqslant -C_0 s^{2/n} \tag{39}$$

hold for all $s \geqslant s_0$.

Proof. Suppose that $\overline{\lambda}_s$ and $\underline{\lambda}_s$ are eigenvalues of the first boundary-value problem in Q for the operators $\overline{\mathscr{L}} = \operatorname{div}(\overline{k}\nabla) - \overline{a} = \overline{k}\Delta - \overline{a}$ and $\underline{\mathscr{L}} = \underline{k}\Delta - \underline{a}$, respectively, where $\overline{k} = \max\limits_{x\in\overline{Q}} k(x)$, $\underline{k} = \min\limits_{x\in\overline{Q}} k(x)$, $\overline{a} = \max\limits_{x\in\overline{Q}} a(x)$, $\underline{a} = \min\limits_{x\in\overline{Q}} a(x)$. Then, by Assertion 2 of Theorem 4, $\overline{\lambda}_s \leqslant \lambda_s \leqslant \underline{\lambda}_s$ for $s = 1, 2, \ldots$.

By K' and K'' we denote the cubes such that $K' \subset Q \subset K''$, and by $\overline{\lambda}_s'$ and $\underline{\lambda}_s''$ the eigenvalues of the first boundary-value problem for the operator $\overline{\mathscr{L}}$ in K' and the operator $\underline{\mathscr{L}}$ in K'', respectively. By Assertion 3 of Theorem 4, $\overline{\lambda}_s' \leqslant \overline{\lambda}_s$ and $\underline{\lambda}_s'' \geqslant \underline{\lambda}_s$ for all s. The conclusion of the theorem now follows from the fact that starting with some $s = s_0$ the inequalities (39) hold for $\overline{\lambda}_s'$ and $\underline{\lambda}_s''$. ■

6. Solvability of Boundary-Value Problems in the Case of Homogeneous Boundary Conditions. In Subsec. 2, we examined the question of existence and uniqueness of generalized solutions of the first and third (second) boundary-value problems for Eq. (1) under the assumption that $a(x) \geqslant 0$ in Q. Now the general case will be discussed.

Let $m = \min\limits_{x \in \bar{Q}} a(x)$. The identities (4) and (6) can be written in the form

$$(u, v)_{\mathring{H}^1(Q)} + (m-1)(u, v)_{L_2(Q)} = -(f, v)_{L_2(Q)}, \tag{4'}$$

$$(u, v)_{H^1(Q)} + (m-1)(u, v)_{L_2(Q)} = -(f, v)_{L_2(Q)}, \tag{6'}$$

where the scalar products in $\mathring{H}^1(Q)$ and $H^1(Q)$ are defined by formulas (17) and (18). By Lemma 1, Sec. 1.3, the identity (4′) is equivalent to the operator equation

$$u + (m-1)Au = -Af \tag{40}$$

in the space $\mathring{H}^1(Q)$, while the identity (6′), by Lemma 1′, to the operator equation

$$u + (m-1)A'u = -A'f \tag{40'}$$

in the space $H^1(Q)$ (recall that $Af \in \mathring{H}^1(Q)$, $A'f \in H^1(Q)$). The operator A from $\mathring{H}^1(Q)$ into $\mathring{H}^1(Q)$ (A' from $H^1(Q)$ into $H^1(Q)$) is completely continuous. Therefore for the investigation of Eq. (40) ((40′)) Fredholm's theorems can be applied (Theorems 1-4, Secs. 4.3-4.7, Chap. II).

(1) If the number $-m+1$ is not a characteristic value of the operator A (A'), then by Fredholm's first theorem Eq. (40) ((40′)) is uniquely solvable with any $f \in L_2(Q)$, and the inequality $\|u\|_{\mathring{H}^1(Q)} \leqslant C_1 \|Af\|_{\mathring{H}^1(Q)} \leqslant C\|f\|_{L_2(Q)}$ ($\|u\|_{H^1(Q)} \leqslant C\|f\|_{L_2(Q)}$) holds, where the constant $C > 0$ does not depend on f. Since $-m+1$ is a characteristic value of the operator A (A') if and only if zero is an eigenvalue of the first (third) boundary-value problem for the operator $\mathscr{L}$, we have established the following theorem.

Theorem 6. *For any* $f \in L_2(Q)$, *there exists a unique generalized solution* $u(x)$ *of each of the boundary-value problems* (1), (2) *and* (1), (3) *with homogeneous boundary conditions* ($\varphi = 0$), *provided that zero is not an eigenvalue for the operator* $\mathscr{L}$. *Moreover, the inequality*

$$\|u\|_{H^1(Q)} \leqslant C\|f\|_{L_2(Q)},$$

where the constant $C > 0$ *does not depend on* f, *holds.*

(2) If $-m+1$ is a characteristic value of A (A') (then, of course, $m \neq 1$), we use Fredholm's third theorem. In this case, in order

that Eq. (40) ((40′)) be solvable it is necessary and sufficient that the equality $(Af, u_p)_{\mathring{H}^1(Q)} = 0$ $((A'f, u_p)_{H^1(Q)} = 0)$ hold for all the eigenfunctions u_p of the operator A (A') corresponding to the characteristic value $-m+1$. Eq. (40) ((40′)) has a unique solution u which is orthogonal in $\mathring{H}^1(Q)$ $(H^1(Q))$ to all the functions u_p, and for this solution the inequality $\| u \|_{\mathring{H}^1(Q)} \leqslant C \| f \|_{L_2(Q)}$ $(\| u \|_{H^1(Q)} \leqslant C \| f \|_{L_2(Q)})$ holds in which the constant $C > 0$ does not depend on f. Any other solution of Eq. (40) ((40′)) is expressed as the sum of the solution u and a linear combination of the functions u_p.

From the definition of operators A and A' ((21) and (21′), respectively) it follows that orthogonality in $\mathring{H}^1(Q)$ $(H^1(Q))$ of functions Af $(A'f)$ and u_p is equivalent to that of functions f and u_p in $L_2(Q)$. What is more, the orthogonality in $\mathring{H}^1(Q)$ $(H^1(Q))$ of the solution u of Eq. (40) ((40′)) to the eigenfunction u_p is equivalent to their orthogonality in $L_2(Q)$, because, in view of (21) ((21′))

$$(u, u_p)_{\mathring{H}^1(Q)} = (1-m)(Au, u_p)_{\mathring{H}^1(Q)} - (Af, u_p)_{\mathring{H}^1(Q)}$$

$$= (1-m)(Au, u_p)_{\mathring{H}^1(Q)} = (1-m)(u, u_p)_{L_2(Q)}$$

$((u, u_p)_{H^1(Q)} = (1-m)(u, u_p)_{L_2(Q)})$.
We thus have the following result.

Theorem 7. *If zero is an eigenvalue of the first or third (second) boundary-value problem for the operator* $\mathcal{L}$, *the necessary and sufficient conditions for the problem* (1), (2) *or* (1), (3) *with homogeneous boundary conditions* $(\varphi = 0)$ *to have a generalized solution are the following:* $(f, u_p)_{L_2(Q)} = 0$ *for all generalized eigenfunctions* u_p *of the respective problem corresponding to the zero eigenvalue. The problem* (1), (2) *or* (1), (3) (*with* $\varphi = 0$) *has a unique solution u which is orthogonal to all the eigenfunctions:* $(u, u_p)_{L_2(Q)} = 0$. *This solution satisfies the inequality*

$$\| u \|_{H^1(Q)} \leqslant C \| f \|_{L_2(Q)},$$

where the constant $C > 0$ *does not depend on f. Any other solution is expressed as a sum of this solution u and a linear combination of the functions* u_p.

From Theorem 3 it follows that zero is an eigenvalue of the second boundary-value problem $(\sigma \equiv 0)$ for the operator $\mathcal{L}$ when $a \equiv 0$; the corresponding unique eigenfunction is $1/\sqrt{|Q|}$. Therefore Theorem 7, in particular, implies

Theorem 8. *For the problem*

$$\operatorname{div}(k(x)\nabla u) = f, \quad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 0,$$

to have a generalized solution it is necessary and sufficient that

$$\int_Q f\,dx = 0.$$

Under this condition, there is a unique solution u satisfying the condition $\int_Q u\,dx = 0$, *and this solution satisfies the inequality*

$$\|u\|_{H^1(Q)} \leqslant C\|f\|_{L_2(Q)},$$

where the constant $C > 0$ *does not depend on* f. *Any other generalized solution* $\tilde{u}$ *of this problem can be expressed as* $\tilde{u} = u + c_1$, *where* c_1 *is a constant.*

Remark. If f is a real-valued function, then the solutions described in Theorem 6 are also real-valued. This assertion is proved just as the corresponding assertion in Remark at the end of Subsec. 2. The solutions mentioned in Theorems 7 and 8 could also be considered real-valued, provided, of course, all the corresponding eigenfunctions are taken real-valued and one considers their linear combinations with real coefficients only.

7. First Boundary-Value Problem for the General Elliptic Equation. The results of the foregoing subsections are easily extended to the case of more general elliptic equations. To illustrate it, we consider the following boundary-value problem:

$$\mathcal{L}u \equiv \sum_{i,j=1}^{n} (a_{ij}(x)\,u_{x_i})_{x_j} + \sum_{i=1}^{n} a_i(x)\,u_{x_i} + a(x)u = f(x), \quad x \in Q, \tag{41}$$

$$u|_{\partial Q} = 0, \tag{42}$$

where the real-valued coefficients $a_{ij}(x) \in C^1(\bar{Q})$, $a_i(x) \in C^1(\bar{Q})$, $a(x) \in C(\bar{Q})$, $i, j = 1, \ldots, n$; the matrix $\|a_{ij}(x)\|$ is assumed to be symmetric and positive-definite (ellipticity of Eq. (41)), that is, it satisfies the inequality

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\xi_j \geqslant \gamma \sum_{i=1}^{n} \xi_i^2 \tag{43}$$

with a constant $\gamma > 0$ for any real vector $(\xi_1, \ldots, \xi_n)$ and any point $x \in \bar{Q}$.

The classical solution $u(x)$ of the problem (41), (42) is defined in the usual manner: this is a function which belongs to $C^2(Q) \cap C(\bar{Q})$ and satisfies (41) and (42). By means of Ostrogradskii's formula, it is easily seen that if $f \in L_2(Q)$, the classical solution of the problem (41), (42) belonging to $\overset{\circ}{H}{}^1(Q)$ satisfies for all $v \in \overset{\circ}{H}{}^1(Q)$ the integral identity

$$\int\limits_Q \sum_{i,j=1}^{n} a_{ij} u_{x_i} \bar{v}_{x_j}\, dx$$

$$+\int\limits_Q u\Big[\sum_{i=1}^{n} a_i \bar{v}_{x_i} + \Big(\sum_{i=1}^{n} a_{ix_i} - a\Big)\bar{v}\Big]\,dx = -\int\limits_Q f\bar{v}\,dx. \qquad (44)$$

The function $u \in \mathring{H}^1(Q)$ is called a generalized solution of the problem (41), (42) if for all $v \in \mathring{H}^1(Q)$ it satisfies the integral identity (44) with $f \in L_2(Q)$.

According to Theorem 6, Sec. 5.6, Chap. III, we can define in $\mathring{H}^1(Q)$ a scalar product

$$(u, v)_{\mathring{H}^1(Q)} = \int\limits_Q \sum_{i,j=1}^{n} a_{ij} u_{x_i} \bar{v}_{x_j}\, dx$$

which is equivalent to the usual scalar product. In view of this, the identity (44) can be written as

$$(u, v)_{\mathring{H}^1(Q)} + \Big(u, \sum_{i=1}^{n} a_i v_{x_i} + \Big(\sum_{i=1}^{n} a_{ix_i} - a\Big) v\Big)_{L_2(Q)} = -(f, v)_{L_2(Q)}. \qquad (45)$$

Lemma 2. 1. *For any functions $a_0(x)$, $a_1(x)$, ..., $a_n(x)$ continuous in $\bar{Q}$ there exists a bounded linear operator A acting from $L_2(Q)$ into $\mathring{H}^1(Q)$ and defined on the whole $L_2(Q)$ such that*

$$\Big(u, \sum_{i=1}^{n} a_i v_{x_i} + a_0 v\Big)_{L_2(Q)} = (Au, v)_{\mathring{H}^1(Q)}$$

for all $v \in \mathring{H}^1(Q)$.

2. *The operator A regarded as an operator from $\mathring{H}^1(Q)$ into $\mathring{H}^1(Q)$ is completely continuous.*

Proof. Since for a fixed $u \in L_2(Q)$ the linear functional $l(v) = \Big(u, \sum_{i=1}^{n} a_i v_{x_i} + a_0 v\Big)_{L_2(Q)}$ $(v \in \mathring{H}^1(Q))$ defined on $\mathring{H}^1(Q)$ is bounded: $|l(v)| \leqslant \|u\|_{L_2(Q)} \Big\|\sum_{i=1}^{n} a_i v_{x_i} + a_0 v\Big\|_{L_2(Q)} \leqslant C\|u\|_{L_2(Q)}\|v\|_{\mathring{H}^1(Q)}$, where the constant $C > 0$ depends only on $\|a_i\|_{C(\bar{Q})}$, $i = 0, 1, \ldots, n$, by Riesz's theorem there is a unique element $U \in \mathring{H}^1(Q)$ such that $l(v) = (U, v)_{\mathring{H}^1(Q)}$ for all $v \in \mathring{H}^1(Q)$, and $\|U\|_{\mathring{H}^1(Q)} = \|l\| \leqslant C\|u\|_{L_2(Q)}$. This means that an operator A (obviously, linear) is defined on

$L_2(Q)$ that maps $L_2(Q)$ into $\overset{\circ}{H}{}^1(Q)$: $Au=U$. This operator is bounded, $\|A\|\leqslant C$, and for any $u\in L_2(Q)$ and $v\in \overset{\circ}{H}{}^1(Q)$ the identity

$$\left(u, \sum_{i=1}^{n} a_i v_{x_i} + a_0 v\right)_{L_2(Q)} = (Au, v)_{\overset{\circ}{H}{}^1(Q)}$$

holds.

Let us demonstrate that A regarded as an operator from $\overset{\circ}{H}{}^1(Q)$ into $\overset{\circ}{H}{}^1(Q)$ is completely continuous. Take any bounded set in $\overset{\circ}{H}{}^1(Q)$. This set is compact in $L_2(Q)$, by Theorem 3, Sec. 5.4, Chap. III. Therefore from any infinite sequence of its elements a subsequence can be chosen which is fundamental in $L_2(Q)$. Since A as an operator from $L_2(Q)$ into $\overset{\circ}{H}{}^1(Q)$ is bounded (and therefore continuous), it maps this subsequence into a fundamental sequence in $\overset{\circ}{H}{}^1(Q)$. Accordingly, the operator A from $\overset{\circ}{H}{}^1(Q)$ into $\overset{\circ}{H}{}^1(Q)$ is completely continuous. ∎

The linear functional $(f, v)_{L_2(Q)}$ defined on $\overset{\circ}{H}{}^1(Q)$ $(v\in \overset{\circ}{H}{}^1(Q))$ is bounded: $|(f, v)_{L_2(Q)}|\leqslant C\,\|f\|_{L_2(Q)}\,\|v\|_{\overset{\circ}{H}{}^1(Q)}$, therefore Riesz's theorem guarantees the existence of a unique element $F\in \overset{\circ}{H}{}^1(Q)$ such that for all $v\in \overset{\circ}{H}{}^1(Q)$ $(f, v)_{L_2(Q)} = (F, v)_{\overset{\circ}{H}{}^1(Q)}$, and $\|F\|_{\overset{\circ}{H}{}^1(Q)}\leqslant$ $\leqslant C\,\|f\|_{L_2(Q)}$.

Thus, by means of Lemma 2 (put $a_0 = \sum_{i=1}^{n} a_{ix_i} - a$), the integral identity (44) defining the generalized solution can be written in the form of an operator equation in the space $\overset{\circ}{H}{}^1(Q)$:

$$u + Au = F, \qquad u\in \overset{\circ}{H}{}^1(Q). \tag{46}$$

Lemma 3. *If* $\frac{1}{2}\sum_{i=1}^{n} a_{ix_i} - a\geqslant 0$ *in* Q, *then the homogeneous equation* (46) *has only a trivial solution.*

Proof. Let u denote the solution of the equation $u+Au=0$. Scalar multiplication in $\overset{\circ}{H}{}^1(Q)$ of this equation by u gives $\|u\|^2_{\overset{\circ}{H}{}^1(Q)} +$ $+ (Au, u)_{\overset{\circ}{H}{}^1(Q)} = 0$, which implies that $\|u\|^2_{\overset{\circ}{H}{}^1(Q)} + \operatorname{Re}(Au, u)_{\overset{\circ}{H}{}^1(Q)} = 0$.

Since $\operatorname{Re} a_i u_{x_i}\bar{u} = \frac{1}{2}(a_i|u|^2)_{x_i} - \frac{a_{x_i}}{2}|u|^2$ and $u|_{\partial Q}=0$, it follows that

$$\operatorname{Re}(Au, u)_{\mathring{H}^1(Q)} = \operatorname{Re}\int\limits_Q \Big(\sum_{i=1}^n a_i u_{x_i}\bar{u} + \Big(\sum_{i=1}^n a_{ix_i} - a\Big)|u|^2\Big)\,dx$$

$$= \int\limits_Q \Big(\sum_{i=1}^n \frac{1}{2}(a_i|u|^2)_{x_i} + \Big(\frac{1}{2}\sum_{i=1}^n a_{ix_i} - a\Big)|u|^2\Big)\,dx$$

$$= \int\limits_Q \Big(\frac{1}{2}\sum_{i=1}^n a_{ix_i} - a\Big)|u|^2\,dx \geqslant 0.$$

Hence $\|u\|^2_{\mathring{H}^1(Q)} \leqslant 0$, that is, $u = 0$. ∎

Lemma 3 coupled with Fredholm's first theorem yields the following

Theorem 9. *If* $\frac{1}{2}\sum_{i=1}^n a_{ix_i} - a \geqslant 0$ *in* Q, *then the generalized solution of the problem* (41), (42) *exists for any* $f \in L_2(Q)$ *and is unique.*

8. Generalized Solutions of Boundary-Value Problems with Nonhomogeneous Boundary Conditions. Let us first examine the problem (1), (2). We recall that the generalized solution of this problem is defined to be a function $u \in H^1(Q)$ satisfying the integral identity (4) and whose trace on the boundary ∂Q equals the boundary function φ.

The definition of a generalized solution imposes a natural condition on the boundary function φ. This function must be required to have an extension into Q which belongs to $H^1(Q)$. In the sequel it will be assumed that this condition is fulfilled, otherwise the generalized solution of the problem (1), (2) cannot exist. From the theorem on traces of functions belonging to $H^1(Q)$ it follows that φ must belong to the space $L_2(\partial Q)$, but this is not enough for the function to have an extension into Q by means of a function which belongs to $H^1(Q)$; what is more, even its continuity is not enough for this purpose. We shall return to this problem at the end of this subsection where we obtain a necessary and sufficient condition for such an extension in the case of a circle.

Note that such an extension exists when $\varphi \in C^1(\partial Q)$. By Theorem 2, Sec. 4.2, Chap. III, there follows the existence of a function $\Phi(x)$ in $C^1(\bar{Q})$, and more so in $H^1(Q)$, such that $\Phi|_{\partial Q} = \varphi$ and $\|\Phi\|_{H^1(Q)} \leqslant C_1\|\varphi\|_{C^1(\partial Q)}$, where the constant $C_1 > 0$ does not depend on φ.

Thus, suppose that there is a function $\Phi \in H^1(Q)$ such that $\Phi|_{\partial Q} = \varphi$. With the aid of the substitution $u - \Phi = w$, the problem of finding the generalized solution u reduces to that of finding a function $w \in \mathring{H}^1(Q)$ which satisfies the integral identity

$$\int\limits_Q (k\nabla w\nabla\bar{v} + aw\bar{v})\,dx = -\int\limits_Q (k\nabla\Phi\nabla\bar{v} + a\Phi\bar{v} + f\bar{v})\,dx \tag{4'}$$

for all $v \in \mathring{H}^1(Q)$.

Note that if $\Phi \in H^2(Q)$ (when $\partial Q \in C^2$, for this it is enough that $\varphi \in C^2(\partial Q)$), the identity (4′) can be written as

$$\int\limits_Q (k\nabla w\nabla\bar{v} + aw\bar{v})\,dx = -\int\limits_Q \tilde{f}\,\bar{v}\,dx,$$

where $\tilde{f} = f - \operatorname{div}(k\nabla\Phi) + a\Phi$, that is, the problem reduces to the one investigated in Subsecs. 2 and 4.

As in Subsec. 2, we confine ourselves to the case when $a(x) \geqslant 0$ in Q. Introducing in $\mathring{H}^1(Q)$ the scalar product according to formula (7), identity (4′) can be written in the form

$$(w, v)_{\mathring{H}^1(Q)} = l(v),$$

where $l(v) = -\int\limits_Q (k\nabla\Phi\nabla\bar{v} + a\Phi\bar{v} + f\bar{v})\,dx$ is a linear functional defined on $\mathring{H}^1(Q)$ $(v \in \mathring{H}^1(Q))$. Since

$$|l(v)| \leqslant \|f\|_{L_2(Q)}\|v\|_{L_2(Q)} + \max_{x\in\bar{Q}} k(x)\,\|\,|\nabla\Phi|\,\|_{L_2(Q)}\,\|\,|\nabla v|\,\|_{L_2(Q)}$$

$$+ \max_{x\in\bar{Q}} a(x)\cdot\|\Phi\|_{L_2(Q)}\|v\|_{L_2(Q)} \leqslant C_2(\|f\|_{L_2(Q)} + \|\Phi\|_{H^1(Q)})\|v\|_{\mathring{H}^1(Q)},$$

where the constant $C_2 > 0$ depends only on the coefficients k and a, the functional l is bounded and $\|l\| \leqslant C_2(\|f\|_{L_2(Q)} + \|\Phi\|_{H^1(Q)})$. Therefore by the Riesz theorem there is a unique function w in $\mathring{H}^1(Q)$ which satisfies (4′) and is such that $\|w\|_{\mathring{H}^1(Q)} = \|l\| \leqslant C_2(\|f\|_{L_2(Q)} + \|\Phi\|_{H^1(Q)})$. Then the function $u = w + \Phi$ is a generalized solution of the problem (1), (2). Furthermore,

$$\|u\|_{H^1(Q)} \leqslant C_3(\|f\|_{L_2(Q)} + \|\Phi\|_{H^1(Q)}),$$

where the constant $C_3 > 0$ does not depend on f or Φ, and hence

$$\|u\|_{H^1(Q)} \leqslant C\Big(\|f\|_{L_2(Q)} + \inf_{\substack{\Phi\in H^1(Q)\\ \Phi|_{\partial Q}=\varphi}} \|\Phi\|_{H^1(Q)}\Big), \tag{47}$$

where the constant does not depend on f or φ. If the boundary function $\varphi \in C^1(\partial Q)$, these inequalities imply the inequality

$$\|u\|_{H^1(Q)} \leqslant C(\|f\|_{L_2(Q)} + \|\varphi\|_{C^1(\partial Q)}). \tag{47'}$$

Let us show that the above solution is unique. Indeed, if there is another generalized solution u', then the difference $\tilde{u} = u - u'$ belongs to $\overset{\circ}{H}{}^1(Q)$ and, by virtue of (4), satisfies the integral identity $\int\limits_Q (k\nabla\tilde{u}\nabla\bar{v} + a\tilde{u}\bar{v})\,dx = 0$ for all $v \in \overset{\circ}{H}{}^1(Q)$. Since the left-hand side of this identity is the scalar product in $\overset{\circ}{H}{}^1(Q)$ of functions $\tilde{u}$ and v, it follows that $\tilde{u} = 0$.

Thus we have proved the following assertion.

Theorem 10. *If* $a(x) \geqslant 0$ *in* Q *and* φ *is the boundary-value of a function belonging to* $H^1(Q)$, *then there exists a unique solution* u *of the problem* (1), (2). *This solution satisfies the inequality* (47), *and thus also the inequality* (47′) *with* $\varphi \in C^1(\partial Q)$.

Remark. As can be easily verified, the set $\mathcal{M}$ of functions φ defined on ∂Q which are traces of some functions Φ belonging to $H^1(Q)$ is a Banach space with the norm $\|\varphi\|_{\mathcal{M}} = \inf\limits_{\substack{\Phi \in H^1(Q)\\ \Phi|_{\partial Q} = \varphi}} \|\Phi\|_{H^1(Q)}$. In view of this, inequality (47) can be written in the form

$$\|u\|_{H^1(Q)} \leqslant C\,(\|f\|_{L_2(Q)} + \|\varphi\|_{\mathcal{M}}).$$

Next we consider the problem (1), (3). We recall that a function $u \in H^1(Q)$ is called the generalized solution of this problem if it satisfies the integral identity (5) for all $v \in H^1(Q)$. In this case it is assumed that the boundary function $\varphi \in L_2(\partial Q)$.

Theorem 11. *If* $a(x) \geqslant 0$ *in* Q *and either* $a(x) \not\equiv 0$ *in* Q *or* $\sigma(x) \not\equiv 0$ *on* ∂Q, *then for all* $f \in L_2(Q)$ *and* $\varphi \in L_2(\partial Q)$ *there exists a unique generalized solution* u *of the problem* (1), (3). *Moreover,*

$$\|u\|_{H^1(Q)} \leqslant C\,(\|f\|_{L_2(Q)} + \|\varphi\|_{L_2(\partial Q)}), \tag{48}$$

where the constant $C > 0$ *does not depend on* f *or* φ.

Proof. By means of formula (10), we introduce in $H^1(Q)$ a scalar product equivalent to the usual scalar product. Then the integral identity (5) can be written in the form

$$(u, v)_{H^1(Q)} = l(v),$$

where

$$l(v) = -\int\limits_Q f\bar{v}\,dx + \int\limits_{\partial Q} k\varphi\bar{v}\,dS$$

is a linear functional on $H^1(Q)$ $(v \in H^1(Q))$.

By Theorem 1, Sec. 5.1, Chap. III,

$$|l(v)| \leqslant \|f\|_{L_2(Q)}\,\|v\|_{L_2(Q)} + \max_{x \in \bar{Q}} k(x) \cdot \|\varphi\|_{L_2(\partial Q)}\,\|v\|_{L_2(\partial Q)}$$
$$\leqslant C\,(\|f\|_{L_2(Q)} + \|\varphi\|_{L_2(\partial Q)})\,\|v\|_{H^1(Q)},$$

where the constant $C > 0$ does not depend on f, φ or v. Therefore the functional $l(v)$ is bounded and $\| l \| \leqslant C(\| f \|_{L_2(Q)} + \| \varphi \|_{L_2(\partial Q)})$. Accordingly, by Riesz's theorem, there is a unique function u in $H^1(Q)$ satisfying the identity (5); moreover, $\| u \|_{H^1(Q)} = \| l \| \leqslant$ $\leqslant C\,(\| f \|_{L_2(Q)} + \| \varphi \|_{L_2(\partial Q)})$. ■

Now suppose that $a(x) \equiv 0$ in Q and $\sigma(x) \equiv 0$ on ∂Q in the problem (1), (3). In $H^1(Q)$ we define the scalar product

$$(u, v)_{H^1(Q)} = \int_Q (k\nabla u \nabla \bar{v} + u\bar{v})\, dx \tag{49}$$

equivalent to the usual scalar product. Then the integral identity (5) defining the generalized solution of the second boundary-value problem for the operator div $(k\nabla)$ assumes the form

$$(u, v)_{H^1(Q)} - (u, v)_{L_2(Q)} = l(v), \tag{50}$$

where

$$l(v) = -\int_Q f\bar{v}\, dx + \int_{\partial Q} k\varphi\bar{v}\, dS \tag{51}$$

is a linear functional on $H^1(Q)$ $(v \in H^1(Q))$. Since, by Theorem 1, Sec. 5.1, Chap. III,

$$| l(v) | \leqslant \| f \|_{L_2(Q)} \| v \|_{L_2(Q)} + \max_{x \in \bar{Q}} k(x) \cdot \| \varphi \|_{L_2(\partial Q)} \| v \|_{L_2(\partial Q)}$$
$$\leqslant C' (\| f \|_{L_2(Q)} + \| \varphi \|_{L_2(\partial Q)}) \| v \|_{H^1(Q)},$$

where the constant $C' > 0$ does not depend on f, φ or v, the functional $l(v)$ is bounded and $\| l \| \leqslant C' (\| f \|_{L_2(Q)} + \| \varphi \|_{L_2(\partial Q)})$. According to Riesz's theorem, there exists a unique element F' in $H^1(Q)$ such that for all $v \in H^1(Q)$

$$l(v) = (F', v)_{H^1(Q)} \tag{52}$$

and

$$\| F' \|_{H^1(Q)} \leqslant C' (\| f \|_{L_2(Q)} + \| \varphi \|_{L_2(\partial Q)}). \tag{53}$$

By Lemma 1′, Subsec. 3 (the scalar product in $H^1(Q)$ is defined by the formula (49)), there is a bounded operator A' from $L_2(Q)$ into $H^1(Q)$, with $L_2(Q)$ as the domain of definition, such that

$$(u, v)_{L_2(Q)} = (A'u, v)_{H^1(Q)} \tag{54}$$

for all $v \in H^1(Q)$. If A' is regarded as an operator from $H^1(Q)$ into $H^1(Q)$, it is selfadjoint and completely continuous.

With the aid of (52) and (54), the identity (50) can be replaced by an equivalent operator equation in the space $H^1(Q)$:

$$u - A'u = F'. \tag{55}$$

Since for the function $u_1(x) \equiv \text{const} = \frac{1}{\sqrt{|Q|}}$ there holds the relation $(u_1, v)_{L_2(Q)} = (u_1, v)_{H^1(Q)}$ (the scalar product in $H^1(Q)$ is defined by the formula (49)) for any $v \in H^1(Q)$, it follows from (54) that $(A'u_1, v)_{H^1(Q)} = (u_1, v)_{L_2(Q)} = (u_1, v)_{H^1(Q)}$. This means that the number 1 is a characteristic value of the operator A' and $u_1 = \frac{1}{\sqrt{|Q|}}$ is the corresponding eigenfunction. Since any eigenfunction u_1' of the operator A' corresponding to the characteristic value 1 satisfies the equality $(u_1', u_1')_{H^1(Q)} = (A'u_1', u_1')_{H^1(Q)} = (u_1', u_1')_{L_2(Q)}$, this eigenfunction satisfies

$$\int_Q k\,|\nabla u_1'|^2\,dx = 0.$$

Accordingly, $u_1' = \text{const}$, that is, the number 1 is a nonrepeated characteristic value of the operator A'.

According to Fredholm's third theorem, in order that Eq. (55) be solvable it is necessary and sufficient that the function F' be orthogonal in $H^1(Q)$ to the function $u_1 = \frac{1}{\sqrt{|Q|}}$: $\left(F', \frac{1}{\sqrt{|Q|}}\right)_{H^1(Q)} = 0$. This condition is equivalent, in view of (52), (51), to the condition

$$-\int_Q f\,dx + \int_{\partial Q} k\varphi\,dS = 0. \tag{56}$$

When this condition is fulfilled, Eq. (55) has a unique solution u which is orthogonal in $H^1(Q)$ to constant functions. This is a generalized solution of the problem under consideration. And, in view of (53), the inequality (48) holds, where the constant C does not depend on f or φ. All other solutions differ from u by constant terms. Since for functions in $H^1(Q)$ the condition of orthogonality to constant functions in the scalar product (49) is equivalent to that of orthogonality to constant functions in $L_2(Q)$ with a scalar product, we have established the following result.

Theorem 12. *For the existence of a generalized solution of the problem* div $(k(x)\ \nabla u) = f$, $\left.\frac{\partial u}{\partial n}\right|_{\partial Q} = \varphi$ *it is necessary and sufficient that equality* (56) *hold. The generalized solution u is orthogonal to constant functions in* $L_2(Q)$ *with a scalar product, is unique and satisfies the inequality* (48). *All other generalized solutions of the problem differ from the function u by constants.*

Remark. If f and φ are real-valued functions, then so are the solutions described by Theorems 10-12.

In the investigation of the first boundary-value problem for Eq. (1) with nonhomogeneous boundary condition there arose the following problem: Under what conditions the function $\varphi \in L_2(\partial Q)$

has an extension into the region Q belonging to $H^1(Q)$. As shown above, a sufficient condition for this is that the function φ belong to the space $C^1(\partial Q)$. Now we shall obtain a necessary and sufficient condition for the case when Q is a disc.

Let $Q(n=2)$ be the disc $\{|x| = \rho < 1\}$, $x = (x_1, x_2) = (\rho\cos\theta, \rho\sin\theta)$. On the circle $\partial Q = \{\rho = 1\}$ we consider a real-valued function φ belonging to the (real) space $L_2(\partial Q)$, $\varphi(\theta) \in L_2(0, 2\pi)$. The Fourier expansion of $\varphi(\theta)$, which converges in the norm of $L_2(0, 2\pi)$, is of the form

$$\varphi(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos k\theta + b_k \sin k\theta),$$

where

$$a_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\theta)\cos k\theta\, d\theta, \quad k = 0, 1, \ldots,$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} \varphi(\theta)\sin k\theta\, d\theta, \quad k = 1, 2, \ldots,$$

are its Fourier coefficients.

According to the Parseval-Steklov equality,

$$\frac{a_0^2}{2} + \sum_{k=1}^{\infty} (a_k^2 + b_k^2) = \frac{1}{\pi}\|\varphi\|^2_{L_2(0,\,2\pi)} < \infty. \tag{57}$$

The following result holds.

Theorem 13. *For the function* $\varphi(\theta) \in L_2(0, 2\pi)$ *to be a trace on the circle* $\{|x| = 1\}$ *of a function belonging to* $H^1(|x| < 1)$ *it is necessary and sufficient that the series*

$$\sum_{k=1}^{\infty} k(a_k^2 + b_k^2) \tag{58}$$

converge.

By (57), the sequences a_k, b_k, $k = 1, 2, \ldots$, are bounded. Therefore the function $\sum_{k=1}^{\infty} (a_k - ib_k) z^k$, where $z = x_1 + ix_2$, is analytic in the disc $\{|z| < 1\}$. This means that the function

$$w(x) = w(\rho, \theta) = \frac{a_0}{2} + \operatorname{Re}\sum_{k=1}^{\infty} (a_k - ib_k)(x_1 + ix_2)^k$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} \rho^k (a_k\cos k\theta + b_k \sin k\theta) \tag{59}$$

belongs to $C^\infty(|x|<1)$, and the series (59) as well as those obtained from it by termwise differentiation converge absolutely and uniformly in the disc $\{|x|<r\}$ for any $r<1$.

To prove Theorem 13, we shall require the following result.

Lemma 4. *In order that the function* $w(x)$ *defined by the series* (59) *may belong to the space* $H^1(|x|<1)$ *it is necessary and sufficient that the series* (58) *converge.*

Proof. By $w_m(x)$ we denote the partial sum of the series (59):

$$w_m(x)=\frac{a_0}{2}+\sum_{k=1}^{m}\rho^k(a_k\cos k\theta+b_k\sin k\theta).$$

Since all the functions of the system $\rho^k\cos k\theta$, $\rho^k\sin k\theta$, $k=0,1,\ldots$, are mutually orthogonal in $L_2(|x|<1)$ and $\|\rho^k\cos k\theta\|^2_{L_2(|x|<1)}=\|\rho^k\sin k\theta\|^2_{L_2(|x|<1)}=\frac{\pi}{2(k+1)}$, $k=1,2,\ldots$, for any p and q, $q>p$, we have

$$\|w_q-w_p\|^2_{L_2(|x|<1)}=\frac{\pi}{2}\sum_{k=p+1}^{q}\frac{a_k^2+b_k^2}{k+1}.$$

Therefore the convergence of the series (57) implies that of the sequence $w_m(x)$, $m=1,2,\ldots$, in $L_2(|x|<1)$. Accordingly, the function $w\in L_2(|x|<1)$, and the series (59) converges to it in $L_2(|x|<1)$.

Assume that the series (58) converges. Then (for $q>p$)

$$\|w_q-w_p\|^2_{H^1(|x|<1)}=\int\limits_{|x|<1}\left[(w_q-w_p)^2+|\nabla(w_q-w_p)|^2\right]dx$$

$$=\int_0^1\rho\,d\rho\int_0^{2\pi}\left[(w_q-w_p)^2+(w_{q\rho}-w_{p\rho})^2+\frac{1}{\rho^2}(w_{q\theta}-w_{p\theta})^2\right]d\theta$$

$$=\frac{\pi}{2}\sum_{k=p+1}^{q}\frac{a_k^2+b_k^2}{k+1}+\pi\sum_{k=p+1}^{q}k(a_k^2+b_k^2)\to 0$$

as $p,q\to\infty$. That is, the sequence w_m, $m=1,2,\ldots$, converges in $H^1(|x|<1)$. Consequently, $w\in H^1(|x|<1)$.

Suppose now that $w\in H^1(|x|<1)$. Since for any $r<1$ the sequence of norms

$$\|w_m\|^2_{H^1(|x|<r)}=\frac{\pi r^2}{2}\left(\frac{a_0^2}{2}+\sum_{k=1}^{m}r^{2k}\frac{a_k^2+b_k^2}{k+1}\right)+2\sum_{k=1}^{m}r^{2k-2}k(a_k^2+b_k^2),$$

$$m=1,2,\ldots,$$

being monotone nondecreasing, converges, as $m \to \infty$, to $\|w\|^2_{H^1(|x|<r)}$, the inequality

$$\sum_{k=1}^{m} k(a_k^2+b_k^2)\, r^{2k} \leqslant \frac{1}{\pi}\,\|w_m\|^2_{H^1(|x|<r)} \leqslant \frac{1}{\pi}\,\|w\|^2_{H^1(|x|<r)} \leqslant \frac{1}{\pi}\,\|w\|^2_{H^1(|x|<1)}$$

holds for all $r<1$ and all m. Consequently, the partial sums of the series (58) are bounded:

$$\sum_{k=1}^{m} k(a_k^2+b_k^2) \leqslant \frac{1}{\pi}\,\|w\|^2_{H^1(|x|<1)}, \qquad m=1,\ 2,\ \ldots,$$

that is, the series (58) converges. ▌

Proof of Theorem 13. That the condition is sufficient is an immediate consequence of Lemma 4, because, if the series (58) converges, the function w in (59) belongs to $H^1(|x|<1)$ and its trace on the circle $\{|x|=1\}$ is φ.

We shall prove the necessity. Assume that there is a function $\Phi \in H^1(|x|<1)$ for which $\Phi|_{\{|x|=1\}} = \varphi$. Then, by Theorem 10, the first boundary-value problem for Eq. (1) with boundary function φ has a generalized solution belonging to $H^1(|x|<1)$. Let u be the generalized solution of the first boundary-value problem for Eq. (1) with $k\equiv 1$, $a\equiv f\equiv 0$, satisfying the condition $u|_{\{|x|=1\}} = \varphi$. It will be shown in Sec. 2.2 that the function u belongs to $C^\infty(|x|<1)$ and satisfies the equation $\Delta u = 0$ in the disc $\{|x|<1\}$. We use this result, and expand the function $u(x) = u(\rho, \theta)$ with fixed $\rho \in (0, 1)$ in a uniformly and absolutely convergent Fourier series in θ:

$$u(\rho,\theta)=\frac{U_0(\rho)}{2}+\sum_{k=1}^{\infty}\left(U_k(\rho)\cos k\theta+V_k(\rho)\sin k\theta\right),$$

where

$$U_k(\rho)=\frac{1}{\pi}\int_0^{2\pi} u(\rho,\theta)\cos k\theta\, d\theta, \qquad k=0,\ 1,\ \ldots,$$

$$V_k(\rho)=\frac{1}{\pi}\int_0^{2\pi} u(\rho,\theta)\sin k\theta\, d\theta, \qquad k=1,\ 2,\ \ldots.$$

The functions $U_k(\rho)$ (and $V_k(\rho)$), $k=0, 1, \ldots$, are infinitely differentiable for $0<\rho<1$ and are bounded as $\rho \to +0$. Since $u \in H^1(|x|<1)$, by the theorem on the traces of functions belonging to $H^1(|x|<1)$ we have, as $\rho \to 1-0$,

$$\int_0^{2\pi} (u(\rho,\theta)-\varphi(\theta))\cos k\theta\, d\theta \to 0 \ \left(\int_0^{2\pi} (u(\rho,\theta)-\varphi(\theta))\sin k\theta\, d\theta \to 0\right)$$

for all $k = 0, 1, \ldots$. This means that all the functions $U_k(\rho)$ $(V_k(\rho))$ are left-continuous at the point $\rho = 1$ and $U_k(1) = a_k$ $(V_k(1) = b_k)$, $k = 0, 1, \ldots$.

Since for $\rho \in (0, 1)$ $\Delta u = u_{\rho\rho} + \frac{1}{\rho} u_\rho + \frac{1}{\rho^2} u_{\theta\theta} = 0$, for such ρ

$$U_k''(\rho) = \frac{1}{\pi} \int_0^{2\pi} u_{\rho\rho}(\rho, \theta) \cos k\theta \, d\theta = -\frac{1}{\pi\rho} \int_0^{2\pi} u_\rho \cos k\theta \, d\theta$$

$$-\frac{1}{\pi\rho^2} \int_0^{2\pi} u_{\theta\theta} \cos k\theta \, d\theta = -\frac{1}{\rho} U_k' + \frac{k^2}{\rho^2} U_k, \quad k = 0, 1, \ldots$$

This means that for any $k = 0, 1, \ldots$ the function $U_k(\rho)$ satisfies the ordinary (Euler's) differential equation $y'' + \frac{1}{\rho} y' - \frac{k^2}{\rho^2} y = 0$ for $0 < \rho < 1$. The general solution of this equation is of the form $B\rho^k + C\rho^{-k}$, $k \neq 0$, and $B + C \ln \rho$ when $k = 0$, where B and C are arbitrary constants, therefore $U_k(\rho) = a_k \rho^k$, $k = 0, 1, \ldots$. It is similarly shown that $V_k(\rho) = b_k \rho^k$, $k = 1, 2, \ldots$.

Thus the function u belonging to $H^1(|x| < 1)$ coincides with the function w in (59). And then, by Lemma 4, the series (58) converges. ∎

Let us utilise this theorem in constructing the function $\varphi(\theta)$ continuous on the circle $\{\rho = 1\}$ which cannot be extended into the disc $\{|x| < 1\}$ by a function belonging to $H^1(|x| < 1)$. Put

$$\varphi(\theta) = \sum_{k=1}^{\infty} \frac{\cos k^3\theta}{k^2}.$$

Since this series is uniformly convergent, the function $\varphi \in C(|x| = 1)$. But at the same time, in view of Theorem 13, it cannot be the trace of any function belonging to $H^1(|x| < 1)$, because the series (58) $\left(\text{which has the form } \sum_{k=1}^{\infty} k^3 \frac{1}{(k^2)^2}\right)$ for it diverges.

9. Variational Method for Solving Boundary-Value Problems. Let H' be any subspace of the real space $H^1(Q)$; in particular, H' may coincide with the whole $H^1(Q)$. We shall assume that a scalar product is defined in H' that is equivalent to the usual scalar product in $H^1(Q)$.

We take a real function $f \in L_2(Q)$ and on H' consider the functional

$$E(v) = \|v\|_{H'}^2 + 2(f, v)_{L_2(Q)}, \qquad v \in H'. \tag{60}$$

Since $|(f, v)_{L_2(Q)}| \leqslant \|f\|_{L_2(Q)} \|v\|_{L_2(Q)} \leqslant C \|f\|_{L_2(Q)} \|v\|_{H'}$, for all $v \in H'$

$$E(v) \geqslant \|v\|^2_{H'} - 2|(f, v)_{L_2(Q)}| \geqslant \|v\|^2_{H'} - 2C\|v\|_{H'}\|f\|_{L_2(Q)}$$
$$= (\|v\|_{H'} - C\|f\|_{L_2(Q)})^2 - C^2\|f\|^2_{L_2(Q)} \geqslant -C^2\|f\|^2_{L_2(Q)}.$$

This means that the range of values of the functional E on H' is bounded below. Let $d = d(H')$ denote the exact lower bound of the functional E on H':

$$d = \inf_{v \in H'} E(v).$$

A function u in H' is said to *realize the minimum* of the functional E on H' if

$$E(u) = d. \tag{61}$$

Like the number d, the function u, of course, depends on the choice of the subspace H'.

By the definition of the exact lower bound, there is a sequence of functions v_m, $m = 1, 2, \ldots$, in H' for which

$$\lim_{m \to -\infty} E(v_m) = d. \tag{62}$$

Any such sequence is called a *minimizing sequence* for the functional E on H'.

Lemma 5. *For any subspace H' of the space $H^1(Q)$ (in particular, H' may coincide with the whole $H'(Q)$) there is a unique function u in H' realizing the minimum of the functional E on H'. Any minimizing sequence for the functional E on H' converges to this function in the norm of $H^1(Q)$.*

Proof. Suppose that a sequence v_m, $m = 1, 2, \ldots$, of elements in H' is a minimizing sequence for the functional E on H'. Then given an $\varepsilon > 0$ a number $N = N(\varepsilon)$ can be found such that for all $m \geqslant N$

$$d \leqslant E(v_m) \leqslant d + \varepsilon. \tag{63}$$

Since $\left\|\frac{v_m \pm v_s}{2}\right\|^2_{H'} = \frac{1}{4}\|v_m\|^2_{H'} + \frac{1}{4}\|v_s\|^2_{H'} \pm \frac{1}{2}(v_m, v_s)_{H'}$, it follows that

$$\left\|\frac{v_m + v_s}{2}\right\|^2_{H'} + \left\|\frac{v_m - v_s}{2}\right\|^2_{H'} = \frac{1}{2}(\|v_m\|^2_{H'} + \|v_s\|^2_{H'}).$$

Taking into account (60), the last relation yields

$$\left\|\frac{v_m - v_s}{2}\right\|^2_{H'} = \frac{1}{2}(\|v_m\|^2_{H'} + \|v_s\|^2_{H'}) - \left\|\frac{v_m + v_s}{2}\right\|^2_{H'}$$
$$= \frac{1}{2}(E(v_m) + E(v_s)) - E\left(\frac{v_m + v_s}{2}\right).$$

But $E\left(\frac{v_m+v_s}{2}\right) \geqslant d$, and the functions v_m and v_s satisfy inequalities (63) for $m, s \geqslant N$; this implies that

$$0 \leqslant \left\| \frac{v_m-v_s}{2} \right\|_{H'}^2 \leqslant \frac{1}{2}(d+\varepsilon+d+\varepsilon)-d=\varepsilon$$

for all $m, s \geqslant N$. Since $\varepsilon > 0$ is arbitrary, this means that the sequence under consideration is fundamental in $H^1(Q)$. Consequently, there is a function u in H' to which this sequence converges in the norm of $H^1(Q)$, but then $\| v_s \|_{H'} \to \| u \|_{H'}$ and $(f, v_s)_{L_2(Q)} \to$ $\to (f, u)_{L_2(Q}$ as $s \to \infty$. Accordingly, $E(v_s) \to E(u)$. The equality (61) now follows from the relation (62).

We shall now show that the function u realizing the minimum of the functional E on H' is unique. Suppose that there are two such functions u_1 and u_2. Then the sequence $u_1, u_2, u_1, u_2, \ldots$ is a minimizing sequence for the functional E on H' and does not converge in H', which contradicts the assertion just established. ■

We now set forth the *Ritz method* for constructing a minimizing sequence for the functional E. In H' take an arbitrary linearly independent system of functions φ_k, $k = 1, 2, \ldots$, whose linear hull is everywhere dense in H'. When $H' = H^1(Q)$, for such a system one can take the set of all monomials $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where α is any n-dimensional vector with nonnegative integer components.

We denote by R_k the k-dimensional subspace of $H' \subset H^1(Q)$ spanned by $\varphi_1, \ldots, \varphi_k$, and find the element which realizes the minimum of the functional E on R_k (such an element exists, by Lemma 5). Since any element in R_k can be expressed in the form $c_1\varphi_1 + \ldots + c_k\varphi_k$ with some real constants c_i, this problem is equivalent to that of finding the minimum of the function

$$F(c_1, \ldots, c_k) = E(c_1\varphi_1 + \ldots + c_k\varphi_k)$$
$$= \sum_{i,\,j=1}^{k} c_i c_j (\varphi_i, \varphi_j)_{H'} + 2\sum_{i=1}^{k} c_i (f, \varphi_i)_{L_2(Q)}$$

with respect to $c_1, \ldots, c_k$.

The vector $(c_1, \ldots, c_k)$ where the function F attains its minimum is a solution of the system of linear equations $\frac{\partial F}{\partial c_i} = 0$, $i = 1, \ldots$ $\ldots, k$, that is, of the system

$$\sum_{j=1}^{k} (\varphi_i, \varphi_j)_{H'} c_j + (f, \varphi_i)_{L_2(Q)} = 0, \qquad i = 1, \ldots, k. \tag{64}$$

The determinant of the system (64), called Gram's determinant of the system $\varphi_1, \ldots, \varphi_k$, is not zero. Indeed, if it were zero, the linear dependence of its rows would imply the existence of constants $\xi_1, \ldots, \xi_k$, $|\xi_1| + \ldots + |\xi_k| \neq 0$, such that $\xi_1(\varphi_1, \varphi_j)_{H'} + \ldots$

$\ldots + \xi_k (\varphi_k, \varphi_j)_{H'} = 0$ for all $j = 1, \ldots, k$. This would mean that the function $\xi_1\varphi_1 + \ldots + \xi_k\varphi_k$ is orthogonal to all the functions $\varphi_1, \ldots, \varphi_k$, that is, $\xi_1\varphi_1 + \ldots + \xi_k\varphi_k = 0$, contradicting the linear independence of the system $\varphi_1, \ldots, \varphi_k$.

Thus the linear system (64) has always a unique solution $c_1^k, \ldots$ $\ldots, c_k^k$. Then the function

$$v_k = c_1^k\varphi_1 + \ldots + c_k^k\varphi_k \tag{65}$$

belonging to R_k realizes the minimum of the functional E on R_k. The sequence of functions v_k, $k = 1, 2, \ldots$, is called the *Ritz sequence* for the functional E with respect to the system $\varphi_1, \varphi_2, \ldots$.

Lemma 6. *The Ritz sequence v_k, $k = 1, 2, \ldots$, of the functional E with respect to an arbitrary linearly independent system of functions φ_k, $k = 1, 2, \ldots$, whose linear hull is everywhere dense in H' is a minimizing sequence for the functional E on H'. The sequence v_k, $k = 1, 2, \ldots$, converges in the norm of $H^1(Q)$ to a function u which realizes the minimum of E on H'.*

Proof. As $R_1 \subset R_2 \subset \ldots \subset R_k \subset \ldots \subset H'$,

$$E(v_1) \geqslant E(v_2) \geqslant \ldots \geqslant E(v_k) \geqslant \ldots \geqslant d. \tag{66}$$

Since the linear hull of the system φ_k, $k = 1, 2, \ldots$, is everywhere dense in H', for any $\varepsilon > 0$ we can find numbers $k = k(\varepsilon)$ and $c_1'(\varepsilon), \ldots, c_k'(\varepsilon)$ such that $\| u_\varepsilon - u \|_{H'} \leqslant \varepsilon$, where $u_\varepsilon = c_1'(\varepsilon)\varphi_1 + \ldots + c_k'(\varepsilon)\varphi_k$ belongs to R_k. Then (60) yields

$$\begin{aligned}
E(u_\varepsilon) &= \| u_\varepsilon \|_{H'}^2 + 2(f, u_\varepsilon)_{L_2(Q)} = \| u_\varepsilon - u + u \|_{H'}^2 \\
&\quad + 2(f, u_\varepsilon - u + u)_{L_2(Q)} = E(u) + E(u_\varepsilon - u) + 2(u_\varepsilon - u, u)_{H'} \\
&\quad \leqslant d + |E(u_\varepsilon - u)| + 2\| u_\varepsilon - u \|_{H'} \| u \|_{H'} \leqslant d + \| u_\varepsilon - u \|_{H'}^2 \\
&\quad + 2C \| f \|_{L_2(Q)} \| u_\varepsilon - u \|_{H'} + 2 \| u_\varepsilon - u \|_{H'} \| u \|_{H'} \\
&\qquad \leqslant d + \varepsilon^2 + 2C \| f \|_{L_2(Q)} \varepsilon + 2 \| u \|_{H'} \varepsilon \leqslant d + C_1\varepsilon
\end{aligned}$$

with a constant $C_1 > 0$. As the minimum of E on R_k is attained on the function v_k, (66) implies that $d \leqslant E(v_s) \leqslant E(v_k) \leqslant d + C_1\varepsilon$ for all $s \geqslant k$. This means that $E(v_s) \to d$ as $s \to \infty$. The convergence of the sequence v_s, $s = 1, 2, \ldots$, to the function u follows from Lemma 5. ▮

We shall now establish an important property of the function u which realizes the minimum of the functional E on H'. Take any function $v \in H'$ and any real number t. The function $w_t = u + tv$ belongs to H', therefore the polynomial (in t) $P(t) = E(w_t) = E(u) + 2t((u, v)_{H'} + (f, v)_{L_2(Q)}) + t^2 \| v \|_{H'}^2 \geqslant d$ for all $t \in (-\infty, +\infty)$. Moreover, $P(0) = E(u) = d$. This means that

$$\left.\frac{dP}{dt}\right|_{t=0} = 2((u, v)_{H'} + (f, v)_{L_2(Q)}) = 0.$$

Thus any function u in H' realizing the minimum of E on H' satisfies the identity

$$(u, v)_{H'} + (f, v)_{L_2(Q)} = 0 \tag{67}$$

for all $v \in H'$.

The manner in which the scalar product equivalent to the scalar product in $H^1(Q)$ is defined in H' was so far immaterial.

Let $H' = H^1(Q)$. Take functions $k(x) \in C(\overline{Q})$, $k(x) \geqslant k_0 = \text{const} > 0$, $a(x) \in C(\overline{Q})$, $a(x) \geqslant 0$ in Q and $\sigma(x) \in C(\partial Q)$, $\sigma(x) \geqslant 0$ on ∂Q, and assume that either $a(x) \not\equiv 0$ in Q or $\sigma(x) \not\equiv 0$ on ∂Q. We define a scalar product in $H^1(Q)$ by the formula (10):

$$(u, v)_{H^1(Q)} = \int_Q (k \nabla u \nabla v + auv)\, dx + \int_{\partial Q} k\sigma uv\, dS.$$

Then identity (67) coincides with the identity (6):

$$\int_Q (k \nabla u \nabla v + auv)\, dx + \int_{\partial Q} k\sigma uv\, dS = -\int_Q fv\, dx,$$

defining the generalized solution of the third or second (if $\sigma \equiv 0$) boundary-value problem for Eq. (1) (with homogeneous boundary conditions).

If $H' = \overset{\circ}{H}{}^1(Q)$ and the scalar product in $\overset{\circ}{H}{}^1(Q)$ is defined by the formula (7):

$$(u, v)_{\overset{\circ}{H}{}^1(Q)} = \int_Q (k \nabla u \nabla v + auv)\, dx$$

($k(x)$, $a(x) \in C(\overline{Q})$, $k(x) \geqslant k_0 > 0$, $a(x) \geqslant 0$), then the identity (67) coincides with the identity (4) defining the generalized solution of the first boundary-value problem for Eq. (1) (with homogeneous boundary condition).

Thus the following result has been established.

Theorem 14. *There is a unique function u in $H^1(Q)$ which realizes the minimum of the functional E on $H^1(Q)$. If the scalar product in $H^1(Q)$ is defined by* (10), *then this function is a generalized solution of the third or second* (*if* $\sigma \equiv 0$) *boundary-value problem for Eq.* (1) (*with homogeneous boundary condition*).

There is a unique function u in $\overset{\circ}{H}{}^1(Q)$ realizing the minimum of the functional E on $\overset{\circ}{H}{}^1(Q)$. If the scalar product in $\overset{\circ}{H}{}^1(Q)$ is defined by formula (7), *then this function is a generalized solution of the first boundary-value problem for Eq.* (1).

Theorem 14 provides an alternative method, the variational method, different from the one employed in Theorems 1 and 2 for establishing the existence and uniqueness of generalized solutions of the boundary-value problems examined in Subsec. 2, and indicates the variational significance of generalized solutions.

If the subspace H' coincides with $H^1(Q)(\overset{\circ}{H}{}^1(Q))$ and the scalar product, equivalent to the usual scalar product, is defined there by the formula (10) ((7)), then, by Lemma 6 and Theorem 14, the Ritz sequence v_k, $k = 1, 2, \ldots$, for the functional E on $H^1(Q)$ $(\overset{\circ}{H}{}^1(Q))$ converges in $H^1(Q)$ to the generalized solution of the third (first) boundary-value problem for Eq. (1). That is, the Ritz sequence may be regarded as an approximation to the generalized solution of the boundary-value problem for Eq. (1).

Thus we have the following result.

Theorem 15. *The Ritz sequence for the functional E defined on $H^1(Q)$ or on $\overset{\circ}{H}{}^1(Q)$ (the scalar product is defined by* (10) *or* (7)) *constructed with respect to any linearly independent system of functions whose linear hull is everywhere dense in $H^1(Q)$ or $\overset{\circ}{H}{}^1(Q)$, respectively, converges in $H^1(Q)$ to the generalized solution of the corresponding boundary-value problem (third or first) for Eq.* (1).

§ 2. SMOOTHNESS OF GENERALIZED SOLUTIONS. CLASSICAL SOLUTIONS

In the preceding section we examined the question of solvability of basic boundary-value problems for the second-order elliptic equations. We shall now investigate the smoothness of solutions of these problems.

We shall assume that the data of the problem under consideration are real-valued. Then, as noted in Sec. 1, the generalized solutions of these problems are also real-valued. Accordingly, by solutions in the sequel we shall mean, unless otherwise stated, real-valued functions—the elements of real spaces $C^k(\bar{Q})$ or $H^k(Q)$, $k = 0, 1, 2, \ldots$.

In the investigation of smoothness of generalized solutions it is advantageous to consider the one-dimensional case separately, since the results obtained in this case are, in general, not true for $n > 1$. Moreover, the investigation in this case is far more simpler. In particular, when $n = 1$, the generalized solutions of the boundary-value problems (they belong to the space $H^1(\alpha, \beta)$) are continuous functions on $[\alpha, \beta]$, by Theorem 3, Sec. 6.2, Chap. III. The question of extension of a function defined on the boundary into the region by a function in $H^1(\alpha, \beta)$ is also easily solved in the one-dimensional case: if $\Phi|_{x=\alpha} = \varphi_0$, $\Phi|_{x=\beta} = \varphi_1$, then for such an extension one can take the linear function $\Phi = \frac{x(\varphi_1 - \varphi_0)}{\beta - \alpha} - \frac{\alpha\varphi_1 - \beta\varphi_0}{\beta - \alpha}$.

1. Smoothness of Generalized Solutions in the One-Dimensional Case. We recall that in the classical formulation the boundary-value problems for the equation

$$\mathscr{L}u \equiv (ku')' - au = f, \quad x \in (\alpha, \beta), \tag{1}$$

consist in finding the solution $u(x)$ of this equation which satisfies the following conditions: in the case of the first boundary-value problem $u(x) \in C^2(\alpha, \beta) \cap C([\alpha, \beta])$ and

$$u|_{x=\alpha} = \varphi_0, \qquad u|_{x=\beta} = \varphi_1, \tag{2}$$

in the case of the third (second) boundary-value problem $u(x) \in \in C^2(\alpha, \beta) \cap C^1([\alpha, \beta])$ and

$$(-u_x + \sigma_0 u)|_{x=\alpha} = \varphi_0, \qquad (u_x + \sigma_1 u)|_{x=\beta} = \varphi_1. \tag{3}$$

Here $k(x) \in C^1([\alpha, \beta])$, $a(x) \in C([\alpha, \beta])$ and $f(x)$ are given functions with $k(x) \geqslant k_0 > 0$; σ_0, σ_1, φ_0, φ_1 are given constants.

The generalized solution of the problem (1), (2) ($f \in L_2(\alpha, \beta)$) is the function $u(x) \in H^1(\alpha, \beta)$ satisfying the integral identity

$$\int_\alpha^\beta (ku'v' + auv)\, dx = -\int_\alpha^\beta fv\, dx \tag{4}$$

for all $v \in \overset{\circ}{H}{}^1(\alpha, \beta)$ and the boundary conditions (2).

The generalized solution of the problem (1), (3) ($f \in L_2(\alpha, \beta)$) is the function $u(x) \in H^1(\alpha, \beta)$ satisfying the integral identity

$$\int_\alpha^\beta (ku'v' + auv)\, dx + k(\beta)\sigma_1 u(\beta) v(\beta) + k(\alpha)\sigma_0 u(\alpha) v(\alpha)$$

$$= -\int_\alpha^\beta fv\, dx + k(\beta)\varphi_1 v(\beta) + k(\alpha)\varphi_0 v(\alpha) \tag{5}$$

for all $v \in H^1(\alpha, \beta)$.

The following auxiliary proposition holds.

Lemma 1. *If $f \in C([\alpha, \beta])$, then the function $u(x) \in H^1(\alpha, \beta)$ satisfying the integral identity (4) for all $v \in \overset{\circ}{H}{}^1(\alpha, \beta)$ belongs to the space $C^2([\alpha, \beta])$ and is a solution of Eq. (1) in the interval (α, β).*

Proof. Since the function $u(x) \in C([\alpha, \beta])$ ($H^1(\alpha, \beta) \subset C([\alpha, \beta])$), $f + au \in C([\alpha, \beta])$ and $\int_0^x (f(\xi) + a(\xi)u(\xi))\, d\xi \in C^1([\alpha, \beta])$. Consider the function $u_0(x) = \int_0^x \frac{d\eta}{k(\eta)} \int_0^\eta (f(\xi) + a(\xi)u(\xi))\, d\xi$. By the conditions imposed on $k(x)$, the function $u_0(x) \in C^2([\alpha, \beta])$, and, moreover, is a solution on (α, β) of the differential equation $(ku_0')' = f(x) + a(x)\,u(x)$. Therefore the function $u_0(x)$ satisfies for any $v \in \overset{\circ}{H}{}^1(\alpha, \beta)$ the identity

$$\int_\alpha^\beta (ku_0'v' + auv)\,dx = -\int_\alpha^\beta fv\,dx,$$

which means that the function $u_1 = u - u_0$ belonging to $H^1(\alpha, \beta)$ satisfies the integral identity

$$\int_\alpha^\beta ku_1'v'\,dx = 0, \quad v \in \overset{\circ}{H}{}^1(\alpha, \beta),$$

thereby implying that on (α, β) the function ku_1' has a generalized derivative equal to zero. Consequently, $ku_1' = \text{const}$, that is, $u_1 \in C^2([\alpha, \beta])$. Therefore also $u(x) \in C^2([\alpha, \beta])$.

Since $\int_\alpha^\beta ku'v'\,dx = -\int_\alpha^\beta (ku')'v\,dx$ for all $v \in \overset{\circ}{H}{}^1(\alpha, \beta)$, (4) implies that the function $(ku')' - a(x)\,u(x) - f(x)$ continuous on $[\alpha, \beta]$ is orthogonal (in $L_2(\alpha, \beta)$ with a scalar product) to any function belonging to $\overset{\circ}{H}{}^1(\alpha, \beta)$. Therefore the function $u(x)$ is a solution of Eq. (1) in (α, β). ■

The generalized solution $u(x)$ of any of the first, second or third boundary-value problems belongs to $H^1(\alpha, \beta)$ and for it identity (4) holds for all $v \in \overset{\circ}{H}{}^1(\alpha, \beta)$. Therefore, by Lemma 1, $u(x) \in C^2([\alpha, \beta])$ and $u(x)$ is the solution of Eq. (1) in (α, β).

Thus we have shown that if $f \in C([\alpha, \beta])$, the generalized solutions of the boundary-value problems under discussion have continuous derivatives up to order 2 on $[\alpha, \beta]$ and satisfy Eq. (1). Moreover, it is evident that in the case of the first boundary-value problem the function $u(x)$ satisfies the boundary conditions (2). In the case of the third (second) boundary-value problem,

$$\int_\alpha^\beta ku'v'\,dx = -\int_\alpha^\beta (ku')'\,v\,dx + k(\beta)\,u'(\beta)\,v(\beta) - k(\alpha)\,u'(\alpha)\,v(\alpha)$$

for any $v(x) \in H^1(\alpha, \beta)$. If account is taken of (5), this gives

$$k(\beta)\,(u'(\beta) + \sigma_1 u(\beta) - \varphi_1)\,v(\beta) + k(\alpha)\,(-u'(\alpha) + \sigma_0 u(\alpha) - \varphi_0)\,v(\alpha) = 0$$

for any $v(\beta)$ and $v(\alpha)$ (recall that for any $v(\alpha)$ and $v(\beta)$ there is a function $v(x)$ in $H^1(\alpha, \beta)$ assuming the values $v(\alpha)$ and $v(\beta)$ at $x = \alpha$ and $x = \beta$). Hence the function $u(x)$ satisfies the boundary conditions (3).

Thus we have proved the following theorem.

Theorem 1. *If the function* $f(x) \in C([\alpha, \beta])$, *then the generalized solutions of the first, second and third boundary-value problems for Eq.* (1) *belong to* $C^2([\alpha, \beta])$ *and are classical solutions of the corresponding problems.*

Suppose that $u_s(x)$ is a generalized eigenfunction of, for example, the third (second) boundary-value problem for the operator $\mathscr{L}$. This means that $u_s(x) \in H^1(\alpha, \beta)$, and hence $u_s(x) \in C([\alpha, \beta])$, and satisfies the equality

$$\int_\alpha^\beta (ku_s'v' + au_sv)\,dx + k(\beta)\,\sigma_1 u_s(\beta)\,v(\beta) + k(\alpha)\,\sigma_0 u_s(\alpha)\,v(\alpha) = -\lambda_s \int_\alpha^\beta u_s v\,dx,$$

where λ_s is an eigenvalue, for all $v \in H^1(\alpha, \beta)$. By Theorem 1, the function $u_s(x)$ belongs to $C^2([\alpha, \beta])$ and is a classical solution of the equation

$$\mathscr{L}u = (ku')' - au = \lambda_s u, \qquad x \in (\alpha, \beta), \tag{6}$$

satisfying the homogeneous boundary conditions (3), that is, $u_s(x)$ is a classical eigenfunction of the third (second) boundary-value problem for the operator $\mathscr{L}$.

It is similarly shown that the generalized eigenfunction $u_s(x)$ of the first boundary-value problem belongs to $C^2([\alpha, \beta])$ and is a classical eigenfunction of the same problem.

Let us show that the eigenvalues of any of the boundary-value problems are nondegenerate. Assume that there is an eigenvalue λ_s of, for example, the third boundary-value problem to which there correspond two linearly independent eigenfunctions $u^{(1)}$ and $u^{(2)}$. We know that the general solution of Eq. (6) is of the form $C_1u^{(1)} + C_2u^{(2)}$, where C_1 and C_2 are arbitrary constants. This means that any solution of Eq. (6) must satisfy the boundary condition $u'(\alpha) - \sigma_0 u(\alpha) = 0$, because this boundary condition is satisfied by both the functions $u^{(1)}$ and $u^{(2)}$. But there is a solution of Eq. (6) that does not satisfy this condition, for example, the solution with initial conditions $u(\alpha) = 0$, $u'(\alpha) = 1$.

Thus we have established the following result.

Theorem 2. *The generalized eigenfunctions of the first, second and third boundary-value problems for the operator* $\mathscr{L}$ *belong to* C^2 $([\alpha, \beta])$ *and are classical eigenfunctions of the corresponding problems. All the eigenvalues are nondegenerate.*

2. Interior Smoothness of Generalized Solutions. We shall now examine the question of smoothness of generalized solutions of boundary-value problems when $n > 1$. In order that the essential points do not get lost in technical details, we confine our discussion to a particular case of Eq. (1) of the preceding section, and investigate the smoothness of the generalized solutions of the boundary-value problems for Poisson's equation ($k \equiv 1$, $a \equiv 0$)

$$\Delta u = f. \tag{7}$$

We recall that the generalized solution of the first boundary-value problem for Eq. (7) is a function $u(x)$ in $H^1(Q)$ satisfying the integral identity

$$\int_Q \nabla u \nabla v \, dx = -\int_Q fv \, dx \tag{8}$$

for all $v \in \mathring{H}^1(Q)$ and the boundary condition $u|_{\partial Q} = \varphi$ (the function $f \in L_2(Q)$ and φ is the trace of a function Φ belonging to $H^1(Q)$, that is, there is a function $\Phi \in H^1(Q)$ such that $\Phi|_{\partial Q} = \varphi$).

The generalized solution of the third (second) boundary-value problem for Eq. (7) is a function $u(x) \in H^1(Q)$ satisfying the integral identity

$$\int_Q \nabla u \nabla v \, dx + \int_{\partial Q} \sigma uv \, dS = -\int_Q fv \, dx + \int_{\partial Q} \varphi v \, dS \tag{9}$$

for all $v \in H^1(Q)$ (the function $f \in L_2(Q)$ and $\varphi \in L_2(\partial Q)$).

The results of the preceding section imply that the generalized solution $u(x)$ of the first boundary-value problem exists, is unique and satisfies the inequality

$$\| u \|_{H^1(Q)} \leqslant C \Big(\| f \|_{L_2(Q)} + \inf_{\substack{\Phi \in H^1(Q) \\ \Phi|_{\partial Q} = \varphi}} \| \Phi \|_{H^1(Q)} \Big), \tag{10}$$

where the constant C does not depend on f or φ.

The generalized solution $u(x)$ of the third boundary-value problem with $\sigma \geqslant 0$, $\sigma \not\equiv 0$, also exists, is unique and satisfies the inequality

$$\| u \|_{H^1(Q)} \leqslant C \left(\| f \|_{L_2(Q)} + \| \varphi \|_{L_2(\partial Q)} \right), \tag{11}$$

where the constant C does not depend on f or φ.

In the case of the second boundary-value problem ($\sigma \equiv 0$) the condition of its solvability is assumed fulfilled: $-\int_Q f \, dx + \int_{\partial Q} \varphi \, dS = 0$. Then the second boundary-value problem has a unique generalized solution $u(x)$ in the class of functions which are orthogonal to the constant functions with respect to $L_2(Q)$ with a scalar product, and for this solution inequality (11) holds. Because all other generalized solutions differ from $u(x)$ by constant terms, we may confine ourselves to the function $u(x)$ in investigating the smoothness of generalized solutions of the second boundary-value problem.

Lemma 2. *Suppose that* $f \in L_2(Q) \cap H^k_{\text{loc}}(Q)$, $k = 0, 1, 2, \ldots$, *and the function* $u \in H^1(Q)$ *and satisfies the integral identity* (8) *for all* $v \in \overset{\circ}{H}{}^1(Q)$. *Then* $u \in H^{k+2}_{\text{loc}}(Q)$ *and for any pair of subregions* Q' *and* Q'' *of* Q, *such that* $Q' \Subset Q'' \Subset Q$, *the inequality*

$$\|u\|_{H^{k+2}(Q')} \leqslant C(\|f\|_{H^k(Q'')} + \|u\|_{H^1(Q'')}) \tag{12}$$

holds with a positive constant $C = C(k, Q', Q'')$.

Proof. Let Q', Q'' be arbitrary subregions of Q such that $Q' \Subset Q'' \Subset Q$. By $\delta > 0$ we denote the distance between the boundaries $\partial Q'$ and $\partial Q''$, and consider the function $\zeta(x)$ such that $\zeta(x) \in C^\infty(R_n)$, $\zeta(x) \equiv 1$ in Q''_δ (and thus in Q'), $\zeta(x) \equiv 0$ outside $Q''_{2\delta/3}$.

For function $v(x)$ in identity (8) take the function $\zeta(x)\, v_0(x)$, where $v_0(x)$ is any function in $H^1(Q'')$ extended as being equal to zero outside Q'' (obviously, $\zeta(x)\, v_0(x) \in \overset{\circ}{H}{}^1(Q)$). Since $\nabla u \nabla v = \nabla u \nabla(\zeta v_0) = \nabla u(\nabla\zeta \cdot v_0 + \zeta \nabla v_0) = \nabla u \nabla \zeta \cdot v_0 + \nabla(\zeta u) \nabla v_0 - u \nabla\zeta \nabla v_0$, the identity (8) assumes the form

$$\int_{Q''} \nabla U \nabla v_0 \, dx = \int_{Q''} F v_0 \, dx + \int_{Q''} u \nabla\zeta \nabla v_0 \, dx, \tag{13_0}$$

where the function

$$U(x) = \zeta(x)\, u(x) \tag{14}$$

belongs to $H^1(Q'')$, vanishes outside $Q''_{2\delta/3}$ and coincides with $u(x)$ in Q''_δ, and the function

$$F(x) = -f\zeta - \nabla u \nabla \zeta \tag{15}$$

belongs to $L_2(Q'')$ and vanishes outside $Q''_{2\delta/3}$.

Note that the integration in (13_0) is, in fact, over $Q''_{2\delta/3}$. Therefore this identity is true not only for any $v_0 \in H^1(Q'')$ but also for any $v_0 \in H^1(Q''_{\delta/2})$ (extended arbitrarily outside $Q''_{\delta/2}$, as an element of $L_2(Q'')$).

Take any function $v_1(x)$ belonging to $H^1(Q'')$ and extended as being equal to zero outside Q''. For any $i = 1, 2, \ldots, n$ and any h, $0 < |h| < \delta/2$, the finite-difference ratio

$$\delta^i_{-h} v_1(x) = \frac{v_1(x_1, \ldots, x_{i-1}, x_i - h, x_{i+1}, \ldots, x_n) - v_1(x)}{-h}$$

belongs to $H^1(Q''_{\delta/2}) \cap L_2(Q'')$. Put in ($13_0$) $v_0 = \delta^i_{-h} v_1(x)$ for some $i = 1, 2, \ldots, n$ and some h, $0 < |h| < \delta/2$. The formula of "integration by parts" ((9), Sec. 3.4, Chap. III) yields the identity

$$\int_{Q''} \nabla \delta^i_h U \nabla v_1 \, dx = - \int_{Q''_{2\delta/3}} F \delta^i_{-h} v_1 \, dx + \int_{Q''} \delta^i_h (u \nabla\zeta) \nabla v_1 \, dx. \tag{16_0}$$

We shall first prove the lemma for $k = 0$. From (15) it readily follows that

$$\| F \|_{L_2(Q'')} \leqslant C'(Q', Q'') (\| f \|_{L_2(Q'')} + \| u \|_{H^1(Q'')}).$$

Therefore from (16_0), by means of Theorem 3, Sec. 3.4, Chap. III, we obtain the inequalities

$$\left| \int_{Q''} \nabla \delta_h^i U \, \nabla v_1 \, dx \right| \leqslant (\| F \|_{L_2(Q'')} + C \| u \|_{H^1(Q'')}) \| | \nabla v_1 | \|_{L_2(Q'')}$$

$$\leqslant C (Q', Q'') (\| f \|_{L_2(Q'')} + \| u \|_{H^1(Q'')}) \| | \nabla v_1 | \|_{L_2(Q'')}.$$

Putting $v_1 = \delta_h^i U$ (U is assumed extended as being zero outside Q), we find that

$$\| | \nabla \delta_h^i U | \|_{L_2(Q'')} \leqslant C (Q', Q'') (\| f \|_{L_2(Q'')} + \| u \|_{H^1(Q'')})$$

for all $i = 1, 2, \ldots, n$, $0 < | h | < \delta/2$.

In view of Theorem 3, Sec. 3.4, Chap III, the last inequality implies that $U \in H^2(Q'')$ and $\| U \|_{H^2(Q'')} \leqslant C(Q', Q'') (\| f \|_{L_2(Q'')} + + \| u \|_{H^1(Q'')})$. Because $U = u$ in Q', it follows that $u \in H^2(Q')$ and the inequality (12) holds for $k = 0$. Thus $u \in H^2_{\mathrm{loc}}(Q)$, since Q' is any subregion of Q.

Now let $f \in H^{m+1}_{\mathrm{loc}}(Q)$. Suppose that u has the following properties: $u \in H^{m+2}_{\mathrm{loc}}(Q)$ for any pair of subregion Q_1 and Q_2 of Q such that $Q_1 \Subset Q_2 \Subset Q$, u satisfies the inequality (12) for $k = m$:

$$\| u \|_{H^{m+2}(Q_1)} \leqslant C (m, Q_1, Q_2) (\| f \|_{H^m(Q_2)} + \| u \|_{H^1(Q_2)}), \qquad (12_m)$$

and for any α, $| \alpha | \leqslant m$, $i = 1, 2, \ldots, n$ and $0 < | h | < \delta/2$ the identity

$$\int_{Q''} \nabla \delta_h^i (D^\alpha U) \, \nabla v_{m+1} \, dx = - \int_{Q''_{2\delta/3}} D^\alpha F \delta^i_{-h} v_{m+1} \, dx + \int_{Q'} \delta_h^i \, (D^\alpha (u \, \nabla \zeta)) \, \nabla v_{m+1} \, dx, \qquad (16_m)$$

where v_{m+1} is any function in $H^1(Q'')$, holds. Note that the above properties have already been established for $m = 0$.

According to the above hypotheses, (14) and (15) imply that $D^\alpha U \in H^2(Q'')$ and $D^\alpha F \in H^1(Q'')$, therefore, by Theorem 3, Sec. 3.4, Chap. III, we may pass to the limit, as $h \to 0$, in (16_m). Thus for any α, $| \alpha | = m$, $i = 1, 2, \ldots, n$, we obtain the equality

$$\int_{Q''} \nabla D^\alpha U_{x_i} \nabla v_{m+1} \, dx = - \int_{Q''_{2\delta/3}} D^\alpha F (v_{m+1})_{x_i} \, dx + \int_{Q''} D^\alpha (u \, \nabla \zeta)_{x_i} \, \nabla v_{m+1} \, dx,$$

whence it follows that

$$\int\limits_{Q''} \nabla D^{\alpha}U_{x_i}\nabla v_{m+1}\, dx = \int\limits_{Q''} D^{\alpha}F_{x_i}v_{m+1}\, dx + \int\limits_{Q''} D^{\alpha}(u\nabla\zeta)_{x_i}\nabla v_{m+1}\, dx \quad (13_{m+1})$$

for all $v_{m+1} \in H^1(Q'')$ (the function $D^{\alpha}F$ vanishes outside $Q''_{2\delta/3}$). This equality coincides with (13_0) if we replace there $D^{\alpha}U_{x_i}$ by U, $D^{\alpha}F_{x_i}$ by F, $D^{\alpha}(u\nabla\zeta)_{x_i}$ by $u\nabla\zeta$ and v_{m+1} by v_0. Moreover, $D^{\alpha}U_{x_i} \in H^1(Q'')$, vanishes outside $Q''_{2\delta/3}$ and coincides with $D^{\alpha}u_{x_i}$ in Q''_{δ}, while $D^{\alpha}F_{x_i} \in L_2(Q'')$ and vanishes outside $Q''_{2\delta/3}$. Since the integration in (13_{m+1}) is, in fact, taken over $Q''_{2\delta/3}$, in the last equality we can put $v_{m+1}(x) = \delta^j_{-h}v_{m+2}(x)$, $j = 1, 2, \ldots, n$, $0 < |h| < \delta/2$, where v_{m+2} is an arbitrary function in $H^1(Q'')$. This results in

$$\int\limits_{Q''} \nabla\delta^j_h(D^{\alpha}U_{x_i})\,\nabla v_{m+2}\, dx$$
$$= -\int\limits_{Q''} D^{\alpha}F_{x_i}\delta^j_{-h}v_{m+2}\, dx + \int\limits_{Q''} \delta^j_h\,(D^{\alpha}(u\,\nabla\zeta)_{x_i})\,\nabla v_{m+2}\, dx. \quad (16_{m+1})$$

Using the inequality (12_m) (with $Q_1 = Q''_{2\delta/3}$, $Q_2 = Q''$), from (15) we obtain

$$\|F\|_{H^{m+1}(Q'')} \leqslant C_1(\|f\|_{H^{m+1}(Q'')} + \|u\|_{H^{m+2}(Q''_{2\delta/3})})$$
$$\leqslant C_2(\|f\|_{H^{m+1}(Q'')} + \|u\|_{H^1(Q'')}).$$

Setting in (16_{m+1}) $v_{m+2} = \delta^j_h\,(D^{\alpha}U_{x_i})$ and again using Theorem 3, Sec. 3.4, Chap. III, we find that $u \in H^{m+3}_{\text{loc}}(Q)$ and the inequality (12) holds for $u(x)$ for $k = m + 1$. ■

As a corollary to Lemma 2, we have the following result.

Corollary. *Suppose that* $f \in L_2(Q)$, *and the function* $u \in H^1(Q)$ *and satisfies the integral identity* (8) *for all* $v \in \overset{\circ}{H}{}^1(Q)$. *Then the function* $u(x)$ *satisfies Eq.* (7) *(a.e.) in* Q.

We must show that the sum of the generalized second derivatives $u_{x_1x_1} + \ldots + u_{x_nx_n}$ (the existence of these derivatives has just been established) equals f a.e. in Q. As $v(x)$ in (8) we take an arbitrary function belonging to $\overset{\circ}{H}{}^1(Q')$, $Q' \Subset Q$, extended outside Q' by assigning to it the value zero. Since $u \in H^2(Q')$, the Ostrogradskii's formula yields that $\int\limits_{Q'} (\Delta u - f)\, v\, dx = 0$, whence it follows that $\Delta u - f = 0$ (a.e.) in Q', and hence (a.e.) in Q.

The generalized solutions $u(x)$ of the first, second and third boundary-value problems for Eq. (7) satisfy the hypotheses of Lemma 2 as well as the inequality (10) or (11) (the solution $u(x)$ of the second boundary-value problem is assumed orthogonal in $L_2(Q)$ to constant functions), therefore from Lemma 2 and Theorem 2, Sec. 6.2, Chap. III, the following result follows.

Theorem 3. *If* $f \in L_2(Q) \cap H^k_{\text{loc}}(Q)$, $k \geqslant 0$, *then the generalized solutions* $u(x)$ *of the first, second and third boundary-value problems for Eq.* (7) *belong to* $H^{k+2}_{\text{loc}}(Q)$ *and satisfy Eq.* (7) (*a.e.*) *in* Q. *For any subregions* Q' *and* Q'' *of* Q, $Q' \Subset Q'' \Subset Q$, *there exists a positive constant* C *depending on* Q', Q'' *and* k *such that*

$$\|u\|_{H^{k+2}(Q')} \leqslant C\Big(\|f\|_{H^k(Q'')} + \|f\|_{L_2(Q)} + \inf_{\substack{\Phi \in H^1(Q) \\ \Phi|_{\partial Q} = \varphi}} \|\Phi\|_{H^1(Q)}\Big)$$

for the first boundary-value problem and

$$\|u\|_{H^{k+2}(Q')} \leqslant C\,(\|f\|_{H^k(Q'')} + \|f\|_{L_2(Q)} + \|\varphi\|_{L_2(\partial Q)})$$

for the second and third boundary-value problems (in the case of the second boundary-value problem it is assumed that $\int\limits_Q u\,dx = 0$).

If $k \geqslant \left[\frac{n}{2}\right] - 1$, *then* $u(x) \in C^{k+1-\left[\frac{n}{2}\right]}(Q)$. *In particular, if* $f \in L_2(Q) \cap C^\infty(Q)$, *then* $u(x) \in C^\infty(Q)$.

Theorem 3 implies that the smoothness of generalized solutions of boundary-value problems for Eq. (7) in the interior of Q does not depend on the type of boundary conditions nor on the smoothness of the boundary or on that of boundary functions. The interior smoothness depends only on the smoothness of the right-hand side $f(x)$ of the equation. The result obtained above is utmost precise: the smoothness of the solution is higher than that of the right-hand side by a number equal to the order of the equation.

Remark. For the one-dimensional case it was shown, in particular, that if the right-hand side of Eq. (7) is continuous, then the generalized solutions have continuous derivatives up to order 2. Analogous result does not hold for the multi-dimensional case. Later (Subsec. 3 of the next section) an example will be given of a function $f(x)$ continuous in $\overline{Q}$ such that the generalized solution of the first boundary-value problem for the Poisson equation (7) does not belong to $C^2(Q)$ (it belongs, of course, to $H^2_{\text{loc}}(Q)$).

3. Smoothness of Generalized Solutions of Boundary-Value Problems. In the preceding subsection the interior smoothness of generalized solutions was established, that is, we established that the solutions belong to the spaces $H^k_{\text{loc}}(Q)$ or $C^l(Q)$ for some k and l. Here we shall examine the smoothness of generalized solutions of boundary-value problems in the whole of Q, that is, the question of the solutions being contained in the spaces $H^k(Q)$ or $C^l(\bar{Q})$. Naturally, the smoothness of solutions "right up to the boundary" depends on the smoothness of the boundary as well as that of boundary functions.

It will be assumed that the boundary $\partial Q \in C^{k+2}$ for some $k \geqslant 0$.

We first examine the case of homogeneous boundary conditions. The generalized solutions of the first or second boundary-value problems* with homogeneous boundary conditions (the function φ, the right-hand side in the boundary conditions, is zero) for Eq. (7) are functions belonging to the spaces $\overset{\circ}{H}{}^1(Q)$ or $H^1(Q)$ and satisfying the integral identity (8) for all v belonging to $\overset{\circ}{H}{}^1(Q)$ or to $H^1(Q)$, respectively (in the case of the second boundary-value problem the functions f and u are assumed orthogonal to constant functions in $L_2(Q)$) with a scalar product.

Theorem 4. *If $f \in H^k(Q)$ and $\partial Q \in C^{k+2}$ for certain $k \geqslant 0$, then the generalized solutions $u(x)$ of the first and second boundary-value problems with homogeneous boundary conditions for the Poisson equation* (7) *belong to $H^{k+2}(Q)$ and satisfy (in the case of the second boundary-value problem it is assumed that $\int\limits_Q u\,dx = 0$) the inequality*

$$\| u \|_{H^{k+2}(Q)} \leqslant C \| f \|_{H^k(Q)}, \tag{17}$$

*where the constant $C > 0$ does not depend on f***.

Proof. Let x^0 be an arbitrary point on the boundary ∂Q. Let the system of coordinates be chosen in such a way that the point x^0 becomes the origin and the normal to the boundary at this point is directed along the Ox_n-axis. We take a small number $r = r(x^0)$ such that the portion of the boundary $\partial Q \cap (|x| < 4r)$ is a connected set which is uniquely projected along the Ox_n-axis onto a region D in the plane $x_n = 0$; the equation of the surface $\partial Q \cap (|x| < 4r)$ has the form

$$x_n = \psi(x'), \qquad x' = (x_1, \ldots, x_{n-1}) \in D,$$

where the function $\psi(x')$ belongs to $C^{k+2}(\bar{D})$ and satisfies $\psi(0) = 0$, $\psi_{x_1}(0) = \ldots = \psi_{x_{n-1}}(0) = 0$ as well as the inequalities

$$|\psi_{x_i}| \leqslant \frac{1}{2n}, \qquad i = 1, \ldots, n-1, \qquad x' \in \bar{D}. \tag{18}$$

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* For the sake of simplicity, we confine our discussion to the solutions of the first and second boundary-value problems. The investigation of smoothness of generalized solutions of the third boundary-value problem with certain conditions on the function $\sigma(x)$ (of (9)) can be carried out by the same method.

** For the function $u(x)$ which is a generalized solution of the third boundary-value problem for Eq. (7) with homogeneous boundary conditions the following result holds: *if $f \in H^k(Q)$, $\partial Q \in C^{k+2}$ and $\sigma(x) \in C^{k+1}(\partial Q)$ ($\sigma \geqslant 0$) for some $k \geqslant 0$, then $u(x) \in H^{k+2}(Q)$ and satisfies the inequality* (17).

Then the region $\Omega = Q \cap (|x| < 3r)$, and thus also its subregion $\Omega' = Q \cap (|x| < r)$ are projected along the Ox_n-axis into D.

Let Γ denote the common part of the boundaries of Q and Ω, $\Gamma = \partial Q \cap (|x| < 3r)$, and by Γ_0 the remaining part of the boundary Ω. The set of functions in $H^1(\Omega)$ whose traces on Γ are zero will be denoted by $\overset{\circ}{H}{}^1_\Gamma(\Omega)$.

Suppose that the function $\zeta(x) \in C^\infty(R_n)$, $\zeta(x) \equiv 1$ when $|x| < r$, $\zeta(x) \equiv 0$ when $|x| > 2r$. Then for any function $v_0(x)$ in $\overset{\circ}{H}{}^1_\Gamma(\Omega)(H^1(\Omega))$ the function $v(x)$, which is equal to $\zeta(x)\,v_0(x)$ in Ω and to zero at the remaining points of Q, belongs to $\overset{\circ}{H}{}^1(Q)$ $(H^1(Q))$. Substituting $v(x)$ into (8), we obtain, as in the preceding subsection,

$$\int_\Omega \nabla U\, \nabla v_0\, dx = \int_\Omega F v_0\, dx + \int_\Omega u \nabla\zeta\, \nabla v_0\, dx, \tag{19}$$

where the function v_0 is an arbitrary function in $\overset{\circ}{H}{}^1_\Gamma(\Omega)$ in the case of the first boundary-value problem, while for the second boundary-value problem v_0 is an arbitrary function in $H^1(\Omega)$, and the functions $F(x)$ and $U(x)$ are determined by (14) and (15) with $\zeta(x)$ just introduced. Evidently, $U(x) \in H^1(\Omega)$ and $F(x) \in L_2(\Omega)$.

The transformation

$$y_i = x_i, \quad i = 1, \ldots, n-1, \quad y_n = x_n - \psi(x') \tag{20}$$

maps the regions Ω and Ω' onto certain regions ω and ω' with the identity Jacobian.

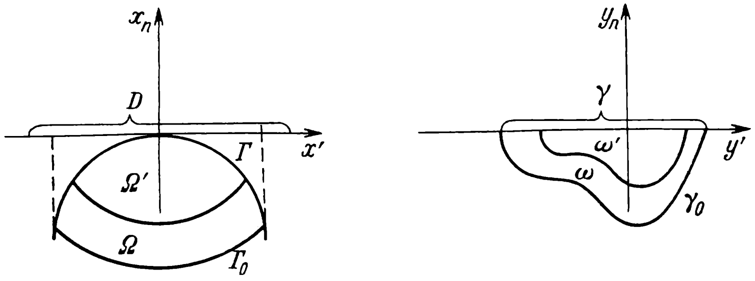

Fig. 1

The images of surfaces Γ and Γ_0 are denoted by γ and γ_0, respectively. The functions $U(x)$, $u(x)$, ... defined in Ω are mapped by transformation (20) into functions $U(y)$, $u(y)$, ... defined in ω (the same notation is retained). Moreover, $U(y) = u(y)$ in ω' and for some $\delta > 0$ the functions $U(y)$ and $F(y)$ vanish at points of the set $\omega \setminus \tilde{\omega}_\delta$, where $\tilde{\omega}_\delta$ is a subregion of ω which consists of those points whose distance from γ_0 is greater than δ.

Since for $x \in \Omega(y \in \omega)$ $U_{x_i} = U_{y_i} - U_{y_n}\psi_{x_i}$ for $i = 1, \ldots, n - 1$ and $U_{x_n} = U_{y_n}$, we see that

$$\nabla_x U \nabla_x v_0 = \nabla_y U \nabla_y v_0 - \sum_{i=1}^{n-1} (U_{y_i} v_{0y_n} + U_{y_n} v_{0y_i}) \psi_{x_i} + U_{y_n} v_{0y_n} \sum_{i=1}^{n-1} \psi_{x_i}^2,$$

therefore in the new variables identity (19) assumes the form

$$\int_\omega \nabla_y U \nabla_y v_0 \, dy = \int_\omega F v_0 \, dy + \int_\omega \sum_{i,\, j=1}^{n} A_{ij} U_{y_i} v_{0y_j} \, dy + \int_\omega \sum_{i,\, j=1}^{n} B_{ij} \zeta_{y_j} u v_{0y_i} \, dy, \tag{21_0}$$

where $A_{in} = A_{ni} = \psi_{x_i}$ for $i = 1, \ldots, n-1$, $A_{nn} = -\sum_{i=1}^{n-1} \psi_{x_i}^2$, $A_{ij} = 0$ for remaining i and j; $B_{ij} = \delta_{ij} - A_{ij}$, $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$, $i = 1, \ldots, n$. Evidently, $A_{ij} \in C^{k+1}(\overline{\omega})$, $B_{ij} \in C^{k+1}(\overline{\omega})$ for all i and j; furthermore, by (18),

$$|A_{ij}(y)| \leqslant \frac{1}{2n}, \quad i, j = 1, 2, \ldots, n, \quad y \in \overline{\omega}. \tag{22}$$

Since the functions $U(y)$ and $F(y)$ vanish in $\omega \diagdown \widetilde{\omega}_{\delta/2}$, equality (21_0) holds not only for any $v_0(y)$ in $\mathring{H}^1_\gamma(\omega)$ or in $H^1(\omega)$ but also for any $v_0(y)$ in $\mathring{H}^1_\gamma(\widetilde{\omega}_{\delta/2})$ (arbitrarily extended outside $\widetilde{\omega}_{\delta/2}$ as a function belonging to $L_2(\omega)$) or in $H^1(\widetilde{\omega}_{\delta/2})$ (arbitrarily extended outside $\widetilde{\omega}_{\delta/2}$ as a function belonging to $L_2(\omega)$), respectively.

Consider any function $v_1(y)$ belonging to $\mathring{H}^1_\gamma(\omega)$ $(H^1(\omega))$ and extended outside ω by assigning to it the value zero, and in (21_0) set $v_0 = \delta^l_{-h} v_1$ for a certain $l < n$ and $0 < |h| < \delta/2$ (the function $v_0(y)$ belongs, obviously, to $\mathring{H}^1_\gamma(\widetilde{\omega}_{\delta/2}) \cap L_2(\omega)$ and to $H^1(\widetilde{\omega}_{\delta/2}) \cap L_2(\omega)$, respectively). The identity (21_0) then becomes

$$\int_\omega \nabla_y (\delta^l_h U) \nabla_y v_1 \, dy = -\int_\omega F \delta^l_{-h} v_1 \, dy + \int_\omega \sum_{i,\, j=1}^{n} \delta^l_h (A_{ij} U_{y_i}) \, v_{1y_j} \, dy + \int_\omega \sum_{i,\, j=1}^{n} \delta^l_h (B_{ij} \zeta_{y_j} u) \, v_{1y_i} \, dy \tag{23_0}$$

(note that the integration here is taken not over whole ω but only over its subregion $\widetilde{\omega}_{\delta/2}$; accordingly, all the expressions under the integral sign in (23_0) are defined).

From Theorem 4, Sec. 3.4, Chap. III, it follows that

$$\left|\int\limits_{\omega} F\delta^{l}_{-h}v_1\,dy\right| \leqslant \|F\|_{L_2(\omega)}\,\|v_{1y_l}\|_{L_2(\omega)} \leqslant \|F\|_{L_2(\omega)}\,\|\,|\nabla v_1|\,\|_{L_2(\omega)}. \qquad (24_0)$$

In order to estimate the second integral in the right-hand side of (23_0), we break it into two terms

$$\int\limits_{\omega}\sum_{i,\,j=1}^{n}\delta^{l}_{h}(A_{ij}U_{y_i})\,v_{1y_j}\,dy = \int\limits_{\omega}\sum_{i,\,j=1}^{n}(A_{ij})^{l}_{h}\,\delta^{l}_{h}U_{y_i}v_{1y_j}\,dy + \int\limits_{\omega}\sum_{i,\,j=1}^{n}\delta^{l}_{h}(A_{ij})\,U_{y_i}v_{1y_j}\,dy. \qquad (25_0)$$

Here we have used the identity

$$\delta^{l}_{h}(fg) = g^{l}_{h}\delta^{l}_{h}f + f\delta^{l}_{h}g,$$

where $g^{l}_{h}(y) = g(y_1, \ldots, y_{l-1}, y_l + h, y_{l+1}, \ldots, y_n)$, true for arbitrary functions f and g.

Using (22), the first term is estimated as follows:

$$\left|\int\limits_{\omega}\sum_{i,\,j=1}^{n}(A_{ij})^{l}_{h}\,\delta^{l}_{h}U_{y_i}v_{1y_j}\,dy\right| \leqslant \frac{1}{2n}\int\limits_{\omega}\Big(\sum_{i=1}^{n}|\delta^{l}_{h}U_{y_i}|\Big) \times \Big(\sum_{j=1}^{n}|v_{1y_j}|\Big)\,dy \leqslant \frac{1}{2n}\int\limits_{\omega}\sqrt{n}\Big(\sum_{i=1}^{n}(\delta^{l}_{h}U_{y_i})^2\Big)^{1/2} \times \sqrt{n}\Big(\sum_{j=1}^{n}v^{2}_{1y_j}\Big)^{1/2}dy \leqslant \frac{1}{2}\|\,|\nabla\delta^{l}_{h}U|\,\|_{L_2(\omega)}\,\|\,|\nabla v_1|\,\|_{L_2(\omega)}. \qquad (26_0)$$

The second term in (25_0) is estimated together with the third integral in the right-hand side of (23_0). Since the functions A_{ij} and B_{ij} are continuously differentiable in $\overline{\omega}$,

$$\left|\int\limits_{\omega}\sum_{i,\,j=1}^{n}(\delta^{l}_{h}A_{ij})\,U_{y_i}v_{1y_j}\,dy + \int\limits_{\omega}\sum_{i,\,j=1}^{n}\delta^{l}_{h}(B_{ij}\zeta_{y_j}u)\,v_{1y_j}\,dy\right| \leqslant C_1\|u\|_{H^1(\Omega)}\,\|\,|\nabla v_1|\,\|_{L_2(\omega)} \leqslant C_1\|u\|_{H^1(Q)}\,\|\,|\nabla v_1|\,\|_{L_2(\omega)}, \qquad (27_0)$$

where the constant C_1 does not depend on u or v_1.

Taking into account (24_0)-(27_0), from (23_0) we have

$$\left|\int\limits_{\omega}\nabla(\delta^{l}_{h}U)\,\nabla v_1\,dy\right| \leqslant \Big(\|F\|_{L_2(\omega)} + \frac{1}{2}\|\,|\nabla\delta^{l}_{h}U|\,\|_{L_2(\omega)} + C_1\|u\|_{H^1(Q)}\Big)\,\|\,|\nabla v_1|\,\|_{L_2(\omega)}. \qquad (28_0)$$

Putting in this inequality $v_1 = \delta_h^l U$ and using the inequality

$$\| F(y) \|_{L_2(\omega)} = \| F(x) \|_{L_2(\Omega)} \leqslant C_2(\| f \|_{L_2(Q)} + \| u \|_{H^1(Q)}), \qquad (29_0)$$

(the constant C_2 depends only on the function ζ, that is, only on the region Q) which follows from (15), and the inequality (10) or (11), where $\varphi = 0$ (then in (10) $\inf \| \Phi \|_{H^1(Q)} = 0$), we obtain the estimate

$$\| \, | \nabla \delta_h^l U | \, \|_{L_2(\omega)} \leqslant C \| f \|_{L_2(Q)}, \qquad l = 1, \ldots, n-1,$$

which in turn implies that (Theorem 4, Sec. 3.4, Chap. III) all the second-order generalized derivatives of U except $U_{y_n y_n}$ belong to $L_2(\omega)$ and the inequality $\| U_{y_i y_j} \|_{L_2(\omega)} \leqslant C \, \| f \|_{L_2(Q)}$ holds for them. This means that for the corresponding derivatives of the function $u(y)$ the inequalities

$$\| u_{y_i y_j} \|_{L_2(\omega')} \leqslant C \| f \|_{L_2(Q)}$$

hold.

To estimate $u_{y_n y_n}$ in ω', we may use Corollary to Lemma 2, according to which $\Delta_x u = f$ a.e. in Q, and thus a.e. in Ω'. In the new variables this equation has the form

$$\Delta_y u(y) - 2 \sum_{i=1}^{n-1} u_{y_i y_n} \psi_{x_i} + u_{y_n y_n} \sum_{i=1}^{n-1} \psi_{x_i}^2 - u_{y_n} \sum_{i=1}^{n-1} \psi_{x_i x_i} = f(y),$$

whence for all $y \in \omega'$

$$\Bigl(1 + \sum_{i=1}^{n-1} \psi_{x_i}^2\Bigr) u_{y_n y_n} = f(y) + 2 \sum_{i=1}^{n-1} u_{y_i y_n} \psi_{x_i} - \sum_{i=1}^{n-1} u_{y_i y_i} + u_{y_n} \sum_{i=1}^{n-1} \psi_{x_i x_i}. \qquad (30)$$

Since $\psi \in C^{k+2}(\bar{D})$ for $k \geqslant 0$, it follows that $u_{y_n y_n} \in L_2(\omega')$ and

$$\| u_{y_n y_n} \|_{L_2(\omega')} \leqslant \text{const} \| f \|_{L_2(Q)}.$$

Thus it is established that for any point $x^0 \in \partial Q$ we can find positive numbers $r = r(x^0)$ and $C = C(x^0)$ such that $u(x) \in H^2 (Q \cap \cap (| x - x^0 | < r(x^0)))$ and

$$\| u \|_{H^2(Q \cap (|x - x^0| < r(x^0)))} \leqslant C(x^0) \| f \|_{L_2(Q)}.$$

From the cover of the boundary ∂Q by the sets $\partial Q \cap (| x - x^0 | < < r(x^0))$ for all possible $x^0 \in \partial Q$, we choose a finite subcover $\partial Q \cap \cap (| x - x^i | < r(x^i))$, $i = 1, 2, \ldots, N$. Then there is a number $\delta_0 > 0$ such that $Q \setminus Q_{\delta_0} \subset \bigcup_{i=1}^{N} Q \cap (| x - x^i | < r(x^i))$.

Therefore $u(x) \in H^2(Q \setminus Q_{\delta_0})$ and $\| u \|_{H^2(Q \setminus Q_{\delta_0})} \leqslant C_1 \| f \|_{L_2(Q)}$, where C_1 is a positive constant. But by Theorem 3, $u(x) \in H^2(Q_{\delta_0/2})$ and $\|u \|_{H^2(Q_{\delta_0/2})} \leqslant C_2 \| f \|_{L_2(Q)}$. Therefore $u \in H^2(Q)$ and $\| u \|_{H^2(Q)} \leqslant$ $\leqslant C \| f \|_{L_2(Q)}$, where the constant $C > 0$ does not depend on f. This proves Theorem 4 for $k = 0$.

Now let k be any natural number. By the theorem on interior smoothness of generalized solutions (Theorem 3), it is enough to establish, as in the case when $k = 0$, that for any boundary point x^0 there are numbers $r = r(x^0) > 0$ and $C = C(x^0) > 0$ such that $u(x) \in$ $\in H^{k+2}(Q \cap (|x - x^0| < r))$ and

$$\| u \|_{H^{k+2}(Q \cap (|x-x^0| < r))} \leqslant C \| f \|_{H^k(Q)}$$

(the point x^0 may be taken as the origin and the axis Ox_n directed along the normal to ∂Q at this point). In view of smoothness of the transformation (20) (smoothness of the boundary), for this it suffices to show that $u(y) \in H^{k+2}(\omega')$ and $\|u\|_{H^{k+2}(\omega')} \leqslant C \| f \|_{H^k(Q)}$.

We have already established that $u(y) \in H^2(\omega')$, $\| u \|_{H^2(\omega')} \leqslant$ $\leqslant C \| f \|_{L_2(Q)}$, and, furthermore, the relation (23_0) holds. Using the device adopted in the proof of Lemma 2 of the preceding subsection, we shall show that for any $m = 1, 2, \ldots, k$ $u(y) \in H^{m+2}(\omega')$ $\| u \|_{H^{m+2}(\omega')} \leqslant C \| f \|_{H^m(Q)}$ and the equality

$$\int\limits_{\omega} \nabla (\delta_h^l D^\alpha U) \nabla v_{m+1}\, dy$$

$$= - \int\limits_{\omega} D^\alpha F \delta_{-h}^l v_{m+1}\, dy + \int\limits_{\omega} \sum_{i,\, j=1}^{n} \delta_h^l (D^\alpha A_{ij} U_{y_i}) (v_{m+1})_{y_j}\, dy$$

$$+ \int\limits_{\omega} \sum_{i,\, j=1}^{n} \delta_h^l (D^\alpha B_{ij} \zeta_{y_j} u) (v_{m+1})_{y_i}\, dy \qquad (23_m)$$

holds for all $\alpha = (\alpha_1, \ldots, \alpha_{n-1}, 0)$, $| \alpha | \leqslant m$, $l = 1, \ldots, n-1$, $0 < | h | < \delta/2$, $v_{m+1} \in \overset{\circ}{H}{}_\gamma^1(\omega)$ in the case of the first boundary-value problem and $v_{m+1} \in H^1(\omega)$ in the case of the second boundary-value problem. We shall prove this for $m = 1$.

In (23_0) we pass on to the limit as $h \to 0$, and integrate by parts the first term of the right-hand side of the resulting equality. This yields

$$\int\limits_{\omega} \nabla U_{y_l} \nabla v_1\, dy = \int\limits_{\omega} F_{y_l} v_1\, dy + \int\limits_{\omega} \sum_{i,\, j=1}^{n} (A_{ij} U_{y_i})_{y_l} v_{1 y_j}\, dy$$

$$+ \int\limits_{\omega} \sum_{i,\, j=1}^{n} (B_{ij} \zeta_{y_j} u)_{y_l} v_{1 y_i}\, dy, \qquad (21_1)$$

which is true for all v_1 in $\overset{\circ}{H}{}^1_\gamma(\omega)$ in the case of the first boundary-value problem and in $H^1(\omega)$ for the second boundary-value problem.

The identity (21_1) for U_{y_l} differs from the identity (21_0) for U in merely that the functions F, $A_{ij}U_{y_i}$, $B_{ij}\zeta_{y_j}u$ have been replaced by F_{y_l}, $(A_{ij}U_{y_i})_{y_l}$, $(B_{ij}\zeta_{y_i}u)_{y_l}$, respectively, and the function v_0 by the function v_1 with the same properties.

Setting in (21_1) $v_1(y) = \delta^s_{-h}v_2(y)$, $s < n$, $0 < |h| < \delta/2$, where $v_2(y)$ is an arbitrary function in $\overset{\circ}{H}{}^1_\gamma(\omega)$ $(H^1(\omega))$ extended as being equal to zero outside ω, we obtain the identity analogous to (23_0)

$$\int_\omega \nabla(\delta^s_h U_{y_l})\,\nabla v_2\,dy = -\int_\omega F_{y_l}\delta^s_{-h}v_2\,dy$$
$$+\int_\omega \sum_{i,j=1}^{n} \delta^s_h((A_{ij}U_{y_i})_{y_l})\,(v_2)_{y_j}\,dy + \int_\omega \sum_{i,j=1}^{n} \delta^s_h((B_{ij}\zeta_{y_j}u)_{y_l})\,(v_2)_{y_i}\,dy. \tag{23_1}$$

As in the previous case, we estimate the integrals in the right-hand side of (23_1). Just as in (24_0), we have (since $u \in H^2(Q)$ and $f \in H^k(Q)$, $k \geqslant 1$, by (15) $F \in H^1(Q)$, and because $\psi \in C^{k+2}(\bar{D})$, $F(y) \in H^1(\omega)$)

$$\left|\int_\omega F_{y_l}\delta^l_{-h}v_2\,dy\right| \leqslant \|F_{y_l}\|_{L_2(\omega)}\,\|\,|\nabla v_2|\,\|_{L_2(\omega)}. \tag{24_1}$$

Analogously to (25_0), the second integral in the right-hand side of (23_1) is divided into two parts:

$$\int_\omega \sum_{i,j=1}^{n} \delta^s_h((A_{ij}U_{y_i})_{y_l})\,(v_2)_{y_j}\,dy$$
$$= \int_\omega \sum_{i,j=1}^{n} (A_{ij})^s_h\,\delta^s_h U_{y_iy_l}\,(v_2)_{y_j}\,dy$$
$$+ \int_\omega \sum_{i,j=1}^{n} [\delta^s_h(A_{ij})\,U_{y_iy_l} + \delta^s_h(A_{ijy_l}U_{y_i})]\,(v_2)_{y_j}\,dy. \tag{25_1}$$

Using (22), we estimate the first term in (25_1):

$$\left|\int_\omega \sum_{i,j=1}^{n} (A_{ij})^s_h\,\delta^s_h U_{y_iy_l}\,(v_2)_{y_j}\,dy\right|$$
$$\leqslant \frac{1}{2}\,\|\,|\nabla\delta^s_h U_{y_l}|\,\|_{L_2(\omega)}\,\|\,|\nabla v_2|\,\|_{L_2(\omega)}. \tag{26_1}$$

Since the functions A_{ij} and B_{ij} belong to $C^{k+1}(\bar{\omega})$, $k \geqslant 1$, the sum of the second integral in (25_1) and the third integral in the right-hand side of (23_1) is estimated in the following manner:

$$\Big| \int_{\omega} \Big[\sum_{i,j=1}^{n} \delta_h^s (A_{ij}) U_{y_i y_l} + \delta_h^s (A_{ij y_l} U_{y_i}) \Big] v_{2y_j}\, dy$$

$$+ \int_{\omega} \sum_{i,j=1}^{n} \delta_h^s ((B_{ij}\zeta_{y_j} u)_{y_l})\, v_{2y_j}\, dy \Big| \leqslant \text{const}\, \| u \|_{H^2(Q)}\, \| |\nabla v_2| \|_{L_2(\omega)}. \quad (27_1)$$

In view of (24_1)-(27_1), (23_1) yields the inequality

$$\Big| \int_{\omega} \nabla \delta_h^s U_{y_l} \nabla v_2\, dy \Big|$$

$$\leqslant \Big(\| F_{y_l} \|_{L_2(\omega)} + \frac{1}{2} \| |\nabla \delta_h^s U_{y_l}| \|_{L_2(\omega)} + \text{const}\, \| u_2 \|_{H^2(Q)} \Big) \| |\nabla v_2| \|_{L_2(\omega)},$$

$$l = 1, \ldots, n-1, \qquad s = 1, \ldots, n-1.$$

Setting in this inequality $v_2(y) = \delta_h^s U_{y_l}(y)$, and using the estimate

$$\| F \|_{H^1(\omega)} \leqslant C\, (\| f \|_{H^1(Q)} + \| u \|_{H^2(Q)}),$$

which follows from (15), and the estimate (17) already established for $k = 0$, we obtain

$$\| |\nabla \delta_h^s U_{y_l}| \|_{L_2(\omega)} \leqslant \text{const}\, \| f \|_{H^1(Q)},$$

$$s,\ l = 1, \ldots, n-1, \quad 0 < |h| < \delta/2.$$

This means that in ω' there exist generalized derivatives $u_{y_p y_s y_l}$, $p = 1, 2, \ldots, n$, $s, l = 1, \ldots, n-1$ which belong to $L_2(\omega')$ and satisfy the inequality $\| u_{y_p y_s y_l} \| \leqslant \text{const}\ \| f \|_{H^1(Q)}$.

To estimate the remaining third order derivatives $u_{y_p y_n y_n}$, $p = 1, 2, \ldots, n$, we use the identity (30). Differentiating it with respect to y_p for $p < n$, we find that for all such p $u_{y_p y_n y_n} \in L_2(\omega')$ and that $\| u_{y_p y_n y_n} \|_{L_2(\omega')} \leqslant C \| f \|_{H^1(Q)}$. Further, differentiating (30) with respect to y_n, we find that $u_{y_n y_n y_n} \in L_2(\omega')$ and $\| u_{y_n y_n y_n} \|_{L_2(\omega')} \leqslant C \| f \|_{H^1(Q)}$.

Thus we have shown that $u(y) \in H^3(\omega')$, $\| u \|_{H^3(\omega')} \leqslant C \| f \|_{H^1(Q)}$ and the identity (23_1) holds for all l, $s = 1, \ldots, n-1$, $0 < |h| < \delta/2$ and $v_2 \in \overset{\circ}{H}{}^1_\gamma(\omega)$ $(v_2 \in H^1(\omega))$. Repeating this process m times, $m \leqslant k$, we find that $u \in H^{m+2}(\omega')$, $\| u \|_{H^{m+2}(\omega')} \leqslant C \| f \|_{H^m(Q)}$ and identity (23_m) holds. ∎

We shall now see in what sense the generalized solutions under consideration satisfy the boundary conditions. For the case of the first boundary-value problem, it immediately follows from the definition ($u \in \overset{\circ}{H}{}^1(Q)$) that the solution has a trace on ∂Q that is equal to zero: $u|_{\partial Q} = 0$.

We shall now demonstrate that in the case of the second boundary-value problem the solution satisfies the boundary condition in the following sense: $\nabla u|_{\partial Q}\cdot n = 0$, where n is outward normal vector to ∂Q, $|n| = 1$, and $\nabla u|_{\partial Q}$ is a vector whose components $u_{x_i}|_{\partial Q}$, $i = 1, \ldots, n$, are the traces on ∂Q of functions u_{x_i} belonging to $H^1(Q)$.

In fact, since $u \in H^2(Q)$, an application of Ostrogradskii's formula to (8) yields the identity

$$\int_{\partial Q} (\nabla u\cdot n)\, v\, dS = \int_{Q} (\Delta u - f)\, v\, dx,$$

valid for any $v \in H^1(Q)$ (here $\nabla u\cdot n = \nabla u|_{\partial Q}\cdot n$). Since $\Delta u = f$ a.e. in Q, we have

$$\int_{\partial Q} (\nabla u\cdot n)\, v\, dS = 0,$$

implying the desired identity, because, by Theorem 2, Sec. 4.2, Chap. III, the set of traces $v|_{\partial Q}$ of functions in $H^1(Q)$ is everywhere dense in $L_2(\partial Q)$.

In the sequel, the expression $\nabla u|_{\partial Q}\cdot n$ will be denoted by $\left.\frac{\partial u}{\partial n}\right|_{\partial Q}$. Note that if $u \in C^1(\bar{Q}) \cap H^2(Q)$, then the function $\left.\frac{\partial u}{\partial n}\right|_{\partial Q}$ as an element of $L_2(\partial Q)$ coincides with the normal derivative $\frac{\partial u}{\partial n}$ of u on the boundary ∂Q. The notation $\left.\frac{\partial u}{\partial n}\right|_{\partial Q}$ is natural also in the sense that there is a function in $H^1(Q)$ whose trace on ∂Q coincides with $\left.\frac{\partial u}{\partial n}\right|_{\partial Q}$.*

Thus, *if $\partial Q \in C^2$, the generalized solution of the second*** *boundary-value problem satisfies the boundary condition*

$$\left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 0.$$

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* It suffices to construct such a function in $Q\setminus Q_\delta$ for some $\delta > 0$. Since $\partial Q \in C^2$, for any point $x \in Q\setminus Q_\delta$ with sufficiently small $\delta > 0$ there is a unique point $y = y(x) \in \partial Q$, $|y - x| < \delta$, such that the vector $y - x$ is directed along the normal $n(y)$ to the boundary ∂Q at the point y. The function $\nabla u(x) \times \times n(y(x))$ belongs to $H^1(Q\setminus Q_\delta)$ and its trace on ∂Q $\nabla u(x)\cdot n(y(x))|_{\partial Q} = = \nabla u|_{\partial Q}\, n(x) = \left.\frac{\partial u}{\partial n}\right|_{\partial Q}$.

** For the case of the third boundary-value problem the generalized solution satisfies the boundary condition $\left.\left(\frac{\partial u}{\partial n} + \sigma u\right)\right|_{\partial Q} = 0$.

From Theorems 4 and 3 and Theorems 2 and 3 of Sec. 6.2, Chap. III, the following result, in particular, follows.

Theorem 5. *Let* $f \in H^{[\frac{n}{2}]+1}(Q)$. *If* $\partial Q \in C^{[\frac{n}{2}]+1}$, *then the generalized solution of the first boundary-value problem for Eq.* (7) *with homogeneous boundary condition is a classical solution of this problem. When* $\partial Q \in C^{[\frac{n}{2}]+2}$, *the generalized solution of the second boundary-value problem for Eq.* (7) *with homogeneous boundary condition is a classical solution of this problem.*

We now examine the question of smoothness of the generalized solutions in the whole region when the boundary conditions are nonhomogeneous. We confine our discussion to the first boundary-value problem.

Suppose that $u(x)$ is a generalized solution of the first boundary-value problem, that is, it belongs to $H^1(Q)$ and for all $v \in \overset{\circ}{H}{}^1(Q)$ satisfies the integral identity (8) as well as the boundary condition $u|_{\partial Q} = \varphi$.

Assume that for a certain $k \geqslant 0$ $f \in H^k(Q)$, $\partial Q \in C^{k+2}$, and that the boundary function φ is the trace on ∂Q of some function Φ in $H^{k+2}(Q)$ (in order that φ be the trace on ∂Q of a function in $H^{k+2}(Q)$ it is sufficient, by Theorem 2, Sec. 4.2., Chap. III, that it belong to $C^{k+2}(\partial Q)$). Let us show that then $u \in H^{k+2}(Q)$.

Consider the function $z = u - \Phi$. It is clear that $z \in \overset{\circ}{H}{}^1(Q)$ and satisfies for all $v \in \overset{\circ}{H}{}^1(Q)$ the integral identity

$$\int_Q \nabla z \nabla v \, dx = -\int_Q \nabla \Phi \nabla v \, dx - \int_Q f v \, dx$$

or, by Ostrogradskii's formula, equivalently the integral identity

$$\int_Q \nabla z \nabla v \, dx = -\int_Q f_1 v \, dx,$$

where $f_1 = f - \Delta\Phi$. Since the function $f_1 \in H^k(Q)$, $z \in H^{k+2}(Q)$ by Theorem 4. Hence the generalized solution $u = z + \Phi \in H^{k+2}(Q)$, establishing the assertion.

4. **Smoothness of Generalized Eigenfunctions.** Let $u(x)$ be generalized eigenfunction of the first, second or third boundary-value problem for the Laplace operator and λ the corresponding eigenvalue. Then for any $v \in \overset{\circ}{H}{}^1(Q)$

$$\int_Q \nabla u \nabla v \, dx = -\lambda \int_Q u v \, dx,$$

which coincides with the identity (8) for $f = \lambda u$. Since $\lambda u \in H^1(Q)$ and hence $\lambda u \in L_2(Q)$, Theorem 3 implies that $u \in H^2_{\rm loc}(Q)$ and

$$\Delta u = \lambda u \tag{31}$$

a.e. in Q. Thus the function λu appearing on the right-hand side of (28) belongs to $H^2_{\rm loc}(Q) \cap L_2(Q)$. Therefore another application of Theorem 3 shows that $u \in H^4_{\rm loc}(Q)$, and so forth.

Consequently, $u \in H^k_{\rm loc}(Q)$ for any k. Theorem 2, Sec. 6.2, Chap. III, $u(x) \in C^\infty(Q)$.

Thus the following result is established.

Theorem 6. *The generalized eigenfunctions of the first, second and third boundary-value problems for the Laplace operator are infinitely differentiable in Q and satisfy Eq.* (31).

The smoothness of generalized eigenfunctions in the whole region is determined by the smoothness of the boundary.

Theorem 7. *If $\partial Q \in C^k$ for some $k \geqslant 2$, then any eigenfunction $u(x)$ of the first or second boundary-value problem for the Laplace operator belongs to $H^k(Q)$ and satisfies the corresponding boundary condition* $\left(u|_{\partial Q} = 0\right.$ *and* $\left.\frac{\partial u}{\partial n}\Big|_{\partial Q} = 0\right.$ *for the first and second boundary-value problems, respectively*$\Big)$. *Then the generalized eigenfunctions of the first boundary-value problem when $k \geqslant \left[\frac{n}{2}\right] + 1$ and of the second boundary-value problem when $k \geqslant \left[\frac{n}{2}\right] + 2$ are classical eigenfunctions* *.

Proof. Since $\partial Q \in C^2$ and $u \in H^1(Q) \subset L_2(Q)$, $u \in H^2(Q)$ by Theorem 4. If $\partial Q \in C^3$, then, by Theorem 4, $u \in H^3(Q)$. When $\partial Q \in C^4$, the inclusion $u \in H^2(Q)$ implies that $u \in H^4(Q)$, and so forth. Thus we find that $u \in H^k(Q)$, provided $\partial Q \in C^k$.

What is more, by Theorem 6, the generalized eigenfunctions belong to $C^\infty(Q)$ and satisfy Eq. (31).

If $k \geqslant \left[\frac{n}{2}\right] + 1$, then, by Theorem 3, Sec. 6.2, Chap. III, $u \in C^{k-\left[\frac{n}{2}\right]-1}(\overline{Q}) \subset C(\overline{Q})$, while $u \in C^1(\overline{Q})$ if $k \geqslant \left[\frac{n}{2}\right] + 2$. Consequently, the results of Sec. 5.1, Chap. III, imply that the eigenfunction u of the first boundary-value problem satisfies for $k \geqslant \left[\frac{n}{2}\right] + 1$ the boundary condition $u|_{\partial Q} = 0$ in the classical sense, and the eigenfunction u of the second boundary-value problem satisfies for $k \geqslant \left[\frac{n}{2}\right]+2$ the boundary condition $\left.\frac{\partial u}{\partial n}\right|_{\partial Q}=0$ in the classical sense. ∎

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* For the third boundary-value problem the following assertion holds: *If $\partial Q \in C^k$ and $\sigma(x) \in C^{k-1}(\partial Q)$ for some $k \geqslant 2$, then any eigenfunction $u(x)$ of the third boundary-value problem for the Laplace operator belongs to $H^k(Q)$. Moreover,* $\left(\frac{\partial u}{\partial n} + \sigma u\right)\Big|_{\partial Q} = 0$, *and if $k \geqslant \left[\frac{n}{2}\right] + 2$, then the generalized eigenfunctions of the third boundary-value problem are classical eigenfunctions.*

5. **On Series Expansions in Eigenfunctions.** Let $u_1, u_2, \ldots$ denote the system of all the generalized eigenfunctions of the first (second) boundary-value problem for the Laplace operator, and $\lambda_1, \lambda_2, \ldots$ the corresponding system of eigenvalues. As shown above (Theorem 3, Sec. 1.3), the system u_s, $s = 1, 2, \ldots$, is an orthonormal basis for $L_2(Q)$. This means that any function $f \in L_2(Q)$ can be represented by a convergent Fourier series in $L_2(Q)$ with respect to any of these systems:

$$f=\sum_{m=1}^{\infty} f_m u_m, \quad f_m=(f, u_m)_{L_2(Q)}. \tag{32}$$

Suppose that the function $f \in H^k(Q)$ for some $k \geqslant 1$. Its Fourier series in eigenfunctions of the first and second boundary-value problems, of course, converge to it in $L_2(Q)$. However, in the norm of $H^k(Q)$ or even in the norms of $H^{k'}(Q)$, $0 < k' < k$, these series, generally speaking, do not converge. For instance, the Fourier series of the function $f_0(x) = 1$ in Q with respect to the system of eigenfunctions of the first boundary-value problem cannot converge in the norm of $H^k(Q)$ for any $k \geqslant 1$. Indeed, if this series converged in the norm of $H^1(Q)$, it would converge only to $f_0(x)$, but this is impossible, because the sum of any series of elements of $\mathring{H}^1(Q)$ which converges in $H^1(Q)$ must belong to $\mathring{H}^1(Q)$.

In order that the Fourier series of any function f in $H^k(Q)$ may converge to it in $H^n(Q)$, the function f must be subjected to some boundary conditions.

Note that for the Fourier series (32) of a function f with respect to the system of eigenfunctions of the first boundary-value problem for the Laplace operator to converge in the norm of $H^1(Q)$ it is sufficient (Theorem 3, Sec. 1.3) and as shown just now, necessary that $f \in \mathring{H}^1(Q)$.

We denote by $H^k_{\mathscr{D}}(Q)$, $k \geqslant 1$, the subspace of $H^k(Q)$ consisting of all the functions f for whi

$$f\,|_{\partial Q}=0, \ldots, \Delta^{\left[\frac{k-1}{2}\right]} f\,|_{\partial Q}=0.$$

By $H^0_{\mathscr{D}}(Q)$ we shall mean the space $L_2(Q)$. Note that by Theorem 2, Sec. 5.3, Chap. III, $H^1_{\mathscr{D}}(Q) = \mathring{H}^1(Q)$.

By $H^k_{\mathscr{N}}(Q)$, $k \geqslant 2$, we designate the subspace of $H^k(Q)$ consisting of all the functions f for which

$$\frac{\partial f}{\partial n}\Big|_{\partial Q}=0, \ \ldots, \ \frac{\partial}{\partial n}\Delta^{\left[\frac{k}{2}\right]-1} f\,|_{\partial Q}=0.$$

By $H^0_{\mathscr{N}}(Q)$ we shall mean the space $L_2(Q)$ and by $H^1_{\mathscr{N}}(Q)$ the space $H^1(Q)$.

Lemma 3. *Suppose that $\partial Q \in C^k$ for a certain $k \geqslant 1$. Then there is a constant $C > 0$ such that any function $f \in H^k_{\mathscr{D}}(Q)$ or $f \in H^k_{\mathscr{N}}(Q)$ orthogonal to constant functions in $L_2(Q)$ with a scalar product satisfies the inequality*

$$\| f \|_{H^k(Q)} \leqslant C \, \| \Delta^{\frac{k}{2}} f \|_{L_2(Q)}, \tag{33}$$

if k is even, and the inequality

$$\| f \|_{H^k(Q)} \leqslant C \, \| \Delta^{\frac{k-1}{2}} f \|_{H^1(Q)} \tag{33'}$$

if k is odd.

Proof. We first consider the case of even k, $k = 2p$. The lemma will be proved by induction on p. We shall establish (33) for $p = 1$. Suppose that $f \in H^2_{\mathscr{D}}(Q)$ $(H^2_{\mathscr{N}}(Q))$. Let F denote the function Δf. Then $f(x)$ satisfies Poisson's equation

$$\Delta f = F \tag{34}$$

a.e. in Q. Moreover, by the definition of $H^2_{\mathscr{D}}(Q)$ $(H^2_{\mathscr{N}}(Q))$ $u\,|_{\partial Q}=0$ $\left(\frac{\partial u}{\partial n}\Big|_{\partial Q}=0\right)$.

Multiplying (34) by an arbitrary function $v \in \mathring{H}^1(Q)$ and applying Ostrogradskii's formula, we find that u is a generalized solution of the first (second) boundary-value problem for Eq. (34). Now inequality (33) for $k = 2$ follows from Theorem 4.

Assume that the inequality (33) has been established for $k = 2p$, and let $f \in H^{2p+2}_{\mathscr{D}}(Q)$ $(H^{2p+2}_{\mathscr{N}}(Q))$. Since in this case the function $F = \Delta f$ belongs to $H^{2p}_{\mathscr{D}}(Q)$ $(H^{2p}_{\mathscr{N}}(Q))$.

$$\| F \|_{H^{2p}(Q)} \leqslant C_1 \, \| \Delta^p F \|_{L_2(Q)} = C_1 \, \| \Delta^{p+1} f \|_{L_2(Q)}.$$

But $f(x)$ is a generalized solution of the first (second) boundary-value problem for Eq. (34), therefore Theorem 4 yields

$$\| f \|_{H^{2p+2}(Q)} \leqslant C_2 \, \| F \|_{H^{2p}(Q)} \leqslant C \, \| \Delta^{p+1} f \|_{L_2(Q)}.$$

We now consider the case of odd k. For $k = 1$ the inequality (33′) is trivial. Assuming that it has been proved for $k = 2p - 1$, $p \geqslant 1$, we shall establish the same for $k = 2p + 1$. Let $f \in H^{2p+1}_{\mathscr{D}}(Q)$ $(H^{2p+1}_{\mathscr{N}}(Q))$. Then $F(x) = \Delta f \in H^{2p-1}_{\mathscr{D}}(Q)$ $(H^{2p-1}_{\mathscr{N}}(Q))$. By Theorem 4 and by induction, we obtain

$$\|f\|_{H^{2p+1}(Q)} \leqslant C_2 \|F\|_{H^{2p-1}(Q)} \leqslant C \|\Delta^{p-1}F\|_{H^1(Q)} = C\|\Delta^p f\|_{H^1(Q)}. \quad \blacksquare$$

We can now prove the following result.

Theorem 8. *Assume that the boundary* $\partial Q \in C^k$, $k \geqslant 1$. *In order that the function* f *may be expanded in a Fourier series* (32), *converging in the norm of* $H^k(Q)$, *with respect to the system of eigenfunctions of the first* (*second*) *boundary-value problem for the Laplace operator it is necessary and sufficient that* f *belong to* $H^k_{\mathscr{D}}(Q)$ $(H^k_{\mathscr{N}}(Q))$. *If* $f \in H^k_{\mathscr{D}}(Q)$ $(H^k_{\mathscr{N}}(Q))$, *the series* $\sum_{s=1}^{\infty} f_s^2 |\lambda_s|^k$ *converges and there is a constant* $C > 0$, *independent of* f, *such that*

$$\sum_{s=1}^{\infty} f_s^2 |\lambda_s|^k \leqslant C\|f\|^2_{H^k(Q)}. \tag{35}$$

Proof. From the identity (31) and Theorem 7 it follows that if $\partial Q \in C^k$, then the generalized eigenfunctions of the first (second) boundary-value problem for the Laplace operator belong to $H^k_{\mathscr{D}}(Q)$ $(H^k_{\mathscr{N}}(Q))$. Therefore if the Fourier series of the function $f \in H^k(Q)$ with respect to the eigenfunctions of the first (second) boundary-value problem converges in the norm of $H^k(Q)$, then $f \in H^k_{\mathscr{D}}(Q)$ $(H^k_{\mathscr{N}}(Q))$. This proves the necessity part.

We now suppose that $f \in H^k_{\mathscr{D}}(Q)$ $(H^k_{\mathscr{N}}(Q))$, and establish the inequality (35). First assume that k is even, $k = 2p$, $p \geqslant 1$. Let γ_s denote the Fourier coefficients of $\Delta^p f$: $\gamma_s = (\Delta^p f, u_s)_{L_2(Q)}$. An application of Green's formula yields

$$\gamma_s = (\Delta^p f, u_s)_{L_2(Q)} = (\Delta^{p-1} f, \Delta u_s)_{L_2(Q)} = \lambda_s (\Delta^{p-1} f, u_s)_{L_2(Q)}$$
$$= \ldots = \lambda_s^p (f, u_s)_{L_2(Q)} = \lambda_s^p f_s, \quad s = 1, 2, \ldots .$$

Since $\Delta^p f \in L_2(Q)$, it follows that $\sum_{s=1}^{\infty} \gamma_s^2 = \|\Delta^p f\|^2_{L_2(Q)}$, hence $\sum_{s=1}^{\infty} f_s^2 |\lambda_s|^k = \|\Delta^{\frac{k}{2}} f\|^2_{L_2(Q)}$ which, obviously, implies the inequality (35). When $k = 2p + 1$, the function $\Delta^p f \in \overset{\circ}{H}{}^1(Q)$ $(H^1(Q))$.

Therefore, by Theorem 3, Sec. 1.3, the inequality $\sum\limits_{s=1}^{\infty} \gamma_s^2 |\lambda_s| \leqslant$ $\leqslant C \| \Delta^p f \|^2_{H^1(Q)}$ holds, thereby implying the inequality (35).

By $S_m(x)$ we denote the partial sum of the series (32). Evidently, $S_m \in H^k_{\mathscr{D}}(Q)$ $(H^k_{\mathscr{N}}(Q))$ for all $m = 1, 2, \ldots$.

In view of (33) and (35) (assume that $m > i \geqslant 1$), we have for $k = 2p$

$$\| S_m - S_i \|^2_{H^k(Q)} \leqslant C \| \Delta^p (S_m - S_i) \|^2_{L_2(Q)}$$

$$= C \Big\| \sum_{s=i+1}^{m} \lambda_s^p f_s u_s \Big\|^2_{L_2(Q)} = C \sum_{s=i+1}^{m} \lambda_s^{2p} f_s^2 = C \sum_{s=i+1}^{m} \lambda_s^k f_s^2 \to 0$$

as $m, i \to \infty$. This means that the series (32) converges to f in $H^k(Q)$.

When $k = 2p + 1$, the proof is similarly carried out by applying the inequality (33′). ▮

Theorem 9. *If the boundary ∂Q of the region Q belongs to C^k for a certain $k \geqslant \left[\frac{n}{2}\right] + 1$, then any function $f \in H^k_{\mathscr{D}}(Q)$ $(H^k_{\mathscr{N}}(Q))$ has a Fourier series expansion* (32) *with respect to eigenfunctions of the first (second) boundary-value problem for the Laplace operator converging in the space $C^{k - \left[\frac{n}{2}\right] - 1}(\overline{Q})$.*

Proof. Theorem 3, Sec. 6.2, Chap. III, implies that the space $C^{k - \left[\frac{n}{2}\right] - 1}(\overline{Q})$ contains the function $f(x)$ and all the eigenfunctions $u_s(x)$ as well as all the partial sums $S_m(x)$ of the series (32). Moreover, the inequality $\| S_m - S_i \|_{C^{k - \left[\frac{n}{2}\right] - 1}(\overline{Q})} \leqslant C \| S_m - S_i \|_{H^k(Q)}$, where the constant C does not depend on m or i, holds. By Theorem 8, $\| S_m - S_i \|_{H^k(Q)} \to 0$ as $m, i \to \infty$, therefore $\| S_m - S_i \|_{C^{k - \left[\frac{n}{2}\right] - 1}(\overline{Q})} \to 0$ as $m, i \to \infty$, thereby implying that the series (32) converges in $C^{k - \left[\frac{n}{2}\right] - 1}(\overline{Q})$. ▮

6. **Generalizations.** The method applied in Subsecs. 2 and 3 to investigate the smoothness of generalized solutions of boundary-value problems for Poisson's equation is equally suitable for investigating the smoothness of generalized solutions of boundary-value problems for more general equations. For instance, suppose that $u(x)$ is a generalized solution of the first boundary-value problem

$$\mathscr{L}u = \operatorname{div}\,(k(x)\nabla u) - a(x)\,u = f, \qquad x \in Q,$$

$$u|_{\partial Q} = 0 \tag{36}$$

$$(f \in L_2(Q), \quad k(x) \in C^1(\overline{Q}), \quad a(x) \in C(\overline{Q}), \quad k(x) \geqslant k_0 > 0).$$

If $k(x) \in C^{p+1}(\bar{Q})$, $a(x) \in C^p(\bar{Q})$ and $f(x) \in L_2(Q) \cap H^p_{\text{loc}}(Q)$ for some $p \geqslant 0$, then $u(x) \in H^{p+2}_{\text{loc}}(Q)$. In particular, every generalized eigenfunction of the first boundary-value problem for the operator $\mathcal{L}$ belongs to $H^{p+2}_{\text{loc}}(Q)$.

If, in addition, the boundary $\partial Q \in C^{p+2}$ and $f \in H^p(Q)$, then $u(x) \in H^{p+2}(Q)$; in particular, every generalized eigenfunction of the first boundary-value problem for the operator $\mathcal{L}$ belongs to $H^{p+2}(Q)$.

Totally analogous results hold also for the generalized solutions of the second and third boundary-value problems for Eq. (36) and the corresponding eigenfunctions for the operator $\mathcal{L}$.

§ 3. CLASSICAL SOLUTIONS OF LAPLACE'S AND POISSON'S EQUATIONS

1. **Harmonic Functions. Potentials.** A real-valued function $u(x)$ is termed *harmonic* in a region Q (or in an open set) in the space R_n if it is twice continuously differentiable in Q and satisfies the Laplace equation

$$\Delta u = 0 \tag{1}$$

at every point $x \in Q$.

An equivalent definition of a harmonic function can easily be given in terms of H^k spaces (as usual, functions are considered equal if they coincide almost everywhere).

A function $u(x)$ belonging to $H^1_{\text{loc}}(Q)$, where Q is a region in R_n, is called harmonic in Q if it satisfies the integral identity

$$\int\limits_Q \nabla u \, \nabla v \, dx = 0 \tag{2}$$

for all $v \in H^1(Q)$ having compact support in Q (that is, equal to zero a.e. in $Q \setminus Q'$ for a certain $Q' \Subset Q$).

If $u(x) \in C^2(Q)$ is harmonic in Q, then it obviously belongs to $H^1_{\text{loc}}(Q)$. Multiplying (1) by an arbitrary $v \in H^1(Q)$ with compact support in Q and integrating the resulting identity over Q, we find, by means of Ostrogradskii's formula, that u satisfies the integral identity (2).

Suppose now that $u \in H^1_{\text{loc}}(Q)$ and satisfies the integral identity (2) for all v belonging to $H^1(Q)$ and having compact support in Q. Take an arbitrary strictly interior subregion Q' of Q. Since $u \in H^1(Q')$ and satisfies the identity $\int\limits_{Q'} \nabla u \nabla v \, dx = 0$ for all $v \in \overset{\circ}{H}{}^1(Q')$, $u \in C^\infty(Q')$ by Lemma 2, Sec. 2.2, and Theorem 2, Sec. 6.2, Chap. III. What is more, u satisfies Eq. (1) in Q' (see Corollary to Lemma 2 of the preceding section). Since Q' is arbitrary, the *function u (x) belongs* *to* $C^\infty(Q)$ and satisfies Eq. (1) in Q, that is, is harmonic.

If a function u harmonic in Q is twice continuously differentiable in $\overline{Q}$, then, integrating (1) over and using Ostrogradskii's formula, we obtain the identity

$$0=\int_Q \Delta u\,dx=\int_Q \operatorname{div}(\nabla u)\,dx=\int_{\partial Q}\frac{\partial u}{\partial n}\,dS \tag{3}$$

which will be frequently used in the sequel.

Let ξ be any point in R_n, and let $r=|x-\xi|$. Since $\Delta f = f_{rr}+\frac{n-1}{r}f_r$ for a function f depending only on r, the harmonic function u depending only on r satisfies the ordinary differential equation $u_{rr}+\frac{n-1}{r}u_r=0$. The general solution of this equation on the half-line $r>0$ has the form $\frac{c_0}{r^{n-2}}+c_1$ if $n>2$ and $c_0\ln r + c_1$ if $n=2$, where c_0 and c_1 are arbitrary constants. Therefore all the functions harmonic in the entire space except at the point $x=\xi$ and depending only on $|x-\xi|$ have the form $\frac{c_0}{|x-\xi|^{n-2}}+c_1$ if $n>2$, and $c_0\ln|x-\xi|+c_1$ if $n=2$ (c_0 and c_1 are arbitrary constants).

The function

$$U(x-\xi)=\begin{cases}-\dfrac{1}{(n-2)\,\sigma_n\,|x-\xi|^{n-2}}, & n>2\\[2mm] \dfrac{1}{2\pi}\ln|x-\xi|, & n=2,\end{cases} \tag{4}$$

harmonic in $R_n\setminus\{x=\xi\}$, where σ_n is the surface area of the unit sphere, is called the *fundamental solution* of the Laplace equation. This function plays a basic role in the study of solutions of the Laplace and Poisson equations.

For any measurable function $\rho_0(\xi)$ bounded in Q the function

$$u_0(x)=\int_Q U(x-\xi)\,\rho_0(\xi)\,d\xi, \tag{5}$$

which is defined for all x, is called the *volume potential with density* ρ_0.

For any function $\rho_1(\xi)$ and $\rho_2(\xi)$ integrable over ∂Q the functions

$$u_1(x)=\int_{\partial Q} U(x-\xi)\,\rho_1(\xi)\,dS_\xi, \tag{6}$$

$$u_2(x)=\int_{\partial Q}\frac{\partial U(x-\xi)}{\partial n_\xi}\,\rho_2(\xi)\,dS_\xi, \tag{7}$$

which are defined for all $x \in R_n \setminus \partial Q$, are, respectively, referred to as *potentials of simple and double layers with densities* ρ_1 *and* ρ_2.

We have already encountered the potentials (5), (6) and (7). In Sec. 6.1, Chap. III (Theorem 1), it was established that any function $u(x) \in C^2(\bar{Q})$ can be expressed as a sum of three terms: the volume potential with density Δu, the potential of simple layer with density $-\frac{\partial u}{\partial n}$ and the potential of double layer with density u:

$$u(x) = \int_Q U(x-\xi)\,\Delta u(\xi)\,d\xi - \int_{\partial Q} U(x-\xi)\,\frac{\partial u(\xi)}{\partial n}\,dS_\xi + \int_{\partial Q} \frac{\partial U(x-\xi)}{\partial n_\xi}\,u(\xi)\,dS_\xi. \qquad (8)$$

If the function $u \in C^2(\bar{Q})$ and is harmonic in Q, then (8) yields

$$u(x) = \int_{\partial Q} \left(\frac{\partial U(x-\xi)}{\partial n_\xi}\,u(\xi) - \frac{\partial u(\xi)}{\partial n}\,U(x-\xi) \right) dS_\xi \qquad (9)$$

for any $x \in Q$.

Lemma 1. *The potentials of single and double layers are harmonic in* $R_n \setminus \partial Q$.

Proof. Let x^0 be any point in $R_n \setminus \partial Q$, and let $\delta > 0$ be the distance from this point to ∂Q. The integrand functions in (6) and (7) regarded as functions of ξ, $\xi \in \partial Q$, belong to $L_1(\partial Q)$ for all x lying in the ball $\{|x - x^0| < \delta/2\}$ and belong to $C^\infty(|x - x^0| \leqslant \delta/2)$ for almost all $\xi \in \partial Q$ if they are regarded as functions of x. Furthermore, for all $\xi \in \partial Q$ and x in the ball $\{|x - x^0| < \delta/2\}$ the estimates $|D_x^\alpha U(x - \xi)| \leqslant C$, $\left| D_x^\alpha \frac{\partial U(x-\xi)}{\partial n_\xi} \right| \leqslant C$ hold in which $\alpha = (\alpha_1, \ldots, \alpha_n)$ is arbitrary and the constant $C > 0$ depends only on δ and α. Therefore

$$|D_x^\alpha U(x-\xi)\,\rho_1(\xi)| \leqslant C\,|\rho_1(\xi)|,$$

$$\left| D_x^\alpha \frac{\partial U(x-\xi)}{\partial n_\xi}\,\rho_2(\xi) \right| \leqslant C\,|\rho_2(\xi)|.$$

Then, by Theorem 7, Sec. 1.7, Chap. II, the functions $u_1(x)$ and $u_2(x)$ are infinitely differentiable in the ball $\{|x - x^0| < \delta/2\}$ and

$$D^\alpha u_1(x) = \int_{\partial Q} \rho_1(\xi)\,D_x^\alpha U(x-\xi)\,dS_\xi$$

and

$$D^\alpha u_2(x) = \int_{\partial Q} \rho_2(\xi)\,D_x^\alpha \frac{\partial U(x-\xi)}{\partial n_\xi}\,dS_\xi$$

for any α. In particular,

$$\Delta u_1 = \int_{\partial Q} \rho_1(\xi)\, \Delta_x U(x-\xi)\, dS_\xi = 0$$

and

$$\Delta u_2(x) = \int_{\partial Q} \rho_2(\xi) \frac{\partial}{\partial n_\xi} \Delta_x U(x-\xi)\, dS_\xi = 0,$$

because $\Delta_x U(x-\xi) = 0$ when $x \neq \xi$.

Lemma 2. *If $\rho_0(\xi) \in C^1(\overline{Q})$, then the volume potential $u_0(x)$ belongs to $C^2(Q) \cap C^1(\overline{Q})$ and satisfies Poisson's equation $\Delta u_0 = \rho_0$ for all $x \in Q$.*

Proof. Since the function ρ_0 is measurable and bounded in Q, it follows (see Sec. 1.12, Chap. II) that $u_0 \in C^1(Q)$ and

$$\frac{\partial u_0}{\partial x_i} = \int_Q \frac{\partial U(x-\xi)}{\partial x_i} \rho_0(\xi)\, d\xi = -\int_Q \frac{\partial U(x-\xi)}{\partial \xi_i} \rho_0(\xi)\, d\xi, \qquad i = 1, \ldots, n.$$

If $\rho_0(\xi) \in C^1(\overline{Q})$, then, by Ostrogradskii's formula,

$$\frac{\partial u_0}{\partial x_i} = \int_Q U(x-\xi) \frac{\partial \rho_0}{\partial \xi_i}\, d\xi - \int_{\partial Q} U(x-\xi)\, \rho_0(\xi)\, n_i(\xi)\, dS_\xi,$$

where the function $n_i(\xi) = \cos(\widehat{n, \xi_i})$ is continuous on ∂Q.

The first term on the right-hand side of this identity is the volume potential with density $\frac{\partial \rho_0}{\partial \xi_i}$ which is continuous in $\overline{Q}$, thereby implying that this term belongs to $C^1(\overline{Q})$. The second term is the potential of the simple layer with density $\rho_0 n_i$ continuous on ∂Q, and, according to Lemma 1, belongs to $C^\infty(Q)$. Therefore $u_0 \in C^1(\overline{Q}) \cap C^2(Q)$. Take an arbitrary function $\psi \in C^2(\overline{Q})$ with compact support in Q. Since $\psi|_{\partial Q} = \left.\frac{\partial \psi}{\partial n}\right|_{\partial Q} = 0$, by (8)

$$\psi(x) = \int_Q U(x-\xi)\, \Delta\psi(\xi)\, d\xi \tag{10}$$

for all $x \in Q$. Applying Green's formula to the functions ψ and u_0 and using successively Fubini's theorem (Theorem 10, Sec. 1.11, Chap. II) and the identity (10), we obtain

$$\begin{aligned}\int_Q \psi(x)\, \Delta u_0(x)\,dx &= \int_Q \Delta\psi(x)\, u_0(x)\,dx \\ &= \int_Q \Delta\psi(x) \left(\int_Q U(x-\xi)\, \rho_0(\xi)\, d\xi \right) dx \\ &= \int_Q \rho_0(\xi) \left(\int_Q U(x-\xi)\, \Delta\psi(x)\,dx \right) d\xi = \int_Q \psi(\xi)\, \rho_0(\xi)\, d\xi.\end{aligned}$$

Thus for any function $\psi \in C^2(\overline{Q})$ with compact support in Q the identity

$$\int_Q \psi(x)(\Delta u_0(x) - \rho_0(x))\,dx = 0$$

holds, implying that $\Delta u_0 = \rho_0$ in Q. ∎

2. Principal Properties of Harmonic Functions. We shall now establish some of the important properties of harmonic functions.

Theorem 1. (First Mean Value Theorem). *Let $u(x)$ be a harmonic function in Q and x any point of Q. Then for any r, $0 < r < d$, where d is the distance of x from the boundary ∂Q,*

$$u(x) = \frac{1}{\sigma_n r^{n-1}} \int_{|x-\xi|=r} u(\xi)\,dS_\xi. \tag{11}$$

Proof. Since $u(\xi) \in C^2(|\xi - x| \leqslant r)$, formula (9) is applicable to this function in the region $\{|\xi - x| < r\}$. By this formula for $n > 2$ (for $n = 2$, the argument is analogous),

$$\begin{aligned} u(x) = &\int_{|\xi-x|=r} \frac{1}{(n-2)\sigma_n r^{n-2}} \frac{\partial u(\xi)}{\partial n}\,dS_\xi \\ &- \int_{|\xi-x|=r} \frac{u(\xi)}{(n-2)\sigma_n} \frac{\partial \frac{1}{|\xi-x|^{n-2}}}{\partial n_\xi}\,dS_\xi \\ &= \frac{1}{(n-2)\sigma_n r^{n-2}} \int_{|\xi-x|=r} \frac{\partial u(\xi)}{\partial n}\,dS_\xi \\ &\qquad + \frac{1}{\sigma_n r^{n-1}} \int_{|\xi-x|=r} u(\xi)\,dS_\xi, \end{aligned}$$

because on the sphere $\{|\xi - x| = r\}$

$$\frac{\partial \frac{1}{|\xi-x|^{n-2}}}{\partial n_\xi} = \frac{\partial \frac{1}{|\xi-x|^{n-2}}}{\partial |\xi - x|} = -\frac{(n-2)}{|x-\xi|^{n-1}} = -\frac{(n-2)}{r^{n-1}}.$$

Formula (11) now follows from (3) ∎.

Theorem 2 (Second Mean Value Theorem). *Let $u(x)$ be a harmonic function in Q and x any point of Q. Then for any r, $0 < r < d$, where d is the distance from x to the boundary ∂Q,*

$$u(x) = \frac{n}{\sigma_n r^n} \int_{|\xi-x|<r} u(\xi)\,d\xi. \tag{12}$$

Proof. According to Theorem 1, the identity

$$\sigma_n \rho^{n-1} u(x) = \int_{|\xi-x|=\rho} u(\xi)\,dS_\xi$$

holds for any ρ, $0 < \rho < d$. Integrating this with respect to ρ from 0 to r, we obtain (12). ■

Theorems 1 and 2 are referred to as mean value theorems because the right-hand sides in (11) and (12) are the average values of the function u over the sphere $\{|\xi - x| = r\}$ and over the ball $\{|\xi - x| < r\}$, respectively $\left(\sigma_n r^{n-1}\right.$ is the surface area of this sphere and $\frac{\sigma_n r^n}{n}$ the volume of the ball$\left.\right)$.

As shown above, a function which is harmonic in Q is infinitely differentiable in Q. The following result often proves useful in the study of harmonic functions.

Lemma 3. *Let the function $u(x)$ be harmonic in Q and be bounded: $|u(x)| \leqslant M$. Then any derivative $D^\alpha u(x)$ of order $|\alpha| = k$, $k = 1, 2, \ldots$, at the point $x \in Q$ satisfies the inequality*

$$|D^\alpha u(x)| \leqslant M\left(\frac{n}{\delta}\right)^k k^k, \tag{13}$$

where δ is the distance from the point x to the boundary ∂Q.

Proof. This lemma will be proved by induction on k.

Suppose first that $k = 1$. We shall demonstrate that $|u_{x_i}| \leqslant Mn/\delta$ for all $i = 1, \ldots, n$. Since the function u_x is harmonic in Q, by Theorem 2, for any $\delta' < \delta$,

$$u_{x_i}(x) = \frac{n}{\sigma_n \delta'^n} \int\limits_{|\xi - x| < \delta'} u_{\xi_i}\, d\xi = \frac{n}{\sigma_n \delta'^n} \int\limits_{|\xi - x| = \delta'} u(\xi) \cos\alpha_i\, dS_\xi,$$

where α_i is the angle between the vector $\xi - x$ and the $O\xi_i$ axis. Therefore

$$|u_{x_i}(x)| \leqslant \frac{n}{\sigma_n \delta'^n} \int\limits_{|\xi - x| = \delta'} |u(\xi)|\, dS_\xi \leqslant M \frac{n}{\sigma_n \delta'^n} \sigma_n \delta'^{n-1} = Mn/\delta'.$$

The desired inequality follows from this by passing to the limit as $\delta' \to \delta$.

Assume that the lemma has been proved for all derivatives $D^\alpha u$, where $|\alpha| \leqslant k - 1$, $k \geqslant 2$. Let us prove the inequality (13).

Take two balls $\{|\xi - x| < \delta'\}$ and $\{|\xi - x| < \delta'/k\}$ with centre at the point x (δ' is any positive number less than δ). By the induction hypothesis the following inequality holds for any point ξ in the ball $\{|\xi - x| < \delta'/k\}$ and any β, $|\beta| = k - 1$:

$$|D_\xi^\beta u(\xi)| \leqslant M\left(\frac{n}{\delta' - \delta'/k}\right)^{k-1} (k-1)^{k-1} = M\left(\frac{n}{\delta'}\right)^{k-1} k^{k-1}.$$

Thus for any β, $|\beta| = k - 1$, the harmonic function $D_\xi^\beta u(\xi)$ is bounded in the ball $\{|\xi - x| < \delta'/k\}$ by a constant $M\left(\frac{n}{\delta'}\right)^{k-1} k^{k-1}$.

Then, by what has been just established, for the first derivatives of this function we have

$$|(D_\xi^\beta u(\xi))_{\xi_i}| \leqslant M\left(\frac{n}{\delta'}\right)^{k-1} k^{k-1}\left(\frac{n}{\delta'/k}\right) = M\left(\frac{n}{\delta'}\right)^k k^k, \quad i = 1, \ldots, n.$$

That is, for any α, $|\alpha| = k$, we have $|D^\alpha u| \leqslant M\left(\frac{n}{\delta'}\right)^k k^k$. Letting $\delta' \to \delta$ in this inequality, we obtain the inequality (13). ∎

Let us take up some of the applications of Lemma 3.

Theorem 3. *From any infinite set of harmonic functions in Q that are bounded in Q by the same constant a sequence can be chosen which converges uniformly on any strictly interior subregion of Q.*

Proof. Let $\mathfrak{M}$ be an infinite set of harmonic functions $u(x)$ in Q which are bounded in Q by the same constant: $|u(x)| \leqslant M$. Consider a sequence of regions $Q_1, Q_2, \ldots$, such that $Q_1 \subset Q_2 \subset \ldots$; $Q_i \Subset Q$, $i = 1, 2, \ldots$; $\bigcup_{i=1}^{\infty} Q_i = Q$.

The set $\mathfrak{M}$ consists of functions belonging to $C(\overline{Q}_1)$ and bounded on Q_1 by the same constant. By Lemma 3, there is a constant $C > 0$, depending only on Q_1, such that for all functions u in $\mathfrak{M}$ $|\nabla u| \leqslant C$ for $x \in \overline{Q}_1$. Therefore the set $\mathfrak{M}$ is equicontinuous in $\overline{Q}_1$. According to Arzela's theorem, it is possible to choose a sequence $u_{11}, u_{12}, \ldots$ from $\mathfrak{M}$ which converges uniformly in $\overline{Q}_1$. Since this sequence is uniformly bounded and, by Lemma 3, is equicontinuous in $\overline{Q}_2$, a subsequence $u_{21}, u_{22}, \ldots$ can be selected from it which converges uniformly in $\overline{Q}_2$, and so forth. It is apparent that the diagonal sequence $u_{11}, u_{22}, \ldots$ is the desired sequence. ∎

Theorem 4. *Suppose that a sequence $u_1(x), u_2(x), \ldots$ of harmonic functions in Q converges to $u(x)$ uniformly in any strictly interior subregion of Q. Then the function $u(x)$ is harmonic in Q and for any $\alpha = (\alpha_1, \ldots, \alpha_n)$ the sequence $D^\alpha u_1, D^\alpha u_2, \ldots$ converges to $D^\alpha u$ uniformly in any strictly interior subregion of Q.*

Proof. Let Q' be any strictly interior subregion of Q. Then the function $u(x) \in C(\overline{Q}')$. Take any region Q'' such that $Q' \Subset Q'' \Subset Q$. It is clear that each $u_m(x)$ is bounded in Q''. According to Lemma 3, for any α there is a constant $C > 0$, depending only on Q', Q'' and $|\alpha|$, such that

$$\| D^\alpha (u_m - u_s) \|_{C(\overline{Q}')} \leqslant C \| u_m - u_s \|_{C(\overline{Q}'')}$$

for all $m, s = 1, 2, \ldots$. Since $\|u_m - u_s\|_{C(\overline{Q}'')} \to 0$ as $m, s \to \infty$, all the sequences $D^\alpha u_m$, $m = 1, 2, \ldots$, are fundamental in the norm of $C(\overline{Q}')$. This means that the function $u(x) \in C^\infty(\overline{Q}')$ and for any $\alpha \| D^\alpha u_m - D^\alpha u \|_{C(\overline{Q}')} \to 0$ as $m \to \infty$.

Passing to the limit as $m \to \infty$ in the equation $\Delta u_m = 0$, $x \in Q'$, we find that $\Delta u = 0$ in Q', that is, $u(x)$ is harmonic in Q' and hence also in Q. ■

Theorem 5. *A function which is harmonic in Q is analytic there.*

Proof. Let the function $u(x)$ be harmonic in Q. Take any point x^0 in Q, and let $\delta > 0$ denote the distance between this point and the boundary ∂Q and $S_{\delta/4}(x^0)$ denote the ball $\{|x - x^0| < \delta/4\}$. The function $u(x) \in C(\overline{Q}_{\delta/2})$, therefore it is bounded in $Q_{\delta/2}$; let $M = \max\limits_{x\in\overline{Q}_{\delta/2}} |u(x)|$.

Since the distance from any point of the ball $\overline{S}_{\delta/4}(x^0)$ to the boundary $\partial Q_{\delta/2}$ is not less than $\delta/4$, by Lemma 3 for any point x in $\overline{S}_{\delta/4}(x^0)$ and any $\alpha = (\alpha_1, \ldots, \alpha_n)$ the inequality

$$|D^\alpha u| \leqslant M(4n/\delta)^{|\alpha|}|\alpha|^{|\alpha|}$$

holds. Since $\lim\limits_{k\to\infty} \frac{k^{k+1/2}}{k!\,e^k} = \frac{1}{\sqrt{2\pi}}$ (Stirling's formula), there is a constant $C > 0$ such that $k^k \leqslant Ce^k k!$, and hence $|\alpha|^{|\alpha|} \leqslant Ce^{|\alpha|}(|\alpha|)!$ for all natural k.

If in the identity $(x_1 + \ldots + x_n)^k = \sum\limits_{|\alpha|=k} \frac{(|\alpha|)!}{\alpha!} x^\alpha$, which is true for any natural k, we set $x_1 = \ldots = x_n = 1$, we obtain the identity $n^k = \sum\limits_{|\alpha|=k} (|\alpha|)!/\alpha!$ which implies the inequality $(|\alpha|)!/\alpha! \leqslant \leqslant n^{|\alpha|}$. Therefore for all $x \in S_{\delta/4}(x^0)$ and all α

$$|D^\alpha u| \leqslant CM(4n^2e/\delta)^{|\alpha|}\alpha!. \tag{14}$$

From the inequality (14) it follows first of all that the Taylor series

$$\sum_\alpha \frac{D^\alpha u(x^0)}{\alpha!}(x - x^0)^\alpha$$

of function $u(x)$ converges absolutely in the ball $S = \left\{|x - x^0| < \frac{\delta}{4n^2e}\right\}$, therefore the sum of this series is an analytic function in S. Let us show that this series converges to $u(x)$ in the ball $S' = \left\{|x - x^0| < \frac{\delta}{8n^3e}\right\}$. For this it is enough to demonstrate that the remainder term in the Taylor's formula

$$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{D^\alpha u(x^0)}{\alpha!}(x - x^0)^\alpha$$

$$= \sum_{|\alpha|=N} \frac{D^\alpha u(x^0 + \theta(x - x^0))}{\alpha!}(x - x^0)^\alpha$$

for function u, where $|\theta| < 1$, approaches zero as $N \to \infty$ at every point of S'. Since for $x \in S'$ the point $x^0 + \theta(x - x^0)$ also lies in S', and thus in the ball $S_{\delta/4}(x^0)$, we find, in view of (14), that for all $x \in S'$

$$|R_N(x)| \leqslant \sum_{|\alpha|=N} CM \left(\frac{4n^2e}{\delta}\right)^N \left(\frac{\delta}{8n^3e}\right)^N \leqslant \frac{CM}{(2n)^N} n^N = \frac{CM}{2^N}.$$

Hence $R_N(x) \to 0$ as $N \to \infty$.

Since the point $x^0 \in Q$ is arbitrary, the function $u(x)$ is analytic in Q. ▮

In the two-dimensional case, apart from Theorem 5 establishing the analyticity of a harmonic function as a function of two variables x_1, x_2, a deeper result holds that connects harmonic functions with analytic functions of a single complex variable $z = x_1 + ix_2$. For simplicity, we confine our discussion to a simply connected region.

Theorem 6. *For the function $u(x_1, x_2)$ to be harmonic in a simply connected region Q it is necessary and sufficient that there exist an analytic function $f(z)$ in Q, $z = x_1 + ix_2$, such that $u(x_1, x_2) = \operatorname{Re} f(z)$.*

Sufficiency. Suppose that $f(z)$ is analytic in Q. Then the functions $u(x_1, x_2) = \operatorname{Re} f(x_1 + ix_2)$ and $v(x_1, x_2) = \operatorname{Im} f(x_1 + ix_2)$ in Q are infinitely differentiable and they satisfy Cauchy-Riemann equations:

$$u_{x_1} = v_{x_2}, \quad u_{x_2} = -v_{x_1}. \tag{15}$$

Differentiating the first of equations (15) with respect to x_1 and the second with respect to x_2 and adding them, we see that $\Delta u = 0$, that is u is a harmonic function.

Necessity. Assume that the function u is harmonic in Q, and consider the function

$$v(x) = \int_{L(x^0, x)} -u_{x_2}\,dx_1 + u_{x_1}\,dx_2,$$

where $x^0 = (x_1^0, x_2^0)$ is a fixed point of Q, $x = (x_1, x_2)$, and $L(x^0, x)$ is any rectifiable curve joining the points x^0 and x and lying in Q (since Q is a simply connected region, Green's formula yields that the function v does not depend on the contour L). The function $v(x) \in C^1(Q)$ and satisfies conditions (15). Consequently, the function $f = u + iv$ is an analytic function of $x_1 + ix_2$ in Q. ▮

Remark 1. The analytic function $f(z)$ corresponding to the harmonic function $u(x_1, x_2)$ is defined up to a pure arbitrary constant. Indeed, suppose that $f_1(z) = u + iv_1$ and $f_2(z) = u + iv_2$ are two analytic functions in Q for which $\operatorname{Re} f_1 = \operatorname{Re} f_2 = u$. Then the function $f_1 - f_2 = iv$, where the (real) function $v = v_1 - v_2$, is analytic in Q. In view of the Cauchy-Riemann equations, $v_{x_1} = v_{x_2} = 0$, that is, $v_2 = v_1 + c$, where c is any real constant. Accordingly, $f_2(z) = f_1(z) + ic$. ▮

Remark 2. An assertion analogous to that of Theorem 6 holds also for any region Q. In this case the analytic function f of $x_1 + ix_2$ in Q, constructed corresponding to the function u harmonic in Q, may be multiple-valued. For instance, to the function $\ln|z|$ harmonic in the annulus $\{1 < |z| < 2\}$ there corresponds a multiple-valued function $\ln z = \ln |z| + i \arg(x_1 + ix_2)$ $(z = x_1 + ix_2)$ analytic in this annulus.

Corollary. *Suppose that the function* $z' = F(z) = F_1(x_1, x_2) + iF_2(x_1, x_2)$, $z = x_1 + ix_2$, *analytic in a simply connected region* Q *maps* Q *one-to-one onto another simply connected region* Q' *of the complex plane* $z' = x'_1 + ix'_2$. *If the function* $u'(x')$ *is harmonic in* Q', *then the function* $u(x) = u'(F_1(x), F_2(x))$ *is harmonic in* Q.

In fact, let the function $f'(z')$ analytic in Q' be such that $u'(x') = \operatorname{Re} f'(z')$. Since the function $f(z) = f'(F(z))$ is analytic in Q, the function $u(x) = u'(F_1(x), F_2(x)) = \operatorname{Re} f'(z') = \operatorname{Re} f(z)$ is analytic in Q.

The following important property of harmonic functions, a consequence of a mean value theorem, is referred to as the maximum principle.

Theorem 7 (The maximum principle). *Let a function* $u(x)$ *harmonic in* Q *be continuous in* $\bar{Q}$. *Then either* $u(x) = \text{const}$ *in* Q *or*

$$\min_{x\in\partial Q} u(x) < u(x) < \max_{x\in\partial Q} u(x) \tag{16}$$

for all $x \in Q$.

Proof. Put $M = \max\limits_{x\in\bar{Q}} u(x)$. Let us show that if the right-hand inequality in (16) is violated at a point in Q, then the function $u(x) = \text{const} = M$ in Q. Indeed, assume that there is such a point. Then in Q there is a point x^0 at which $u = M$. Taking any point y in Q, we shall demonstrate that $u(y) = M$. Join the points y and x^0 by a polygonal line L (without self-intersection) lying wholly in Q. By $d > 0$ denote the distance between L and ∂Q. We cover L by a finite number of balls $S_i = \left\{|x - x^i| < \frac{d}{2}\right\}$, $i = 0, 1, \ldots, N$, with centres at $x^i \in L \cap S_{i-1}$, $i = 1, \ldots, N$. The point x^0 is the centre of the ball S_0 and the point $y \in S_N$.

By the second mean value theorem (Theorem 2),

$$u(x^0) = \frac{n}{\sigma_n (d/2)^n} \int\limits_{|x-x^0|<d/2} u(x)\, dx,$$

that is,

$$\int\limits_{S_0} (u(x) - u(x^0))\, dx = 0.$$

Since the integrand function $u(x) - u(x^0)$ is nonpositive $u(x) = u(x^0) = M$ in S_0, and, in particular, $u(x^1) = M$. Repeating the same arguments for the point x^1 and the ball S_1, it can be shown that $u(x) = M$ in S_1, and, in particular, $u(x^2) = M$, and so on. Finally we find that $u(x) = M$ in S_N, and, in particular, $u(y) = M$.

Thus it is established that either $u(x) = \text{const}$ in Q or the right-hand inequality in (16) holds for all x in Q. Applying this result to $-u(x)$, we obtain that either $u(x) = \text{const}$ in Q or the left-hand inequality in (16) holds for all x in Q. ■

Corollary. *It readily follows from Theorem 7 that a function $u(x)$ harmonic in Q and continuous in $\bar{Q}$ satisfies the inequality*

$$\| u \|_{C(\bar{Q})} \leqslant \| u \|_{C(\partial Q)}. \tag{17}$$

Theorem 8. *Let the functions $u_k(x)$, $k = 1, 2, \ldots$, belong to $C(\bar{Q})$ and be harmonic in Q. If the sequence $u_k|_{\partial Q}$, $k = 1, 2, \ldots$, converges uniformly on ∂Q, then the sequence $u_k, k = 1, 2, \ldots$, converges uniformly in $\bar{Q}$ to a function harmonic in Q.*

Proof. In fact, we have by formula (17)

$$\| u_s - u_m \|_{C(\bar{Q})} \leqslant \| u_s - u_m \|_{C(\partial Q)}$$

for all s and m. Since the sequence $u_k|_{\partial Q}$, $k = 1, 2, \ldots$, converges uniformly on ∂Q, $\| u_s - u_m \|_{C(\partial Q)} \to 0$ as $m, s \to \infty$; hence $\|u_s - - u_m \|_{C(\bar{Q})} \to 0$ as $m, s \to \infty$. The completeness of the space $C(\bar{Q})$ implies that there is a continuous function $u(x)$ to which the sequence $u_k(x)$, $k = 1, 2, \ldots$, converges uniformly in $\bar{Q}$. That the function $u(x)$ is harmonic in Q follows from Theorem 4. ■

3. On Classical Solutions of the Dirichlet Problem for Poisson's Equation. Let us recall that a function $u(x)$ is called the classical solution of the Dirichlet problem (the first boundary-value problem) for Poisson's equation

$$\Delta u = f, \qquad x \in Q, \tag{18}$$

$$u|_{\partial Q} = \varphi, \tag{19}$$

if $u(x) \in C^2(Q) \cap C(\bar{Q})$ and satisfies (18) and (19).

First of all we shall establish uniqueness of the solution.

Theorem 9. *The Dirichlet problem for Poisson's equation cannot have more than one classical solution.*

Proof. Let $u_1(x)$ and $u_2(x)$ be two solutions of the problem (18), (19). Then the function $u(x) = u_1(x) - u_2(x)$ is harmonic in Q, continuous in $\bar{Q}$ and vanishes on ∂Q. Therefore, by the inequality (17), $u = 0$ in Q, that is, $u_1 = u_2$. ■

The existence of the classical solution of the problem (18), (19) was established in Subsec. 3 of the preceding section under the assumption that $\partial Q \in C^{[\frac{n}{2}]+1}$, $f \in H^{[\frac{n}{2}]+1}(Q)$, $\varphi \in C^{[\frac{n}{2}+1]}(\partial Q)$. We have, in fact, established there a stronger result: under the assumptions made regarding ∂Q, f and φ the generalized solution u of the problem (18), (19) belongs to the space $H_{\text{loc}}^{[\frac{n}{2}]+3}(Q) \cap H^{[\frac{n}{2}]+1}(Q)$. This, in view of the embedding theorem, implies that $u(x) \in C^2(Q) \cap C(\overline{Q})$, that is, is a classical solution. But the conditions of functions being contained in the spaces $H_{\text{loc}}^{[\frac{n}{2}]+3}(Q)$ and $H^{[\frac{n}{2}]+1}(Q)$ are more stronger than their being contained in the spaces $C^2(Q)$ and $C(\overline{Q})$, respectively. Therefore it seems plausible that the classical solutions exist under milder conditions on ∂Q, f and φ.

Theorem 10. *If* $\partial Q \in C^2$, $f \in C^1(\overline{Q})$, $\varphi \in C(\partial Q)$, *then the problem* (18), (19) *has a classical solution.*

We start by proving Theorem 10 for the case of homogeneous equation (18), that is, for the problem (1), (19).

Lemma 4. *If* $\partial Q \in C^2$ *and* $\varphi \in C(\partial Q)$, *then the problem* (1), (19) *has a classical solution.*

Proof. Suppose first that $\partial Q \in C^{[\frac{n}{2}]+1}$. Since $\varphi \in C(\partial Q)$, there exists a sequence of functions φ_k, $k = 1, 2, \ldots$, in $C^{[\frac{n}{2}]+1}(\partial Q)$ converging uniformly on ∂Q to the function φ. (Indeed, the continuous extension of φ into $\overline{Q}$ can be approximated in $C(\overline{Q})$ by functions belonging to $C^{\infty}(\overline{Q})$ and their values on the boundary belong to $C^{[\frac{n}{2}]+1}(\partial Q)$.) But for any φ_k there is a function $u_k(x)$ harmonic in Q which is a classical solution of the problem (1), (19) with this boundary function. By Theorem 8, the sequence $u_k(x)$, $k = 1, 2, \ldots$, converges uniformly in $\overline{Q}$. Moreover, the limit function $u(x)$ is harmonic in Q, continuous in $\overline{Q}$ and satisfies the boundary condition (19), that is, is a classical solution of the problem (1), (19).

Assume now that $\partial Q \in C^2$. By Φ denote the continuous extension into $\overline{Q}$ of the boundary function φ, and set $M = \max\limits_{x \in \overline{Q}} |\Phi(x)|$. Take a sequence of regions Q_i, $i = 1, 2, \ldots$, having the following properties: $Q_i \Subset Q_{i+1}$ for all $i = 1, 2, \ldots$; $\bigcup\limits_{i=1}^{\infty} Q_i = Q$; $\partial Q_i \in C^{[\frac{n}{2}]+1}$, $i = 1, 2, \ldots$. By what has been proved, for any $i = 1, 2, \ldots$ there exists in Q_i a classical solution $v_i(x)$ of the problem (1), (19) satisfying the boundary condition $v_i|_{\partial Q_i} = \Phi|_{\partial Q_i}$. Moreover, for all $i = 1, 2, \ldots$

$$\max_{x \in \overline{Q}_i} |v_i(x)| \leqslant M.$$

Let $u_i(x)$ denote a function defined in $\overline{Q}$ that is equal to $v_i(x)$ in $\overline{Q}_i$ and vanishes outside $\overline{Q}_i$, $i = 1, 2, \ldots$. By Theorem 3, the sequence of functions $u_2(x)$, $u_3(x)$, . . . harmonic in Q_2 contains a subsequence u_{11}, u_{12}, . . . which converges uniformly in $\overline{Q}_1$. The sequence u_{11}, u_{12}, . . . is composed of functions harmonic in Q_3 (if the function $u_2(x)$, possibly contained in this, is rejected). Therefore, according to Theorem 3, from it a subsequence u_{21}, u_{22}, . . . can be chosen which converges uniformly in $\overline{Q}_2$, and so forth.

Consider the diagonal sequence $u_{11}, u_{22}, \ldots, u_{pp}, \ldots$. The corresponding subsequence of the sequence of regions Q_m, $m = 1, 2, \ldots$, will be denoted by Q_{ii}, $i = 1, 2 \ldots$: the function u_{ii} equals v_{ii} in $\overline{Q}_{ii}$ and vanishes outside $\overline{Q}_{ii}$. The sequence $\{u_{pp}$, $p = 1, 2, \ldots$, converges, obviously, in Q and this convergence is uniform on any $\overline{Q}_{ii}$. Consequently, by Theorem 4, the limit function $u(x)$ is harmonic in Q. What is more, $|u(x)| \leqslant M$ for all $x \in Q$.

We shall show that $u(x)$ is continuous in $\overline{Q}$ and satisfies the boundary condition (19), that is, $u(x)$ is a classical solution of the problem (1), (19).

Take an arbitrary point $x^0 \in \partial Q$. Since $\partial Q \in C^2$, there are a point $x^1 \notin \overline{Q}$ and a number $r > 0$ such that the ball $\{|x - x^1| < r\}$ touching the boundary ∂Q at x^0 does not contain points of Q and the sphere $\{|x - x^1| = r\}$ has only one common point x^0 with ∂Q. Fix an $\varepsilon > 0$. Since $\Phi(x)$ is continuous at x^0, a number $\delta = \delta(\varepsilon) > 0$ can be found such that $|\Phi(x) - \Phi(x^0)| < \varepsilon$ for all points of the ball $\{|x - x^0| < \delta\}$ lying in $\overline{Q}$. Since for $x \neq x^1$ the harmonic function

$$w(x) = \frac{1}{r^{n-2}} - \frac{1}{|x - x^1|^{n-2}}$$

(for the sake of definiteness the case $n > 2$ is considered; when $n = 2$, $w(x) = -\ln r + \ln|x - x^1|$) is nonnegative for all $x \in \overline{Q}$ and vanishes at only one point x^0 in $\overline{Q}$, we can find a $C = C(\delta) > 0$ such that the inequalities

$$\Phi(x^0) - \varepsilon - Cw(x) < \Phi(x) < \Phi(x^0) + \varepsilon + Cw(x)$$

hold for all $x \in \overline{Q}$.

The functions $u_{pp}(x) + Cw(x)$ and $u_{pp}(x) - Cw(x)$ are harmonic in Q_{pp}, continuous in $\overline{Q}_{pp}$ and $(u_{pp} + Cw)|_{\partial Q_{pp}} = (\Phi + Cw)|_{\partial Q_{pp}} > \Phi(x^0) - \varepsilon$ and $(u_{pp} - Cw)|_{\partial Q_{pp}} = (\Phi - Cw)|_{\partial Q_{pp}} < \Phi(x^0) + \varepsilon$. Therefore, according to the maximum principle, $u_{pp}(x) + Cw(x) > \Phi(x^0) - \varepsilon$ and $u_{pp}(x) - Cw(x) < \Phi(x^0) + \varepsilon$ in Q_{pp}, that is,

$$\Phi(x^0) - \varepsilon - Cw(x) \leqslant u_{pp}(x) \leqslant \Phi(x^0) + \varepsilon + Cw(x)$$

for all $x \in Q_{pp}$. Accordingly, for any $x \in Q$

$$\Phi(x^0) - \varepsilon - Cw(x) \leqslant u(x) \leqslant \Phi(x^0) + \varepsilon + Cw(x).$$

Since $w(x) \to 0$ as $x \to x^0$, these inequalities, in turn, imply that

$$\Phi(x^0) - \varepsilon \leqslant \varliminf_{x \to x^0} u(x) \leqslant \varlimsup_{x \to x^0} u(x) \leqslant \Phi(x^0) + \varepsilon,$$

whence it follows, as $\varepsilon > 0$ is arbitrary, that $u(x)$ is continuous at the point x^0 and $u(x^0) = \Phi(x^0) = \varphi(x^0)$. ▌

Proof of Theorem 10. Consider the function $u_0(x) = \int_Q U(x - y) f(y)\,dy$, which is a volume potential with density f. Applying Lemma 2, we find that $u_0(x) \in C^2(Q) \cap C^1(\bar{Q})$ and is a solution of Eq. (18) in Q. According to Lemma 4, there exists a classical solution $v(x)$ of the problem $\Delta v = 0$ in Q, $v|_{\partial Q} = \varphi - u_0|_{\partial Q}$. Then the function $u = u_0 + v$ is a classical solution of the problem (18), (19). ▌

By means of Theorem 10, we can establish the following important property of harmonic functions.

Theorem 11 (On removal of singularity). *Let the function $u(x)$ be harmonic in the region $Q \setminus \{x^0\}$, where x^0 is a point of Q. If $u(x) = o\,(U\,(x - x^0))$, as $x \to x^0$, where U is the fundamental solution of the Laplace equation, then $\lim\limits_{x \to x^0} u\,(x) = A$ exists and the function $u(x)$ redefined at x^0 by A is harmonic in Q.*

Proof. Consider the ball $S_R(x^0) = \{|\,x - x^0\,| < R\}$ lying strictly inside Q. Let the function $v(x)$, satisfying the boundary condition $v|_{\partial S_R(x^0)} = u|_{\partial S_R(x^0)}$, be the classical solution of the Dirichlet problem for the Laplace equation in the ball $S_R(x^0)$. The function $u(x) - v(x) = w(x)$ is harmonic in $S_R(x^0) \setminus \{x^0\}$ and $w|_{\partial S_R(x^0)} = 0$. To establish the theorem, it suffices to show that at every point of the set $S_R(x^0) \setminus \{x^0\}$ the function $w = 0$: in this case the function $u(x)$ coincides with $v(x)$ for all $x \in S_R(x^0) \setminus \{x^0\}$, and, consequently, the function $u\,(x)$ redefined at the point x^0 by $A = v(x^0)$ coincides with the harmonic function $v(x)$ throughout the ball $S_R(x^0)$.

With arbitrary $\varepsilon > 0$, consider the two functions

$$z_{\pm}(x) = \frac{\varepsilon}{|\,x - x^0\,|^{n-2}} \pm w(x)$$

$\Big($to be definite, let the dimension of the space be $n > 2$; when $n = 2$, $z_{\pm}(x) = \varepsilon \ln \dfrac{2R}{|\,x - x^0\,|} \pm w(x)\Big)$. The functions $z_{\pm}(x)$ are harmonic in $S_R(x^0) \setminus \{x^0\}$ and $z_{\pm}(x)\,|_{\partial S_R(x^0)} = \varepsilon / R^{n-2} > 0$. Since, by hypothesis, $u(x) = o\left(\dfrac{1}{|\,x - x^0\,|^{n-2}}\right)$ as $x \to x^0$, we have $z_{\pm}(x)\,|_{|x - x^0| = \rho} =$

$= \frac{\varepsilon}{\rho^{n-2}} \pm w|_{|x-x^0|=\rho} = \frac{\varepsilon}{\rho^{n-2}} + o\left(\frac{1}{\rho^{n-2}}\right)$. Therefore $z_{\pm}(x)|_{|x-x^0|=\rho} > 0$ for sufficiently small $\rho > 0$. According to the maximum principle, $z_{\pm}(x) > 0$ for all x in the spherical annulus $\rho \leqslant |x-x^0| \leqslant R$. Let x^1 be any point in $S_R(x^0) \setminus \{x^0\}$. This point belongs to the spherical annulus $\rho \leqslant |x-x^0| \leqslant R$ for sufficiently small ρ. Consequently, $z_{\pm}(x^1) > 0$, that is, $|w(x^1)| < \frac{\varepsilon}{|x^1-x^0|^{n-2}}$, implying $w(x^1) = 0$ since $\varepsilon > 0$ is arbitrary. ▮

In Theorem 10 the existence of a classical solution of the Dirichlet problem (18), (19) has been established for any $f \in C^1(\bar{Q})$, $\varphi \in C(\partial Q)$, $\partial Q \in C^2$. It can be asked whether for the solvability of this problem it is enough to require merely that $f \in C(\bar{Q})$. The condition that $f \in C^1(\bar{Q})$ is really more than what is required: it can be shown that for the solvability of the problem it is enough to assume that the function f satisfies in $\bar{Q}$ the Hölder condition with some positive exponent *. But, as demonstrated by the following example, this condition cannot be replaced by the condition that $f \in C(\bar{Q})$.

In the ball $Q = \{|x| < R\}$ of radius $R < 1$ consider Poisson's equation

$$\Delta u = \frac{x_2^2 - x_1^2}{2|x|^2}\left(\frac{n+2}{(-\ln|x|)^{1/2}} + \frac{1}{2(-\ln|x|)^{3/2}}\right), \tag{20}$$

where the function on the right-hand side (set zero at the origin) is continuous in $\bar{Q}$. The function

$$u(x) = (x_1^2 - x_2^2)(-\ln|x|)^{1/2} \tag{21}$$

belongs to $C(\bar{Q}) \cap C^{\infty}(\bar{Q} \setminus \{0\})$ (the point $\{0\}$ is the origin) and, as is easily verified, satisfies in $Q \setminus \{0\}$ the equation (20) as well as the boundary condition

$$u|_{|x|=R} = \sqrt{-\ln R}\,(x_1^2 - x_2^2)|_{|x|=R}. \tag{22}$$

Nevertheless, the function $u(x)$ cannot be a classical solution of the problem (20), (22): since

$$\lim_{|x|\to 0} u_{x_1x_1} = \lim_{|x|\to 0}\left(2(-\ln|x|)^{1/2} + \frac{x_1^2(x_1^2 - x_2^2)}{|x|^4(-\ln|x|)^{1/2}} - \frac{2x_1^2}{|x|^2(-\ln|x|)^{1/2}} - \frac{x_1^2 - x_2^2}{2|x|^2(-\ln|x|)^{1/2}} - \frac{x_1^2(x_1^2 - x_2^2)}{4|x|^4(-\ln|x|)^{3/2}}\right) = \infty,$$

it follows that $u(x) \notin C^2(Q)$.

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* A function $f(x)$ is said to satisfy in Q the Hölder condition with some exponent $\alpha > 0$ if there is a constant M such that $|f(x') - f(x'')| \leqslant M|x' - x''|^{\alpha}$ for any points x', x'' of Q.

Let us show that the problem (20), (22) has no classical solution at all.

Assume, on the contrary, that the classical solution $v(x)$ of this problem exists. Then the funcion $w(x) = u(x) - v(x)$ is harmonic and bounded in $Q \setminus \{0\}$. According to the theorem on removal of singularity, the function $w(x)$ may be redefined at the origin so that it will become harmonic in Q and therefore belong to $C^2(Q)$. Therefore, in particular, the (finite) limit $\lim\limits_{|x|\to 0} w_{x_1x_1}$ exists. The existence of finite limit $\lim\limits_{|x|\to 0} v_{x_1x_1}$ follows from the fact that $v(x)$ belongs to $C^2(Q)$. Accordingly, the finite limit $\lim\limits_{|x|\to 0} u_{x_1x_1} = \lim\limits_{|x|\to 0} w_{x_1x_1} + \lim\limits_{|x|\to 0} v_{x_1x_1}$ must exist. This contradiction establishes the assertion.

We have more than once used the formula (8) expressing an arbitrary function $u(x)$ in $C^2(Q)$ in terms of the values in Q of its Laplace operator and the values u and $\frac{\partial u}{\partial n}$ on the boundary ∂Q. In the sequel we shall require another formula of the same sort.

First of all note that for an arbitrary function $u(x) \in C^2(\overline{Q})$ and any point $y \notin \overline{Q}$ the following formula holds:

$$0 = \int\limits_Q U(y-\xi)\,\Delta u(\xi)\,d\xi + \int\limits_{\partial Q} \left[u(\xi)\frac{\partial U(y-\xi)}{\partial n_\xi} - \frac{\partial u(\xi)}{\partial n} U(y-\xi)\right] dS_\xi, \tag{23}$$

where $U(y - \xi)$ is the fundamental solution of the Laplace equation.

To prove this identity, it is enough to apply Green's formula to the functions $u(\xi)$ and $U(y - \xi)$ in Q:

$$\int\limits_Q [u(\xi)\,\Delta_\xi U(y-\xi) - U(y-\xi)\,\Delta u(\xi)]\,d\xi = \int\limits_{\partial Q} \left[u(\xi)\frac{\partial U(y-\xi)}{\partial n_\xi} - U(y-\xi)\frac{\partial u(\xi)}{\partial n} \right] dS,$$

and use the fact that the function $U(y - \xi)$ regarded as a function of ξ is harmonic in Q.

Now take any points $x \in Q$, $y \notin \overline{Q}$, and consider a function $d(y)$ continuous outside $\overline{Q}$. Multiplying (23) by $d(y)$ and subtracting the resulting identity from (8), we find that for any function $u \in C^2(\overline{Q})$

the representation

$$u(x) = \int_Q [U(x-\xi) - d(y)U(y-\xi)]\, \Delta u(\xi) d\xi$$
$$+ \int_{\partial Q} \Big[\frac{\partial u(\xi)}{\partial n} (d(y)U(y-\xi) - U(x-\xi))$$
$$+ u(\xi) \frac{\partial}{\partial n_\xi} (U(x-\xi) - d(y)U(y-\xi)) \Big] dS_\xi \qquad (24)$$

holds for all $x \in Q$, $y \notin \bar{Q}$ and any function $d(y)$ which is continuous outside $\bar{Q}$.

It may be shown that under fairly wide assumptions regarding region Q there exists a transformation $y = y(x)$ which associates with every point $x \in Q$ a point $y \notin \bar{Q}$ and there is a function $d(y(x))$ such that

$$d(y(x))\, U(y(x) - \xi) - U(x - \xi) \equiv 0, \qquad \xi \in \partial Q \qquad (25)$$

for all $x \in Q$. Formula (24) gives then representation in Q of an arbitrary function $u(x) \in C^2(\bar{Q})$ in terms of its value on the boundary and the value of the Laplace operator of this function in Q. We shall confine our discussion to the case when Q is a ball; in this case it is possible to find explicit expressions for the functions $y(x)$ and $d(y(x))$.

Thus, suppose that $Q = \{|\xi| < R\}$, and, to be definite, let the dimension of the space be $n > 2$. Then the condition (25) assumes the form

$$\frac{1}{|x-\xi|^{n-2}} - \frac{d(y(x))}{|y(x)-\xi|^{n-2}} \equiv 0, \qquad |\xi| = R,$$

or, if b denotes $d^{1/(n-2)}$, the form

$$\frac{1}{|x-\xi|} = \frac{b(y(x))}{|y(x)-\xi|}, \qquad |\xi| = R. \qquad (26)$$

The mapping $y = y(x)$ will be sought in the form

$$y = a(x)\, x, \qquad (27)$$

where the function $a(x)$ is to be determined. The identity (26) will hold if the functions $a(x)$ and $b(y(x))$ are connected by the relation

$$|y(x) - \xi|^2 \equiv b^2(y(x))\, |x - \xi|^2, \qquad |\xi| = R,$$

or by

$$(a^2(x) - b^2(y(x)))\, |x|^2 + (1 - b^2(y(x)))\, R^2$$
$$\equiv 2\,(x, \xi)(a(x) - b^2(y(x))), \qquad |\xi| = R.$$

Set $b(y(x)) = \frac{R}{|x|}$, $a(x) = b^2(y(x)) = \frac{R^2}{|x^2|}$. Then the identity (26) holds and for any $x \in Q$ the point

$$y = y(x) = a(x)\,x = \frac{R^2}{|x|^2}\,x \tag{28}$$

is situated outside $\overline{Q}$ because $|y| = R^2/|x| > R$ for $|x| < R$.

For the sphere $\{|\xi| = R\}$ the normal is given by $n_\xi = \frac{\xi}{|\xi|} = \frac{\xi}{R}$, therefore

$$\frac{\partial}{\partial n_\xi}\left(\frac{1}{|x-\xi|^{n-2}}\right) = \left(\nabla_\xi \frac{1}{|x-\xi|^{n-2}},\ n_\xi\right) = \frac{n-2}{|x-\xi|^n}(x-\xi,\ n_\xi)$$
$$= \frac{(n-2)(x-\xi,\ \xi)}{R\,|x-\xi|^n} = \frac{(n-2)((x,\ \xi)-R^2)}{R\,|x-\xi|^n}. \tag{29}$$

Analogously $\frac{\partial}{\partial n_\xi}(1/|y(x)-\xi|^{n-2})$ is computed. Therefore, by (26), we have for $|\xi| = R$

$$\frac{\partial}{\partial n_\xi}\left(\frac{1}{|x-\xi|^{n-2}} - \frac{b^{n-2}(y(x))}{|y(x)-\xi|^{n-2}}\right)$$
$$= \frac{(n-2)}{R\,|x-\xi|^n}\left[(x,\ \xi) - R^2 - \frac{(y(x),\ \xi)-R^2}{b^2(y(x))}\right] = \frac{|x|^2-R^2}{R\,|x-\xi|^n}(n-2).$$

Thus, if $u(x) \in C^2(|x| \leqslant R)$, for any point x, $|x| < R$, we have the equality

$$u(x) = \int\limits_{|\xi|=R} P_R(x,\ \xi)\,u(\xi)\,dS_\xi - \int\limits_{|\xi|<R} G_R(x,\ \xi)\,\Delta u(\xi)\,d\xi, \tag{30}$$

where

$$P_R(x,\ \xi) = \frac{R^2-|x|^2}{\sigma_n R\,|x-\xi|^n}, \tag{31}$$

and

$$G_R(x,\ \xi) = \frac{1}{\sigma_n}\left(\frac{1}{|x-\xi|^{n-2}} - \frac{(R/|x|)^{n-2}}{|(R/|x|)^2 x - \xi|^{n-2}}\right). \tag{32}$$

Exactly in the same manner the representation (30) is established in the two-dimensional case, $n = 2$. The function $P_R(x,\ \xi)$ has the form (31) and

$$G_R(x,\ \xi) = \frac{1}{2\pi}\ln\frac{|x|\left|\xi - \frac{R^2}{|x|^2}x\right|}{R\,|x-\xi|}. \tag{32'}$$

The function $P_R(x,\ \xi)$ defined for $|\xi| = R$, $|x| \leqslant R$, by formula (31) is called the *Poisson kernel* for the first boundary-value problem (the Dirichlet problem) for the Laplace operator in the ball $\{|x| < R\}$.

The function $G_R(x,\ \xi)$ defined for $|\xi| \leqslant R$, $|x| \leqslant R$, by formula (32) when $n > 2$ or by (32′) when $n = 2$ is known as *Green's function* of the first boundary-value problem (the Dirichlet problem) for the Laplace operator in the ball $\{|x| < R\}$.

Lemma 5. *The function* $G_R(x, \xi)$ *defined in the region* $\{x \neq \xi, x \neq \xi R^2/|\xi|^2\}$ *of the space* R_{2n} *by formula* (32) *when* $n > 2$ *or by* (32′) *when* $n = 2$ *is continuous there and has the following properties*:

(a) $G_R(x, \xi) \equiv 0$ *for* $|x| = R$,

(b) $G_R(x, \xi) = G_R(\xi, x)$,

(c) $G_R(x, \xi)$ *is a harmonic function of* x *and of* ξ,

(d) *for* $|x| \leqslant R$, $|\xi| \leqslant R$ $0 \leqslant G_R(x, \xi) \leqslant 1/(\sigma_n |x-\xi|^{n-2})$ *for* $n > 2$ *and* $0 \leqslant G_R(x, \xi) \leqslant \frac{1}{2\pi}\ln\frac{2R}{|x-\xi|}$ *for* $n = 2$.

Proof. Property (a) of the function $G_R(x, \xi)$ follows at once from (32) (or from (32′) if $n = 2$).

For any points x and ξ the identity $|R^2\xi - x|\xi|^2|^2|x|^2 = |R^2x - \xi|x|^2|^2|\xi|^2$ holds, whence it readily follows that the condition $\xi = xR^2/|x|^2$ is equivalent to the condition $x = \xi R^2/|\xi|^2$; therefore if the point (x, ξ) (in R_{2n}) belongs to the domain of definition of the function $G_R(x, \xi)$, then so does the point (ξ, x). Furthermore, the same identity implies the identity $\frac{R}{|x||R^2x/|x|^2-\xi|} = \frac{R}{|\xi||R^2\xi/|\xi|^2-x|}$, hence also the identity $G_R(x, \xi) = G_R(\xi, x)$. This proves Property (b).

It follows from (32) (from (32′) if $n = 2$) that the function $G_R(x, \xi)$ is a harmonic function of ξ. Since the function $G_R(x, \xi)$ is symmetric (Property (b)), it is also a harmonic function of x. This proves Property (c).

The right-hand inequality in Property (d) for $n > 2$ follows from (32). To prove this for $n = 2$, note that for $|x| \leqslant R$, $|\xi| \leqslant R$ $|x||\xi - R^2x/|x|^2| = |\xi|x| - R^2x/|x|| \leqslant |\xi||x| + R^2 \leqslant 2R^2$. Therefore for $|x| \leqslant R$, $|\xi| \leqslant R$ $0 \leqslant \ln\frac{2R^2}{|x||\xi-R^2x/|x|^2|}$, and hence

$$G_R(x, \xi) = \frac{1}{2\pi}\ln\frac{2R}{|x-\xi|} - \frac{1}{2\pi}\ln\frac{2R^2}{|x||\xi-R^2x/|x|^2|} \leqslant \frac{1}{2\pi}\ln\frac{2R}{|x-\xi|}.$$

We shall now establish the left-hand inequalities in (d). First take $x = 0$. By virtue of Property (b), $G_R(0, \xi) = G_R(\xi, 0)$, therefore $G_R(0, \xi) = \frac{1}{\sigma_n}\left(\frac{1}{|\xi|^{n-2}} - \frac{1}{R^{n-2}}\right) \geqslant 0$ if $n > 2$ and $G_R(0, \xi) = \frac{1}{2\pi}\ln\frac{R}{|\xi|} \geqslant 0$ if $n = 2$.

Now take any point x^0, $0 < |x^0| < R$, and the ball $\{|\xi - x^0| < \varepsilon\}$ of radius ε, $0 < \varepsilon < R - |x^0|$, lying in the ball $\{|\xi| < R\}$. According to Properties (a) and (b), $G_R(x^0, \xi) \equiv 0$ when $|\xi| = R$.

With a sufficiently small ε, we have on the sphere $\{|\xi - x^0| = \varepsilon\}$

$$G_R(x^0, \xi) = \frac{1}{\sigma_n \varepsilon^{n-2}} - \frac{(R/|x^0|)^{n-2}}{\sigma_n |x^0 R^2/|x^0|^2 - \xi|^{n-2}}$$

$$\geqslant \frac{1}{\sigma_n}\left(\frac{1}{\varepsilon^{n-2}} - \frac{1}{(R-|x^0|)^{n-2}}\right) > 0 \text{ if } n > 2$$

and

$$G_R(x^0, \xi) = \frac{1}{2\pi} \ln \frac{1}{\varepsilon} + \frac{1}{2\pi} \ln \frac{|x^0|\,|\xi - R^2 x^0/|x^0|^2|}{R}$$

$$\geqslant \frac{1}{2\pi}\left(\ln \frac{1}{\varepsilon} + \ln (R - |x^0|)\right) > 0 \text{ if } n = 2.$$

Therefore, by the maximum principle, the function $G_R(x^0, \xi)$ harmonic in ξ is positive in $\{|\xi| < R\} \setminus \{|\xi - x^0| \leqslant \varepsilon\}$. From this inequality the left-hand inequality in (d) follows, since $\varepsilon > 0$ is arbitrarily small. ∎

The integral representation (30) has been obtained under the assumption that $u(x) \in C^2(|x| \leqslant R)$. Lemma 5 enables us to obtain the same representation with conditions of u relaxed.

Lemma 6. *Let the function* $u(x) \in C(|x| \leqslant R) \cap C^2(|x| < R)$, *and let the function* $\Delta u(x)$ *be bounded in the ball* $\{|x| < R\}$. *Then representation* (30) *holds for any point* x, $|x| < R$.

Proof. Let x^0 be any point of the ball $\{|x| < R\}$ and ρ_0, ρ be numbers such that $|x^0| < \rho_0 \leqslant \rho < R$. Since $u(x) \in C^2(|x| \leqslant \rho)$, in view of (30) for all x, $|x| < \rho$, and, in particular, for $x = x^0$, we have

$$u(x^0) = \int\limits_{|\xi|=\rho} P_\rho(x^0, \xi)\, u(\xi)\, dS_\xi - \int\limits_{|\xi|<\rho} G_\rho(x^0, \xi)\, \Delta u(\xi)\, d\xi. \tag{33}$$

In (33), in the integral over the sphere $\{|\xi| = \rho\}$ we change the variables by putting $\xi = \frac{\eta}{R}\rho$:

$$\int\limits_{|\xi|=\rho} P_\rho(x^0, \xi)\, u(\xi)\, dS_\xi = \left(\frac{\rho}{R}\right)^{n-1} \int\limits_{|\eta|=R} P_\rho\left(x^0, \frac{\eta\rho}{R}\right) u\left(\frac{\eta\rho}{R}\right) dS_\eta.$$

Since the function $(\rho/R)^{n-1} P_\rho(x^0, \eta\rho/R)\, u(\eta\rho/R)$ is continuous in the variables $\eta_1, \ldots, \eta_n$, ρ on the set $\{|\eta| = R, \rho_0 \leqslant \rho \leqslant R\}$ and $(\rho/R)^{n-1} P_\rho(x^0, \eta\rho/R)\, u(\eta\rho/R) \to P_R(x^0, \eta)\, u(\eta)$ as $\rho \to R$, we obtain

$$\lim_{\rho \to R} \int\limits_{|\xi|=\rho} P_\rho(x^0, \xi)\, u(\xi)\, dS_\xi = \int\limits_{|\xi|=R} P_R(x^0, \xi)\, u(\xi)\, dS_\xi. \tag{34}$$

Next consider the second term on the right-hand side of (33). Let $\tilde{G}_\rho(x^0, \xi)$ denote the function equal to $G_\rho(x^0, \xi)$ for $|\xi| < \rho$ and to zero for $|\xi| \geqslant \rho$. Then

$$\int\limits_{|\xi|<\rho} G_\rho(x^0, \xi)\, \Delta u(\xi)\, d\xi = \int\limits_{|\xi|<R} \widetilde{G}_\rho(x^0, \xi)\, \Delta u(\xi)\, d\xi.$$

Clearly, $\widetilde{G}_\rho(x^0, \xi) \to G_R(x^0, \xi)$ as $\rho \to R$ for all $\xi \neq x^0$, $|\xi| < R$. Further, by Property (d) of Lemma 5, the function $\widetilde{G}_\rho(x^0, \xi)\, \Delta u(\xi)$ has a majorant independent of ρ and integrable in the ball $\{|\xi| < R\}$:

$$|\widetilde{G}_\rho(x^0, \xi)\, \Delta u(\xi)| \leqslant \frac{M}{\sigma_n |x^0 - \xi|^{n-2}} \quad \text{if} \quad n > 2$$

and

$$|\widetilde{G}_\rho(x^0, \xi)\, \Delta u(\xi)| \leqslant \frac{M}{2\pi} \ln \frac{2R}{|x^0 - \xi|} \quad \text{if} \quad n = 2,$$

where $M = \sup\limits_{|x| \leqslant R} |\Delta u(x)|$. Therefore, according to the Lebesgue theorem,

$$\lim_{\rho \to R} \int\limits_{|\xi|<\rho} G_\rho(x^0, \xi)\, \Delta u(\xi)\, d\xi = \int\limits_{|\xi|<R} G_R(x^0, \xi)\, \Delta u(\xi)\, d\xi. \tag{35}$$

Letting $\rho \to R$ in (33) and noting (34) and (35), we obtain the representation (30) for any point of the ball $\{|x| < R\}$.

It follows from Lemma 6 that the classical solution (if it exists) of the Dirichlet problem

$$\begin{gathered} \Delta u = f, \quad |x| < R, \\ u|_{\{|x|=R\}} = \varphi, \end{gathered} \tag{36}$$

where the function φ is continuous on the sphere $\{|x| = R\}$ and the function f is bounded and continuous in the ball $\{|x| < R\}$, can be expessed in the form

$$u(x) = \int\limits_{|\xi|=R} P_R(x, \xi)\, \varphi(\xi)\, dS_\xi - \int\limits_{|\xi|<R} G_R(x, \xi)\, f(\xi)\, d\xi. \tag{37}$$

Note that these conditions (as illustrated by the above example) do not guarantee the existence of a classical solution.

According to Theorem 10, in order that a classical solution of the problem (36) may exist it is enough to require that $f(x) \in C^1(|x| \leqslant R)$. Thus Theorem 10 in conjunction with Lemma 6 yields the following result.

Theorem 12. *If* $f(x) \in C^1(|x| \leqslant R)$ *and* $\varphi(x) \in C(|x| = R)$ *then a classical solution of the Dirichlet problem* (36) *exists and can be represented in the form* (37).

Remark. Let Q be a simply connected region in the (x_1, x_2)-plane, and let $z' = F(z)$, $z = x_1 + ix_2$, $z' = x_1' + ix_2'$ be a function analytic in Q and continuously differentiable (with respect to x_1, x_2) in $\bar{Q}$ that performs a one-to-one mapping of Q onto the disc $\{|z'|<R\}$ of radius $R(R=|F(z)|_{z\in\partial Q})$.

By $u(z)=u(x_1,x_2)$ we denote a classical solution of the Dirichlet problem

$$\begin{aligned} \Delta u &= 0, \quad z\in Q, \\ u|_{z\in\partial Q} &= \varphi(z), \end{aligned} \tag{38}$$

where $\varphi(z)\in C(\partial Q)$, and by $u'(z')=u'(x_1', x_2')$ a classical solution of the Dirichlet problem

$$\begin{aligned} \Delta u' &= 0, \quad |z'|<R, \\ u'|_{\{|z'|=R\}} &= \psi(z'), \end{aligned}$$

where $\psi(z')=\varphi(F_{-1}(z'))$ $(F_{-1}(F(z))\equiv z,\ z\in Q)$.

By Theorem 12

$$u'(z')=\frac{1}{2\pi R}\int\limits_{|\zeta'|=R}\frac{R^2-|z'|^2}{|z'-\zeta'|^2}\psi(\zeta')\,|d\zeta'|.$$

From the uniqueness theorem regarding the classical solution of the Dirichlet problem (Theorem 9) and Corollary to Theorem 6 it follows that $u(z)=u(F_{-1}(z'))=u'(z')$. Hence the solution of the problem (38) has the form

$$\begin{aligned} u(z) &= \frac{1}{2\pi R}\int\limits_{\partial Q}\frac{R^2-|F(z)|^2}{|F(z)-F(\zeta)|^2}|F'(\zeta)|\,\varphi(\zeta)\,|d\zeta| \\ &= \frac{1}{2\pi}\int\limits_{\partial Q}\frac{|F(\zeta)|^2-|F(z)|^2}{|F(\zeta)-F(z)|^2}\,\frac{|F'(\zeta)|}{|F(\zeta)|}\,\varphi(\zeta)\,|d\zeta|. \end{aligned}$$

4. Harmonic Functions in Unbounded Regions. Let Q be an unbounded region of the space R_n, and let its complement $R_n\diagdown Q$ contain at least one interior point; we take this point as the origin.

Consider a one-to-one mapping

$$x'=\frac{x}{|x|^2} \tag{39}$$

of the region $R_n\diagdown\{0\}$ onto itself. This mapping is known as the *inversion mapping* (with respect to the sphere $\{|x|=1\}$); we have already used this in the preceding subsection. Under the mapping (39) the sphere $\{|x|=1\}$ is transformed into itself, the region $\{0<|x|<1\}$ into the region $\{|x|>1\}$ and vice-versa. It is clear that the inverse of mapping (39) has the form

$$x=\frac{x'}{|x'|^2},$$

that is, is also an inversion mapping.

As a result of the inversion mapping, the rigion Q is transformed into a bounded region Q'. Note that the origin becomes a boundary point of Q'. When the boundary ∂Q is unbounded, the origin is a boundary point of the set $\overline{Q}'$ also. If, however, ∂Q is bounded, that is, if Q is exterior to some bounded set, then the origin is an isolated boundary point of Q', and therefore is an interior point of the set $\overline{Q}'$.

Let a function $u(x)$ be defined in the region Q. The function $u'(x')$ defined in Q' by the relation

$$u'(x')=\frac{1}{|x'|^{n-2}}\,u\left(\frac{x'}{|x'|^2}\right) \tag{40}$$

is known as the *Kelvin transform* of the function u.

It follows from (39) and (40) that

$$u(x)=\frac{1}{|x|^{n-2}}\,u'\left(\frac{x}{|x|^2}\right), \tag{41}$$

that is, the transformation inverse to (40) is also a Kelvin transformation.

Lemma 7. *If a function $u(x)$ is harmonic in the region Q, then the function $u'(x')$ is harmonic in the region Q'.*

Proof. Let Q_1' be any strictly interior subregion of Q', and let Q_1 be its original under the inversion mapping. Then Q_1 is a strictly interior bounded subregion of Q. Since the function u is harmonic in Q_1 and belongs to $C^2(\overline{Q}_1)$, by formula (9) for all $x\in Q_1$

$$u(x)=\int\limits_{\partial Q_1}\left[\frac{\mu(\xi)}{|x-\xi|^{n-2}}+\nu(\xi)\,\frac{\partial}{\partial n_\xi}\left(\frac{1}{|x-\xi|^{n-2}}\right)\right]dS_\xi,$$

where $\mu(\xi)=\frac{1}{(n-2)\,\sigma_n}\left.\frac{\partial u(\xi)}{\partial n}\right|_{\partial Q_1}$, $\nu(\xi)=-\left.\frac{u(\xi)}{(n-2)\,\sigma_n}\right|_{\partial Q_1}$ are continuous functions on ∂Q_1 (to be definite, we are taking the case $n>2$; when $n=2$, arguments are exactly the same). Therefore, by (40),

$$u'(x')=\int\limits_{\partial Q_1}\left[\frac{\mu(\xi)}{|x'|^{n-2}\left|\frac{x'}{|x'|^2}-\xi\right|^{n-2}}\right.$$
$$\left.+\nu(\xi)\,\frac{\partial}{\partial n_\xi}\left(\frac{1}{|x'|^{n-2}\left|\frac{x'}{|x'|^2}-\xi\right|^{n-2}}\right)\right]dS_\xi \tag{42}$$

for all $x'\in Q_1'$.

From Property (c) in Lemma 5 of the preceding subsection it follows that the function $|x'|^{2-n}\left|\frac{x'}{|x'|^2}-\xi\right|^{2-n}$, and hence the function $\frac{\partial}{\partial n_\xi}\left(|x'|^{2-n}\left|\frac{x'}{|x'|^2}-\xi\right|^{2-n}\right)$ are harmonic functions of x' for

$\frac{x'}{|x'|^2} \neq \xi$. Thus the integrand function in (42) and any of its derivatives with respect to x' are jointly continuous in ξ, x' and are harmonic functions of x' for $\xi \in \partial Q_1$ and $x' \in Q'$ (the condition $\frac{x'}{|x'|^2} \notin \partial Q_1$ is equivalent to the condition $x' \notin \partial Q_1'$).

Consequently, for any point $x^1 \in Q_1'$ the identity (42) can be differentiated under the integral sign with respect to x' any number of times, and, moreover, $\Delta u' = 0$. Since Q_1' is arbitrary, the conclusion of lemma follows.

Thus, by means of Lemma 7, the investigation of harmonic functions in an unbounded region whose complement contains interior points is reduced to that of harmonic functions in a bounded region.

Let the complement $R_n \setminus Q$ of the region Q in R_n be bounded. A harmonic function $u(x)$ defined in Q is called *regular at infinity* if $u(x) = o(1)$ when $n > 2$ or $u(x) = o(\ln |x|)$ when $n = 2$ as $|x| \to \infty$.

Suppose that the complement of Q contains interior points (this includes, as above, the origin). As a result of inversion mapping (39) the region Q is transformed into a bounded region Q' having an isolated boundary point, the origin. If the harmonic function defined in Q is regular at infinity, then, by (40), the Kelvin transform $u'(x')$ of this function is, as $x' \to 0$, $o(|x'|^{2-n})$ if $n > 2$ and $o(\ln |x'|)$ if $n = 2$, that is, as $x' \to 0$, $u'(x') = o(U(x'))$, where U is the fundamental solution of the Laplace equation. Then, according to the theorem on removal of singularity, $\lim\limits_{x' \to 0} u'(x') = A$ exists and the function $u'(x')$ redefined at the origin by A (the same notation $u'(x')$ is used) is harmonic in the region $Q_0' = Q' \cup \{0\}$.

Thus we have obtained the following result.

Lemma 8. *Let the function $u(x)$ be harmonic in an unbounded region Q whose complement is bounded and contains interior points, and let this function be regular at infinity. Then its Kelvin transform is harmonic in Q_0'.*

According to Theorem 5, the function $u'(x')$ is analytic in x' in $Q' \cup \{0\}$. Therefore, in particular, there is a number R_0 such that the function $u'(x')$ has a Taylor series expansion

$$u'(x') = \sum_{\alpha} A_\alpha x'^\alpha$$

in the ball $\{|x'| < R_0\}$ that converges (together with all the derivatives) absolutely (and uniformly); here $A_\alpha = \frac{1}{\alpha!} D^\alpha u'(0)$, $A_0 = A$. But then, in view of (39) and (41), for all x, $|x| > 1/R_0$,

$$u(x) = \sum_{\alpha} A_\alpha \frac{x^\alpha}{|x|^{2|\alpha|+n-2}}, \qquad (43)$$

where the series on the right-hand side of this equality converges for $|x| > 1/R_0$ absolutely and uniformly together with all its derivatives.

Let $T(x)$ denote the function $u(x)-\frac{A_0}{|x|^{n-2}}$, that is, for $|x| > 1/R_0$ $T(x)=\sum\limits_{|\alpha|\geqslant 1} A_\alpha \frac{x^\alpha}{|x|^{2|\alpha|+n-2}}$. Since for any $\alpha=(\alpha_1, \ldots, \alpha_n)$ $D^\alpha u = D^\alpha \frac{A_0}{|x|^{n-2}} + D^\alpha T(x)$ and since $|D^\alpha T| \leqslant \frac{C_\alpha}{|x|^{n+|\alpha|-1}}$, where C_α is a positive constant, it follows that

$$\left| D^\alpha u - D^\alpha \frac{A_0}{|x|^{n-2}} \right| \leqslant \frac{C_\alpha}{|x|^{n+|\alpha|-1}}. \tag{44}$$

In particular,

$$\begin{gathered} \left| u(x) - \frac{A_0}{|x|^{n-2}} \right| \leqslant \frac{\text{const}}{|x|^{n-1}}, \\ \left| \nabla u - \frac{A_0 (2-n) x}{|x|^n} \right| \leqslant \frac{\text{const}}{|x|^n}. \end{gathered} \tag{45}$$

The assertions established just now regarding the behaviour for large $|x|$ of the function $u(x)$ harmonic in Q and regular at infinity in the case when the complement of Q is bounded and contains interior points remain always valid if the complement of Q is bounded (in particular, Q may coincide with the whole R_n). Indeed, since we are interested in the values of the function $u(x)$ for sufficiently large $|x|$ only, it may be assumed to be defined only on $Q_1 = \{|x| > R_1\}$, which is a subregion of Q for sufficiently large R_1. And Q_1 is the complement of the set $\{|x| \leqslant R_1\}$ for which the origin is an interior point.

Thus we have proved the following result.

Theorem 13. *Let the complement of a region Q be bounded. Then for any function $u(x)$ harmonic in Q and regular at infinity there is a constant $R > 0$ such that for all x, $|x| > R$, the function $u(x)$ has a series expansion* (43) *which converges absolutely and uniformly together with all its derivatives, and the inequalities* (44) *hold.*

From Theorem 13 it follows, in particular, that if the function $u(x)$ is harmonic in an n-dimensional, $n > 2$, region Q, which is exterior of a bounded set, and decreases at infinity, then it decreases not slower than the fundamental solution of the Laplace equation, and, moreover, the limit of $u(x)\,|x|^{n-2}$ exists as $|x| \to \infty$. When $n = 2$, the function $u(x)$ harmonic in Q which grows slower than the fundamental solution is, in fact, bounded, and its limit exists as $|x| \to \infty$.

Remark. If the complement of Q is unbounded, then for a function $u(x)$ which is harmonic in Q and satisfies the condition

$$u(x) = o(1) \text{ as } |x| \to \infty, \qquad x \in Q, \text{ for } n > 2$$

or

$$u(x) = o(\ln|x|) \text{ as } |x| \to \infty, \quad x \in Q, \text{ for } n = 2,$$

the conclusions of Theorem 13 are, generally speaking, not true. For instance, when $n = 2$, the function $\arg(x_1 + ix_2)$ is harmonic and bounded in $R_2 \diagdown \{x_2 = 0,\ x_1 \geqslant 0\}$ but does not have a limit as $|x| \to \infty$.

Suppose that a function $u(x)$ is harmonic throughout the space R_n. We shall say that $u(x)$ is semibounded if it is bounded above or below, that is, if for all $x \in R_n$ the inequality $u(x) \geqslant M$ or $u(x) \leqslant M$ holds, respectively, with some constant M.

Theorem 14. *A semibounded function which is harmonic in R_n is constant.*

Proof. Since for a function $u(x)$ bounded above the function $-u(x)$ is bounded below, it suffices to establish the theorem for the case $u(x) \geqslant M$ in R_n. In this case the harmonic function $v(x) = u(x) - M \geqslant 0$ in R_n. The theorem will be proved if we show that $v(x) \equiv \text{const}$.

Take an arbitrary point $x^0 \in R_n$ and the ball $\{|x| < R\}$ of radius $R > |x^0|$. The Dirichlet problem for the Laplace equation in the ball $\{|x| < R\}$ with boundary function $v|_{\{|x|=R\}}$ has a unique classical solution, so for all x, $|x| < R$,

$$v(x) = \int\limits_{|\xi|=R} P_R(x, \xi)\, v(\xi)\, dS_\xi,$$

where $P_R(x, \xi)$ is the Poisson kernel of the Dirichlet problem for the Laplace equation in the ball $\{|x| < R\}$ (formula (31)). In particular, for $x = x^0$ we have

$$v(x^0) = \int\limits_{|\xi|=R} P_R(x^0, \xi)\, v(\xi)\, dS_\xi = \frac{R^2 - |x^0|^2}{\sigma_n R} \int\limits_{|\xi|=R} \frac{v(\xi)}{|x^0 - \xi|^n}\, dS_\xi.$$

Since for $|\xi| = R$

$$R - |x^0| \leqslant |x^0 - \xi| \leqslant R + |x^0|,$$

we have (note that the function $v(\xi) \geqslant 0$)

$$\frac{1}{\sigma_n R^{n-1}} \int\limits_{|\xi|=R} v(\xi)\, dS_\xi \cdot \frac{(R^2 - |x^0|^2)\, R^{n-2}}{(R + |x^0|)^n} \leqslant v(x^0)$$

$$\leqslant \frac{1}{\sigma_n R^{n-1}} \int\limits_{|\xi|=R} v(\xi)\, dS_\xi \cdot \frac{(R^2 - |x^0|^2)\, R^{n-2}}{(R - |x^0|)^n}$$

or, in view of the first mean value theorem,

$$\frac{R^{n-2}(R^2 - |x^0|^2)}{(R + |x^0|)^n}\, v(0) \leqslant v(x^0) \leqslant \frac{R^{n-2}(R^2 - |x^0|^2)}{(R - |x^0|^n}\, v(0).$$

Letting $R \to \infty$ in this inequality, we find that $v(x^0) = v(0)$. Consequently, since the point x^0 is arbitrary, $v(x) \equiv \text{const.}$ ∎

Corollary. *If the function $u(x)$ harmonic in R_n satisfies the inequality $|u(x)| \leqslant C(1 + |x|)^k$, where C is a constant and k a non-negative integer, for all $x \in R_n$, then $u(x)$ is a polynomial of degree not exceeding k.*

Proof. When $k = 0$, this result is contained in Theorem 14. Assume that $k > 0$, and take an arbitrary number $R > 1$. By Lemma 3, Subsec. 2, for any $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| = k$

$$\max_{|x| \leqslant R} |D^\alpha u| \leqslant k^k \left(\frac{n}{R}\right)^k \max_{|x| \leqslant 2R} |u(x)|$$

$$\leqslant Ck^k \left(\frac{n}{R}\right)^k (1 + 2R)^k \leqslant Ck^k \left(\frac{n}{R}\right)^k (3R)^k = C\,(3kn)^k.$$

From this inequality it follows that for any α, $|\alpha| = k$, the function $D^\alpha u$ harmonic in R_n is bounded in R_n. According to Theorem 14, the functions $D^\alpha u$, $|\alpha| = k$, are constant in R_n. Consequently, $u(x)$ is a polynomial of degree not exceeding k. ∎

We have established above some properties of harmonic functions in unbounded regions. It was shown, in particular, that the Kelvin transformation reduces the investigation of a harmonic function in an unbounded region (whose complement contains interior points) to that of a harmonic function in a bounded region.

Let us now examine boundary-value problems for the Laplace equation in unbounded regions. First of all note that for an unbounded region the usual conditions on the solution (imposed in the case of a bounded region) are not enough to guarantee its uniqueness. For instance, all the functions $c \ln r$, $c(r^k - r^{-k}) \cos k\theta$, $c(r^k - r^{-k}) \sin k\theta$, $k = 1, 2, \ldots$, where c is any constant, $x_1 = r \cos\theta$, $x_2 = r \sin\theta$, are harmonic in the region $\{r > 1\} \subset R_2$, continuous in its closure and vanish on the boundary $\{r = 1\}$. Therefore in defining a solution some additional condition characterizing its behaviour at infinity should naturally be imposed.

Let the region $Q = R_n \setminus \bigcup_{i=1}^{N} Q_i$, where Q_i, $i = 1, \ldots, N$, are bounded regions with disjoint boundaries.

A function $u(x) \in C^2(Q)$ is called the (classical) *solution of the Dirichlet problem for the Laplace equation in Q*:

$$\begin{gathered} \Delta u = 0, \quad x \in Q, \\ u|_{\partial Q} = \varphi \end{gathered} \tag{46}$$

if it is harmonic in Q, continuous in $\bar{Q}$, satisfies the boundary condition in (46) and is regular at infinity.

A function $u(x) \in C^2(Q)$ is called the (classical) *solution of the third boundary-value problem for the Laplace equation in* Q:

$$\Delta u = 0, \quad x \in Q,$$

$$\left(\frac{\partial u}{\partial n} + \sigma(x)\, u\right)\Big|_{\partial Q} = \varphi \tag{47}$$

if it is harmonic in Q, continuously differentiable in $\bar{Q}$, satisfies the boundary condition in (47) and is regular at infinity.

If $\sigma \equiv 0$, the third boundary-value problem is designated the *second boundary-value problem* or the *Neumann problem.*

Let Q' denote a bounded region which is the image of the region Q under the inversion mapping (the origin is an interior point of the complement of Q).

Assume that $u(x)$ is a solution of the problem (46). It follows from Lemma 8 that the function $u'(x')$, the Kelvin transform of $u(x)$ (redefined with respect to continuity at the origin), is harmonic in $Q_0' = Q' \cup \{0\}$. Furthermore, it is clear that $u'(x') \in C(\bar{Q}_0')$ and $u'(x')|_{x' \in \partial Q_0'} = \varphi'(x')$, where $\varphi'(x') = \frac{1}{|x'|^{n-2}}\varphi\left(\frac{x'}{|x'|^2}\right)$. This means that $u'(x')$ is a classical solution of the Dirichlet problem for the Laplace equation in the (bounded) region Q_0' with boundary function $\varphi'(x')$.

Conversely, if $u'(x')$ is a classical solution of the Dirichlet problem for the Laplace equation in Q_0' with boundary function $\varphi'(x')$, then the function $u(x)$, its Kelvin transform, is harmonic in Q, continuous in $\bar{Q}$, satisfies the boundary condition $u|_{\partial Q} = \varphi$ and is clearly regular at infinity, that is, $u(x)$ is a classical solution of the problem (46).

Therefore the existence and uniqueness theorems regarding classical solution of the Dirichlet problem in a bounded region (Theorems 9 and 10) imply the following result.

Theorem 15. *There exists a unique classical solution of the Dirichlet problem* (46) *with any continuous boundary function* φ.

The Kelvin transformation similarly reduces the investigation of the third boundary-value problem in an unbounded region Q to that in a bounded region Q_0'. We confine to the proof of the uniqueness theorem.

Theorem 16. *The third boundary-value problem, with* $\sigma(x) \geqslant 0$, $\sigma(x) \not\equiv 0$, *for the Laplace equation in* Q *cannot have more than one solution.*

The second boundary-value problem for the Laplace equation in Q *cannot have more than one solution when* $n > 2$, *while in the case* $n = 2$ *the solution* (*if it exists*) *is determined up to a constant term.*

Proof. Suppose that the third (second) boundary-value problem has two solutions $u_1(x)$ and $u_2(x)$. Then the function $u(x) = u_1(x) -$

—$u_2(x)$ is harmonic in Q, continuously differentiable in $\overline{Q}$, satisfies the boundary condition $\left(\frac{\partial u}{\partial n}+\sigma u\right)\Big|_{\partial Q}=0$ and is regular at infinity. We take a number $R>0$ so large that the region $\{|x|>R\}$ is contained in Q, and in the region $Q_R=Q\cap\{|x|<R\}$ apply Green's formula

$$0=\int\limits_{Q_R}u\Delta u\,dx=-\int\limits_{Q_R}|\nabla u|^2\,dx+\int\limits_{\partial Q}\frac{\partial u}{\partial n}u\,dS$$

$$+\int\limits_{|x|=R}\frac{\partial u}{\partial n}u\,dS=-\int\limits_{Q_R}|\nabla u|^2\,dx-\int\limits_{\partial Q}\sigma u^2\,dS+\int\limits_{|x|=R}\frac{\partial u}{\partial n}u\,dS.$$

This yields the identity

$$\int\limits_{Q_R}|\nabla u|^2\,dx+\int\limits_{\partial Q}\sigma u^2\,ds=\int\limits_{|x|=R}\frac{\partial u}{\partial n}u\,dS. \tag{48}$$

By Theorem 13, $u\,|_{\{|x|=R\}}=O\left(\frac{1}{R^{n-2}}\right)$ and $\frac{\partial u}{\partial n}\Big|_{\{|x|=R\}}=O\left(\frac{1}{R^{n-1}}\right)$ when $n>2$ and $\frac{\partial u}{\partial n}\Big|_{\{|x|=R\}}=O\left(\frac{1}{R^2}\right)$ when $n=2$. Therefore

$$\int\limits_{|x|=R}\frac{\partial u}{\partial n}u\,dS=O\left(\frac{1}{R^{n-2}}\right)\quad\text{if } n>2$$

and

$$\int\limits_{|x|=R}\frac{\partial u}{\partial n}u\,dS=O\left(\frac{1}{R}\right)\quad\text{if n}=2.$$

Passing to the limit in (48) as $R\to\infty$, we obtain

$$\int\limits_{Q}|\nabla u|^2\,dx+\int\limits_{\partial Q}\sigma u^2\,dS=0$$

Since $\sigma\geqslant 0$, this identity is equivalent to two identities

$$\int\limits_{Q}|\nabla u|^2\,dx=0\quad\text{and}\quad\int\limits_{\partial Q}\sigma u^2\,dS=0. \tag{49}$$

The first identity in (49) implies that $u\equiv c_0=\text{const}$ in Q. If $n>2$, we find, noting the regularity of $u(x)$ at infinity, $c_0=0$, that is, $u_1=u_2$ in Q.

If $n=2$ and $\sigma(x)\geqslant 0$, $\sigma(x)\not\equiv 0$ (the third boundary-value problem), then the relation $c_0=0$ is a consequence of the second identity in (49).

If, however, $n=2$ and $\sigma(x)\equiv 0$ (the second boundary-value problem), then the function $u(x)\equiv c_0$, with an arbitrary constant c_0, is harmonic and regular in Q and satisfies the homogeneous boundary condition $\frac{\partial u}{\partial n}\Big|_{\partial Q}=0$. ▮

PROBLEMS ON CHAPTER IV

1. Show that a function $u(x)$ belonging to $L_{2,\,loc}(Q)$ and satisfying the identity $\int\limits_Q u\,\Delta v\,dx = 0$ for all $v \in \overset{\circ}{C}{}^{\infty}(\overline{Q})$ is harmonic in Q.

2. Complete the set of functions belonging to $L_2\,(Q)$ and harmonic in Q in the norm of $L_2\,(Q)$.

3. Suppose that the function $u \in H^2_{loc}(Q) \cap \overset{\circ}{C}\,(\overline{Q})$ and $\partial Q \in C^2$. Prove that $u \in H^2_{\mathscr{D}}(Q)$ provided $\Delta u \in L_2\,(Q)$.

Note that the result of Problem 3 implies that the classical solution of the Dirichlet problem for Poisson's equation $\Delta u = f$, $u\,|_{\partial Q} = 0$ with right-hand side f belonging to $L_2(Q)$ is a generalized solution and even a solution almost everywhere. Accordingly, the classical eigenfunctions of the first boundary-value problem for the Laplace operator are generalized eigenfunctions.

4. Let $\partial Q \in C^2$. On the set of all functions $u(x) \in C^2(Q) \cap \overset{\circ}{C}\,(\overline{Q})$ such that $\Delta u(x) \in L_2(Q)$, we define a scalar product $\int\limits_Q \Delta u \cdot \Delta v\,dx$. Complete this set in the norm generated by this scalar product.

5. Suppose that the boundary ∂Q of a region Q belongs to C^k. Establish the following results.

(a) In the Hilbert space $H^k_{\mathscr{D}}\,(Q)$ the following scalar products equivalent to the usual scalar product may be defined:

$$(f,\,g)'_{H^k_{\mathscr{D}}(Q)} = \begin{cases} (\Delta^{k/2}f,\ \Delta^{k/2}g)_{L_2(Q)} & \text{for even } k, \\ (\Delta^{(k-1)/2}f,\ \Delta^{(k-1)/2}g)_{H^1(Q)} & \text{for odd } k \end{cases}$$

and

$$(f,\,g)''_{H^k_{\mathscr{D}}(Q)} = \sum_{s=1}^{\infty} f_s\bar{g}_s|\,\lambda_s\,|^k,$$

where $f_s = (f,\ u_s)_{L_2(Q)}$, while u_s and λ_s are sth eigenfunction and the corresponding eigenvalue of the Dirichlet problem for the Laplace operator in Q.

(b) In the Hilbert space $H^k_{\mathscr{N}}\,(Q)$ one may introduce the following scalar products equivalent to the usual scalar product:

$$(f,\,g)'_{H^k_{\mathscr{N}}(Q)} = \begin{cases} (\Delta^{k/2}f,\ \Delta^{k/2}g)_{L_2(Q)} + (f,\,g)_{L_2(Q)} & \text{for even } k, \\ (\Delta^{(k-1)/2}f,\ \Delta^{(k-1)/2}g)_{H^1(Q)} + (f,\,g)_{L_2(Q)} & \text{for odd } k, \end{cases}$$

and

$$(f,\,g)''_{H^k_{\mathscr{N}}(Q)} = \sum_{s=1}^{\infty} f_s\bar{g}_s\,(\,|\,\lambda_s\,|^k + 1),$$

where $f_s = (f,\ u_s)_{L_2(Q)}$, while u_s and λ_s are sth eigenfunction and the corresponding eigenvalue of the Neumann problem for the Laplace operator in Q.

6. Suppose that a function $u(x) \in C(\overline{Q})$, and for any point $x \in Q$ there is a number $r = r\,(x) > 0$ such that the ball $S_r(x) = \{\,|\,\xi - x\,| < r\} \subset Q$ and $u(x) = \frac{1}{\sigma_n r^{n-1}} \int\limits_{\partial S_r(x)} u\,(\xi)\,dS_\xi$. Show that $u(x)$ is harmonic in Q.

7. Suppose that the function $u(x) \in C^1(Q)$ and $\int\limits_S \frac{\partial u}{\partial n} dS = 0$ for any sphere S lying in Q. Show that $u(x)$ is harmonic in Q.

8. Show that the first eigenvalue of the first boundary-value problem for the Laplace operator in the region Q, $\partial Q \in C^2$, is nondegenerate and the corresponding eigenfunction does not vanish in Q.

9. Show that a function $u(x) \in C^2(Q)$ and satisfying in Q the Helmholtz equation $\Delta u + \lambda u = 0$, where λ is a constant, is analytic in Q.

Note that the result of Problem 9 implies that eigenfunctions of any boundary-value problem for the Laplace operator in Q are analytic in Q.

10. Let $\lambda_k(Q_1)$ and $\lambda_k(Q_2)$ be kth eigenvalues of the first boundary-value problem for the Laplace operator in regions Q_1 and Q_2, $Q_1 \Subset Q_2$. Prove that $\lambda_k(Q_1) < \lambda_k(Q_2)$ for all $k = 1, 2, \ldots$.

11. Let $\widetilde{L}_2(\partial Q)$ (∂Q is the boundary of an n-dimensional region Q) be the subspace of the space $L_2(\partial Q)$ that contains all the functions orthogonal (in $L_2(\partial Q)$ with a scalar product) to constant functions. For any function $\psi(x) \in \widetilde{L}_2(\partial Q)$ there exists a unique generalized solution $u(x)$ of the Neumann problem for the Laplace equation in Q with boundary function ψ whose trace on ∂Q $u|_{\partial Q} = \varphi \in \widetilde{L}_2(\partial Q)$. Thus an operator A is defined on $\widetilde{L}_2(\partial Q)$ that associates with every function $\psi \in \widetilde{L}_2(\partial Q)$ a function $\varphi \in \widetilde{L}_2(\partial Q)$: $A\psi = \varphi$.

Establish the following results.

(a) The eigenvalues λ_k, $k = 1, 2, \ldots$, of the operator A are positive; the eigenfunctions e_k, $Ae_k = \lambda_k e_k$, $k = 1, 2, \ldots$, constitute an orthonormal basis for $\widetilde{L}_2(\partial Q)$.

(b) There exists a generalized solution $u_k(x)$ of the Dirichlet problem for the Laplace equation in Q with boundary function $\sqrt{\lambda_k}\, e_k$, $k = 1, 2, \ldots$. The system $u_k(x)$, $k = 1, 2, \ldots$, constitutes an orthonormal basis for the space with scalar product $\int\limits_Q \nabla u \nabla \bar{v}\, dx$ which consists of all the harmonic functions in Q belonging to $H^1(Q)$ whose trace on ∂Q lies in $\widetilde{L}_2(\partial Q)$.

(c) For any function $\psi \in \widetilde{L}_2(\partial Q)$ the series $\sum\limits_{k=1}^{\infty} \psi_k \sqrt{\lambda_k}\, u_k(x)$, where $\psi_k = (\psi, e_k)_{L_2(\partial Q)}$, converges in $H^1(Q)$ and represents a generalized solution of the Neumann problem for the Laplace equation in Q with the boundary function ψ.

(d) Let $\varphi \in L_2(\partial Q)$. In order that there may exist a generalized solution $u(x)$ of the Dirichlet problem for the Laplace equation in Q with boundary function φ it is necessary and sufficient that the series $\sum\limits_{k=1}^{\infty} \frac{|\varphi_k|^2}{\lambda_k}$, where $\varphi_k = (\varphi, e_k)_{L_2(\partial Q)}$, converge. Moreover, $u(x) = \frac{1}{|\partial Q|} \int\limits_{\partial Q} \varphi\, dS + \sum\limits_{k=1}^{\infty} \varphi_k u_k(x)$.

(e) Find the eigenvalues and eigenfunctions of the operator A when the region Q is the disc $\{|x| < 1\}$ (a two-dimensional case), and show that the condition of part (d) in this case coincides with the condition in Theorem 13, Sec. 1.8.

12. In order that the function $f(\varphi)$ defined on the boundary $\{r = 1\}$ of the unit disc $\{r < 1\}$ in the plane $x_1 = r \cos \varphi$, $x_2 = r \sin \varphi$ and belonging to $L_2(0, 2\pi)$ may be the boundary value of some function in $H^1(r < 1)$ it is

necessary and sufficient that the integral $\int\limits_0^{2\pi} \frac{dt}{t^2} \int\limits_0^{2\pi} (f(\varphi+t) - f(\varphi))^2 \, d\varphi$ converge.

A function $u(x) \in C^4(Q) \cap C^1(\overline{Q})$ satisfying the equation

$$\Delta^2 u = f, \qquad x \in Q, \tag{1}$$

and the boundary conditions

$$u|_{\partial Q} = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_{\partial Q} = 0 \tag{2}$$

is called the classical solution of the Dirichlet problem for the equation $\Delta^2 u = f$ in Q. A function $u(x) \in C^4(Q) \cap C^2(\overline{Q})$ satisfying Eq. (1) and the boundary conditions

$$u|_{\partial Q} = 0, \qquad \Delta u|_{\partial Q} = 0 \tag{3}$$

is known as the classical solution of the Riquet problem for the equation $\Delta^2 u = f$ in Q.

Let the function $f \in L_2(Q)$. A function u belonging to $\overset{\circ}{H}{}^2(Q)$ and satisfying the integral identity

$$\int\limits_Q \Delta u \Delta \bar{v} \, dx = \int\limits_Q f \bar{v} \, dx \tag{4}$$

for all $v \in \overset{\circ}{H}{}^2(Q)$ is called the generalized solution of the Dirichlet problem (1), (2). A function u belonging to $H^2_{\mathscr{D}}(Q)$ and satisfying the integral identity (4) for all $v \in H^2_{\mathscr{D}}(Q)$ is designated as the generalized solution of the Riquet problem (1), (3).

13. Let $\partial Q \in C^2$. Prove the following results.

(a) The classical solutions $u(x)$ of the problems (1), (2) and (1), (3) belonging to $C^4(\overline{Q})$ are generalized solutions of these problems.

(b) The generalized solutions of the problems (1), (2) and (1), (3) exist for all $f \in L_2(Q)$ and are unique.

Let Q be a ball of radius R: $Q = \{|x| < R\}$. By S_1 denote the hemisphere $\{|x| = R\} \cap \{x_1 > 0\}$ and by S_2 the hemisphere $\{|x| = R\} \cap \{x_1 \leqslant 0\}$. A function $u(x)$ belonging to $C^2(Q) \cap C^1(Q \cup S_1) \cap C(\overline{Q})$ and satisfying Poisson's equation

$$\Delta u = f, \qquad x \in Q, \tag{5}$$

and the boundary condition

$$\left.\frac{\partial u}{\partial n}\right|_{S_1} = 0, \qquad u|_{S_2} = 0 \tag{6}$$

is termed the classical solution of the problem (5), (6).

By $\widetilde{H}^1(Q)$ denote the subspace of the space $H^1(Q)$ which contains all the functions $u \in H^1(Q)$ whose trace on S_2 is zero. Let $f \in L_2(Q)$. By a generalized solution of the problem (5), (6) is meant the function $u \in \widetilde{H}^1(Q)$ which satisfies the integral identity

$$\int\limits_Q \nabla u \nabla \bar{v} \, dx = -\int\limits_Q f \bar{v} \, dx$$

for all $v \in \widetilde{H}^1(Q)$.

14. Prove that for any function $f \in L_2(Q)$ the generalized solution of the problem (5), (6) exists and is unique.

Let Q be a bounded two-dimensional region with boundary $\partial Q \in C^3$, and let $l(x)$ be a twice continuously differentiable vector defined on ∂Q, $|l(x)| = 1$, that makes an angle $\alpha(x)$, $|\alpha(x)| < \frac{\pi}{2}$, with the (outward) normal vector to ∂Q ($\alpha(x) = (\widehat{n, l})$). A function $u(x)$ belonging to $C^1(\overline{Q}) \cap C^2(Q)$ and satisfying the equation

$$\Delta u - u = f, \qquad x \in Q, \tag{7}$$

as well as the boundary condition

$$\left.\frac{\partial u}{\partial l}\right|_{\partial Q} = 0 \tag{8}$$

is called the classical solution of the directional derivative problem (7), (8).

Let $A(x)$ be a function belonging to $C^2(\overline{Q})$ whose value on the boundary ∂Q is $\tan \alpha(x)$. Let $f \in L_2(Q)$. By a generalized solution of the problem (7), (8) is meant a function $u \in H^1(Q)$ satisfying the integral identity

$$\int_Q (\nabla u \nabla \bar{v} + u\bar{v})\,dx + \int_Q A\,(u_{x_2}\bar{v}_{x_1} - u_{x_1}\bar{v}_{x_2})\,dx$$
$$+ \int_Q (A_{x_1}u_{x_2} - A_{x_2}u_{x_1})\,\bar{v}\,dx = -\int_Q f\bar{v}\,dx$$

for all $v \in H^1(Q)$.

15. Prove the following assertions.

(a) A classical solution of the problem (7), (8) is a generalized solution.

(b) If $\alpha(x) \equiv \text{const}$, then for any $f \in L_2(Q)$ there exists a unique generalized solution of the problem (7), (8); this solution does not depend on the method of extending the function $\tan \alpha(x)$ by $A(x)$ into Q.

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CHAPTER V

HYPERBOLIC EQUATIONS

In this chapter we shall study the Cauchy problem and the mixed problems for a hyperbolic equation of the form

$$u_{tt} - \operatorname{div}(k(x)\nabla u(x, t)) + a(x)u(x, t) = f(x, t).$$

Here $(x, t) = (x_1, \dots, x_n, t)$ is a point of the $(n+1)$-dimensional space R_{n+1}, $x \in R_n$, $t \in R_1$, $\nabla v(x, t) = \left(\frac{\partial v}{\partial x_1}, \dots, \frac{\partial v}{\partial x_n}\right)$ and $\operatorname{div}(w_1(x, t), \dots, w_n(x, t)) = \frac{\partial w_1}{\partial x_1} + \dots + \frac{\partial w_n}{\partial x_n}$; by $\Delta v(x, t)$ we shall mean $\operatorname{div} \nabla v(x, t) = \frac{\partial^2 v}{\partial x_1^2} + \dots + \frac{\partial^2 v}{\partial x_n^2}$. The data of the problem will be assumed real-valued functions and we shall examine only real-valued solutions of these problems. Therefore H^k, C^k, $k = 0, 1, \dots$, will henceforth mean corresponding real spaces.

§ 1. PROPERTIES OF SOLUTIONS OF WAVE EQUATION. THE CAUCHY PROBLEM FOR WAVE EQUATION

1. Properties of the Solutions of the Wave Equation. Let us examine the simplest hyperbolic equation of the second order, the wave equation,

$$\Box u(x, t) \equiv u_{tt} - \Delta u \equiv u_{tt} - \sum_{i=1}^{n} u_{x_i x_i} = f(x, t). \tag{1}$$

First of all we shall obtain some special solutions of the homogeneous wave equation ($\Box u = 0$) depending only on $t/|x|$. The function $v(x, t) = w(t/|x|)$ which for $|x| \neq 0$ is a solution of the homogeneous wave equation satisfies the ordinary differential equation

$$(z^2 - 1)\frac{d^2 w}{dz^2} + (3 - n)z\frac{dw}{dz} = 0.$$

The general solution of this equation in each of the intervals $(-\infty, -1)$ $(-1, +1)$, $(+1, +\infty)$ is given by the formula

$$c_1 \int || z^2 - 1 |^{\frac{n-3}{2}} dz + c_2,$$

where c_1 and c_2 are arbitrary constants. This, in particular, implies that for $0 < |x| < -t$ $(z = t/|x| < -1)$ the function $v(x, t)$ has the form

$$v(x, t) = c_1 \ln \left| \frac{t - |x|}{t + |x|} \right| + c_2 \qquad \text{when } n = 1,$$

$$v(x, t) = c_1 \ln \frac{|t + \sqrt{t^2 - |x|^2}|}{|x|} + c_2 \qquad \text{when } n = 2,$$

$$v(x, t) = c_1 \frac{t}{|x|} + c_2 \qquad \text{when } n = 3,$$

and so on.

By K_{x', t', t^0} we shall denote the cone $\{|x - x'| < t' - t, t^0 < t < t'\}$ of "height" $t' - t^0$ with vertex at the point (x', t'), by Γ_{x', t', t^0} its lateral surface $\{|x - x'| = t' - t, \ t^0 \leqslant t \leqslant t'\}$, which is a characteristic (see Sec. 2, Chap. I) for the wave equation, by D_{x', t', t^0} the base of the cone $\{|x - x'| < t' - t^0, \ t = t^0\}$ and by S_{x', t', t^0} the boundary of the base which is the sphere $\{|x - x'| = t' - t^0, \ t = t^0\}$.

Let (x^1, t^1) be a point of R_{n+1}, K the cone K_{x^1, t^1, t^0}, $t^0 < t^1$, and $D = D_{x^1, t^1, t^0}$ the base of this cone (see Fig. 2). We shall demonstrate that if the function $u(x, t)$ is sufficiently smooth in $K \cup D$, then its value at any point (x, t) of the cone K is determined by the value of $\square u$ in the cone $\overline{K}_{x, t, t^0}$ and those of u and u_t on the base of this cone $\overline{D}_{x, t, t^0}$.

Consider first the case of three space variables, $n = 3$. It is assumed that $u(x, t) \in C^2(K) \cap C^1(K \cup D)$ and $\square u \in C(K \cup D)$. Let (ξ, τ) be any point of K and ε any positive number less than $\tau - t^0$, $0 < \varepsilon < \tau - t^0$. By K_ε denote the region $\{\varepsilon < |x - \xi| < \tau - t, \ t^0 < t < \tau - \varepsilon\}$ lying in K. The boundary of K_ε is divided into three parts: $\Gamma_\varepsilon = \{|x - \xi| = \tau - t, \ t^0 \leqslant t \leqslant \tau - \varepsilon\}$, $D_\varepsilon = \{\varepsilon < |x - \xi| < \tau - t^0, \ t = t^0\}$, $\gamma_\varepsilon = \{|x - \xi| = \varepsilon, \ t^0 \leqslant t < \tau - \varepsilon\}$.

We examine the special solution, depending only on $\frac{t - \tau}{|x - \xi|}$, of the homogeneous wave equation:

$$v(x - \xi, t - \tau) = \frac{t - \tau}{|x - \xi|} + 1. \tag{2}$$

Since the functions $u(x, t)$ and $v(x-\xi, t-\tau)$ belong to $C^2(K_\varepsilon)$,

$$v\,\square\,u-u\,\square\,v=-\sum_{i=1}^{3}(u_{x_i}v-uv_{x_i})_{x_i}+(u_tv-uv_t)_t$$

in K_ε.

Integrating this equality over K_ε and taking into account that $\square v=0$ in K_ε, we obtain, by Ostrogradskii's formula,

$$\int_{K_\varepsilon} v\,\square\,u\,dx\,dt=\int_{\Gamma_\varepsilon\cup D_\varepsilon\cup\gamma_\varepsilon}\left[-\sum_{i=1}^{3}(u_{x_i}v-uv_{x_i})\,n_i+(u_tv-uv_t)\,n_4\right]dS$$
$$=I_{\Gamma_\varepsilon}+I_{D_\varepsilon}+I_{\gamma_\varepsilon}, \qquad (3)$$

where $n=(n_1, n_2, n_3, n_4)$ is a unit vector normal to ∂K_ε directed outwards, and I_{Γ_ε}, I_{D_ε}, I_{γ_ε} are integrals over Γ_ε, D_ε, γ_ε.

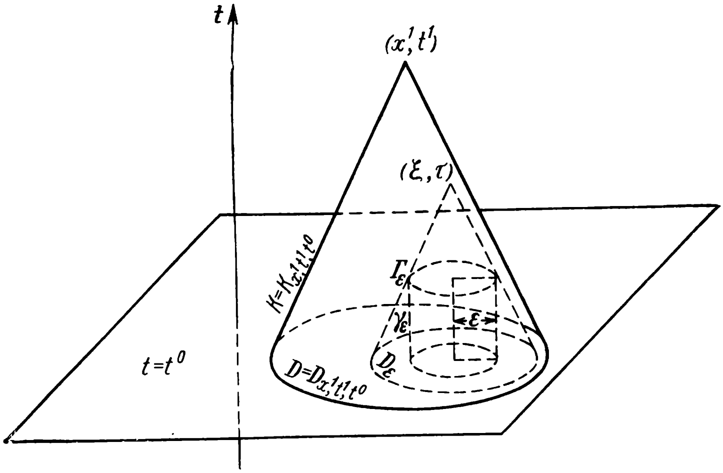

Fig. 2

Consider the integral over Γ_ε. It follows from (2) that $v|_{\Gamma_\varepsilon}=0$. Furthermore, since

$$\nabla v=(t-\tau)\frac{\xi-x}{|\xi-x|^3}, \qquad v_t=\frac{1}{|x-\xi|} \qquad (4)$$

and the normal on Γ_ε $n=\frac{1}{\sqrt{2}}\left(\frac{x_1-\xi_1}{|x-\xi|}, \frac{x_2-\xi_2}{|x-\xi|}, \frac{x_3-\xi_3}{|x-\xi|}, 1\right)=$ $=\frac{1}{\sqrt{2}}\left(\frac{x-\xi}{\tau-t}, 1\right)$, we have

$$\sum_{i=1}^{3}(v_{x_i}n_i)-v_tn_4\Big|_{\Gamma_\varepsilon}=\frac{1}{\sqrt{2}}\left(\frac{(x-\xi,\, x-\xi)}{|x-\xi|^3}-\frac{1}{|x-\xi|}\right)\equiv 0.$$

Consequently,

$$I_{\Gamma_\varepsilon}=\int\limits_{\Gamma_\varepsilon}\left[u\left(\sum_{i=1}^{3}v_{x_i}n_i-v_tn_4\right)+v\left(u_tn_4-\sum_{i=1}^{3}u_{x_i}n_i\right)\right]dS=0. \qquad (5)$$

Since the normal on D_ε is $n=(0,\ 0,\ 0,\ -1)$, we have by (2) and (4)

$$I_{D_\varepsilon}=\int\limits_{\varepsilon<|x-\xi|<\tau-t^0}\frac{u(x,\ t^0)}{|x-\xi|}dx+\int\limits_{\varepsilon<|x-\xi|<\tau-t^0}\left(\frac{\tau-t^0}{|x-\xi|}-1\right)u_t\,(x,\ t^0)\,dx.$$

Since the functions $u(x,\ t^0)$ and $u_t(x,\ t^0)$ are continuous ($u\in C^1(K\cup D)$), the limit of the integral I_{D_ε} exists as $\varepsilon\to 0$ and

$$\lim_{\varepsilon\to 0}I_{D_\varepsilon}=\int\limits_{|x-\xi|<\tau-t^0}\frac{u(x,\ t^0)}{|x-\xi|}dx$$
$$+\int\limits_{|x-\xi|<\tau-t^0}\left(\frac{\tau-t^0}{|x-\xi|}-1\right)u_t\,(x,\ t^0)\,dx. \qquad (6)$$

On the surface γ_ε the normal $n=\left(\frac{-x+\xi}{|x-\xi|},\ 0\right)=\left(\frac{\xi-x}{\varepsilon},\ 0\right)$, therefore, taking (4) into account, we have

$$I_{\gamma_\varepsilon}=-\int\limits_{t^0}^{\tau-\varepsilon}dt\int\limits_{|x-\xi|=\varepsilon}\frac{\partial u}{\partial n}v\,dS_x+\int\limits_{t^0}^{\tau-\varepsilon}dt\int\limits_{|x-\xi|=\varepsilon}\frac{\partial v}{\partial n}u\,dS_x$$
$$=-\int\limits_{t^0}^{\tau-\varepsilon}\left(\frac{t-\tau}{\varepsilon}+1\right)dt\int\limits_{|x-\xi|=\varepsilon}\frac{\partial u}{\partial n}\,dS_x+\int\limits_{t^0}^{\tau-\varepsilon}\frac{t-\tau}{\varepsilon^2}\,dt\int\limits_{|x-\xi|=\varepsilon}u\,dS_x$$

Since for $|x-\xi|=\varepsilon$, $t^0\leqslant t\leqslant\tau-\varepsilon$ the inequalities $\left|\frac{\partial u}{\partial n}\right|\leqslant M$ and $|u(x,\ t)-u(\xi,\ t)|\leqslant M\varepsilon$, where M is a certain constant ($M=\max\limits_{\substack{|x-\xi|\leqslant\tau-t\\ t^0\leqslant t\leqslant\tau}}|\nabla u|$), hold, it follows that

$$\left|\int\limits_{|x-\xi|=\varepsilon}\frac{\partial u}{\partial n}\,dS_x\right|\leqslant 4\pi\varepsilon^2M$$

and

$$\left|\int\limits_{|x-\xi|=\varepsilon}u(x,\ t)\ dS_x-\int\limits_{|x-\xi|=\varepsilon}u(\xi,\ t)\ dS_x\right|$$
$$\leqslant\int\limits_{|x-\xi|=\varepsilon}|u(x,\ t)-u(\xi,\ t)|\,dS_x\leqslant 4\pi\varepsilon^3M.$$

Therefore the limit of the integral I_{γ_ε} exists as $\varepsilon\to 0$ and

$$\lim_{\varepsilon\to 0} I_{\gamma_\varepsilon}=4\pi\int_{t^0}^{\tau}(t-\tau)\,u(\xi,t)\,dt. \tag{7}$$

Passing to the limit in (3) as $\varepsilon\to 0$ and taking (5), (6), (7) into account, we find that for any point (ξ,τ) of K

$$4\pi\int_{t^0}^{\tau}(t-\tau)\,u(\xi,t)\,dt=-\int\limits_{|x-\xi|<\tau-t^0}\frac{u(x,t^0)}{|x-\xi|}\,dx$$
$$-\int\limits_{|x-\xi|<\tau-t^0}\left(\frac{\tau-t^0}{|x-\xi|}-1\right)u_t(x,t^0)\,dx$$
$$+\int_{t^0}^{\tau}dt\int\limits_{|x-\xi|<\tau-t}\left(\frac{t-\tau}{|x-\xi|}+1\right)\Box u(x,t)\,dx.$$

Differentiate this identity with respect to τ:

$$\int_{t^0}^{\tau}u(\xi,t)\,dt=\frac{1}{4\pi(\tau-t^0)}\int\limits_{|x-\xi|=\tau-t^0}u(x,t^0)\,dS_x$$
$$+\frac{1}{4\pi}\int\limits_{|x-\xi|<\tau-t^0}\frac{u_t(x,t^0)}{|x-\xi|}\,dx+\frac{1}{4\pi}\int_{t^0}^{\tau}dt\int\limits_{|x-\xi|<\tau-t}\frac{\Box u(x,t)}{|x-\xi|}\,dx,$$

whence we have

$$u(\xi,\tau)=\frac{\partial}{\partial\tau}\left(\frac{1}{4\pi(\tau-t^0)}\int\limits_{|x-\xi|=\tau-t^0}u(x,t^0)\,dS_x\right)$$
$$+\frac{1}{4\pi(\tau-t^0)}\int\limits_{|x-\xi|=\tau-t^0}u_t(x,t^0)\,dS_x$$
$$+\frac{1}{4\pi}\int_{t^0}^{\tau}dt\int\limits_{|x-\xi|=\tau-t}\frac{\Box u(x,t)}{|x-\xi|}\,dS_x.$$

Since

$$\int_{t^0}^{\tau}dt\int\limits_{|x-\xi|=\tau-t}\frac{\Box u(x,t)}{|x-\xi|}\,dS_x=\int_{0}^{\tau-t^0}d\lambda\int\limits_{|x-\xi|=\lambda}\frac{\Box u(x,\tau-\lambda)}{|x-\xi|}\,dS_x$$
$$=\int\limits_{|x-\xi|<\tau-t^0}\frac{\Box u(x,\tau-|x-\xi|)}{|x-\xi|}\,dx,$$

for any point (x, t) of the cone K_{x^1, t^1, t^0} the following *Kirchhoff formula* holds:

$$u(x, t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi(t-t^0)}\int\limits_{|x-\xi|=t-t^0} u(\xi, t^0)\,dS_\xi\right)+\frac{1}{4\pi(t-t^0}\int\limits_{|x-\xi|=t-t^0} u_t(\xi, t^0)\,dS_\xi+\frac{1}{4\pi}\int\limits_{|x-\xi|<t-t^0}\frac{\Box u(\xi, t-|x-\xi|)}{|x-\xi|}\,d\xi. \qquad (8)$$

Since

$$\frac{1}{4\pi(t-t^0)}\int\limits_{|x-\xi|=t-t^0} u(\xi, t^0)\,dS_\xi=\frac{t-t^0}{4\pi}\int\limits_{|\eta|=1} u(x+\eta(t-t^0), t^0)\,dS_\eta,$$

it follows that

$$\frac{\partial}{\partial t}\left(\frac{1}{4\pi(t-t^0)}\int\limits_{|x-\xi|=t-t^0} u(\xi, t^0)\,dS_\xi\right)=\frac{1}{4\pi}\int\limits_{|\eta|=1} u(x+\eta)(t-t^0), t^0)\,dS_\eta+\frac{t-t^0}{4\pi}\int\limits_{|\eta|=1}(\nabla u(x+\eta(t-t^0), t^0), \eta)\,dS_\eta$$
$$=\frac{1}{4\pi(t-t^0)^2}\int\limits_{|x-\xi|=t-t^0}[u(\xi, t^0)+(\xi-x)\cdot\nabla u(\xi, t^0)]\,dS_\xi.$$

Therefore Kirchhoff's formula can be written in the form

$$u(x, t)=\frac{1}{4\pi(t-t^0)^2}\int\limits_{|x-\xi|=t-t^0}[u(\xi, t^0)+(\xi-x)\cdot\nabla u(\xi, t^0)]\,dS_\xi+\frac{1}{4\pi(t-t^0)}\int\limits_{|x-\xi|=t-t^0} u_t(\xi, t^0)\,dS_\xi+\frac{1}{4\pi}\int\limits_{|x-\xi|<t-t^0}\frac{\Box u(\xi, t-|x-\xi|)}{|x-\xi|}\,d\xi. \qquad (9)$$

Formulas (8) and (9) show that the value of the function u at any point (x, t) of K is expressed in terms of values of $\Box u$ in $\bar{K}_{x, t, t^0}$ and of u and u_t in $\bar{D}_{x, t, t^0}$. Note that the value of the function u at the point $(x, t) \in K$ is determined (when $n = 3$) by the values of the function $\Box u$ not on the whole cone $\bar{K}_{x, t, t^0}$ but merely on its lateral surface Γ_{x, t, t^0} and those of u, u_t and ∇u not on the whole base $\bar{D}_{x, t, t^0}$ but only on its boundary, the sphere S_{x, t, t^0}. In particular, if at some point $(x, t) \in K$ $\square u = 0$ on $\Gamma_{x,\,t,\,t^0}$ and $u = u_t = |\nabla u| = 0$ on $S_{x,\,t,\,t^0}$, then at this point $u(x, t) = 0$.

The above fact at once implies the following theorem, true for $n = 3$, which states that the solution u of Eq. (1) in the cone $K_{x^1,\,t^1,\,t^0} = K$ is uniquely determined by the values of u and u_t on the base $D_{x^1,\,t^1,\,t^0} = D$ of this cone.

Theorem 1. *Suppose that the functions* $u_1(x, t)$ *and* $u_2(x, t)$ *belong to* $C^2(K) \cap C^1(K \cup D)$, $\square u_1 = \square u_2$ *in* K *and* $u_1(x, t^0)|_D = u_2(x, t^0)|_D$, $\left.\frac{\partial u_1(x, t^0)}{\partial t}\right|_D = \left.\frac{\partial u_2(x, t^0)}{\partial t}\right|_D$. *Then* $u_1 \equiv u_2$ *in* K.

Proof. Indeed, the function $u = u_1 - u_2$ belongs to $C^2(K) \cap C^1(K \cup D)$, $\square u = 0$ in K and for $x \in D$ $u(x, t^0) = u_t(x, t^0) = 0$. From (9) it follows that $u(x, t) \equiv 0$ in K, that is, $u_1(x, t) \equiv u_2(x, t)$ in K. ■

For any number of space variables, the corresponding representation and the proof of Theorem 1 can be obtained by the same method. If the function $u(x, t) \in C^2(K) \cap C^1(K \cup D)$, the function $\square u$ and all its derivatives with respect to the space variables up to order $m = \max\left(\left[\frac{n}{2}\right] - 1, 0\right)$ are continuous in $K \cup D$, $u(x, t^0) \in C^{\left[\frac{n}{2}\right]}(D)$ and $u_t(x, t^0) \in C^m(D)$, then the value of $u(x, t)$ at any point $(x, t) \in K$ is expressed in terms of the function $\square u$ (and its derivatives with respect to space variables up to order m) in $\bar{K}_{x,\,t,\,t^0}$ and the functions u and u_t (and their derivatives up to order $\left[\frac{n}{2}\right]$ and m, respectively) in $\bar{D}_{x,\,t,\,t^0}$. For instance, for $n > 3$ the representation is obtained in exactly the same manner as for the case $n = 3$; as a special solution $v(x - \xi, t - \tau)$, with $\frac{t-\tau}{|x-\xi|} < -1$, of the homogeneous wave equation we have to take the function $\int_{-1}^{z} (\zeta^2 - 1)^{\frac{n-3}{2}}\, d\zeta$, $z = \frac{t-\tau}{|x-\xi|}$.

Note that in the case of even n, $n \geqslant 2$, the value of the function u at the point (x, t) is determined by the values of the function $\square u$ (and its derivatives with respect to space variables) on the whole cone $K_{x,\,t,\,t^0}$ and those of the functions u and u_t (and their derivatives with respect to the space variables) on the whole base $D_{x,\,t,\,t^0}$. In the case of odd number of variables $n > 3$, the value of u at the point (x, t) is defined, as in the case $n = 3$, by the values of the function $\square u$ and its derivatives with respect to the space variables only on the lateral surface $\Gamma_{x,\,t,\,t^0}$ of the cone and those of u, u_t and their derivatives with respect to the space variables only on the boundary $S_{x,\,t,\,t^0}$ of the base.

For two and one space variables the corresponding representations together with the proof of Theorem 1 are most easily obtained directly from formula (9) (or (8)).

Suppose that a function $u(x, t)$, $x = (x_1, x_2)$, is defined in the cone $K = K_{x^1, t^1, t^0}$, $x^1 = (x_1^1, x_2^1)$, and belongs to $C^2(K) \cap C^1(K \cup D)$, $D = D_{x^1, t^1, t^0}$ and $\Box\, u = u_{tt} - u_{x_1x_1} - u_{x_2x_2} \in C(K \cup D)$. The function $u(x, t)$ may be regarded as a function of four variables x_1, x_2, x_3, t, which is independent of x_3, defined in the four-dimensional cone $K_{x_1^1, x_2^1, x_3^1, t^1, t^0}$ in which x_3^1 is arbitrary; moreover, $u(x_1, x_2, t) \in C^2(K_{x_1^1, x_2^1, x_3^1, t^1, t^0}) \cap C^1(K_{x_1^1, x_2^1, x_3^1, t^1, t^0} \cup D_{x_1^1, x_2^1, x_3^1, t^1, t^0})$, and $u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} \in C(K_{1, \dots, t^0} \cup D_{x_1^1, \dots, t^0})$. For all points (x_1, x_2, t) belonging to K, by formula (9) we have

$$u(x_1, x_2, t)$$

$$= \frac{1}{4\pi (t-t^0)^2} \int\limits_{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2=(t-t^0)^2} [u(\xi_1, \xi_2, t^0)$$

$$+ (\xi_1 - x_1)\, u_{\xi_1}(\xi_1, \xi_2, t^0) + (\xi_2 - x_2)\, u_{\xi_2}(\xi_1, \xi_2, t^0)]\, dS_\xi$$

$$+ \frac{1}{4\pi (t-t^0)} \int\limits_{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2=(t-t^0)^2} u_t(\xi_1, \xi_2, t^0)\, dS_\xi$$

$$+ \frac{1}{4\pi} \int\limits_0^{t-t^0} d\rho \int\limits_{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2=\rho^2} \frac{\Box u(\xi_1, \xi_2, t-\rho)}{\rho}\, dS_\xi.$$

Since

$$\int\limits_{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2=\rho^2} g(\xi_1, \xi_2)\, dS_\xi$$

$$= 2\rho \int\limits_{(x_1-\xi_1)^2+(x_2-\xi_2)^2<\rho^2} \frac{g(\xi_1, \xi_2)\, d\xi_1\, d\xi_2}{\sqrt{\rho^2 - (x_1-\xi_1)^2 - (x_2-\xi_2)^2}}, \tag{10}$$

it follows that

$$u(x, t) = \frac{1}{2\pi (t-t^0)} \int\limits_{|x-\xi|<t-t^0} \frac{u(\xi, t^0) + (\xi - x)\cdot\nabla u(\xi, t^0)}{\sqrt{(t-t^0)^2 - |x-\xi|^2}}\, d\xi$$

$$+ \frac{1}{2\pi} \int\limits_{|x-\xi|<t-t^0} \frac{u_t(\xi, t^0)\, d\xi}{\sqrt{(t-t^0)^2 - |x-\xi|^2}}$$

$$+ \frac{1}{2\pi} \int\limits_0^{t-t^0} d\rho \int\limits_{|x-\xi|<\rho} \frac{\Box\, u(\xi, t-\rho)}{\sqrt{\rho^2 - |x-\xi|^2}}\, d\xi, \tag{11}$$

where $\xi = (\xi_1, \xi_2)$, $x = (x_1, x_2)$, and the point (x, t) is any point in the cone K_{x^1, t^1, t^0}. This formula gives the desired representation of the function when $n = 2$.

Note that for any point (x, t) in the cone K

$$\frac{1}{2\pi(t-t^0)} \int\limits_{|x-\xi|<t-t^0} \frac{u(\xi, t^0)+(\xi-x)\cdot\nabla u(\xi, t^0)}{\sqrt{(t-t^0)^2-|x-\xi|^2}}\, d\xi$$
$$= \frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int\limits_{|x-\xi|<t-t^0} \frac{u(\xi, t^0)\, d\xi}{\sqrt{(t-t^0)^2-|x-\xi|^2}}\right).$$

Therefore representation (11) can be written in the form

$$u(x, t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int\limits_{|x-\xi|<t-t^0} \frac{u(\xi, t^0)\, d\xi}{\sqrt{(t-t^0)^2-|x-\xi|^2}}\right) + \frac{1}{2\pi} \int\limits_{|x-\xi|<t-t^0} \frac{u_t(\xi, t^0)\, d\xi}{\sqrt{(t-t^0)^2-|x-\xi|^2}} + \frac{1}{2\pi} \int\limits_0^{t-t^0} d\rho \int\limits_{|x-\xi|<\rho} \frac{\Box u(\xi, t-\rho)}{\sqrt{\rho^2-|x-\xi|^2}}\, d\xi. \quad (12)$$

Formula (12) is referred to as *Poisson's formula*. Similarly, for $n = 1$ the corresponding representation is easily obtained from formula (11) (or (12)). If $u(x, t) \in C^2(K) \cap C^1(K \cup D)$, where $K = K_{x^1, t^1, t^0}$ (triangle $\{t - t^1 < x - x^1 < -t + t^1,\ t^0 < t < t^1\}$), $D = D_{x^1, t^1, t^0}$ (interval $(x^1 + t^0 - t^1,\ x^1 + t^1 - t^0)$), and $\Box u = u_{tt} - u_{xx} \in C(K \cup D)$, then the value of the function u at any point $(x, t) \in K$ is given by the following *D'Alembert formula*:

$$u(x, t) = \frac{u(x-t+t^0, t^0)+u(x+t-t^0, t^0)}{2} + \frac{1}{2}\int\limits_{x-t+t^0}^{x-t^0+t} u_t(\xi, t^0)\, d\xi + \frac{1}{2}\int\limits_{t^0}^{t} d\tau \int\limits_{x-t+\tau}^{x+t-\tau} \Box u(\xi, \tau)\, d\xi, \quad (x, t) \in K_{x^1, t^1, t^0}. \quad (13)$$

2. The Cauchy Problem for the Wave Equation. For brevity, we shall denote the set of points $\{x \in R_n,\ t > t^0\}$, $\{x \in R_n,\ t \geqslant t^0\}$, $\{x \in R_n,\ t^0 \leqslant t \leqslant t^1\}$, $\{x \in R_n,\ t = t^0\}$ by $\{t > t^0\}$, $\{t \geqslant t^0\}$, $\{t^0 \leqslant t \leqslant t^1\}$, $\{t = t^0\}$, respectively, and the spaces $C^k(\{t > t^0\})$, $C^k(\{t \geqslant t^0\})$ by $C^k(t > t^0)$ and $C^k(t \geqslant t^0)$.

A function $u(x, t)$ belonging to $C^2(t > 0) \cap C^1(t \geqslant 0)$ is called the (*classical*) *solution of the Cauchy problem for the wave equation in the half-space* $\{t > 0\}$ if for all $x \in R_n$, $t > 0$ it satisfies the equation

$$\Box u = f, \quad (14)$$

and for $t = 0$ the initial conditions

$$u|_{t=0} = \varphi(x), \qquad u_t|_{t=0} = \psi(x), \quad (15)$$

with functions φ, ψ and f given.

According to Theorem 1 of the preceding subsection, the solution $u(x, t)$ of the problem (14), (15) is uniquely determined in any cone $K_{x^1, t^1, 0}$ $(x^1 \in R_n,\ t^1 > 0)$, and hence in the whole half-space $\{t > 0\}$, by the given functions f, φ and ψ. Thus we have the following theorem.

Theorem 2. *The Cauchy problem* (14), (15) *cannot have more than one solution.*

We now proceed to discuss the question of existence of a solution of the Cauchy problem.

Assume that the solution $u(x, t)$ of the problem (14), (15) exists. The results of the foregoing subsection imply that if $f \in C$ $(t \geqslant 0)$, then in the case of three space variables $(n = 3)$ the solution is given by Kirchhoff's formula

$$u(x, t) = \frac{\partial}{\partial t}\left(\frac{1}{4\pi t} \int\limits_{|x-\xi|=t} \varphi(\xi)\, dS_\xi\right) + \frac{1}{4\pi t} \int\limits_{|x-\xi|=t} \psi(\xi)\, dS_\xi$$
$$+ \frac{1}{4\pi} \int\limits_{|x-\xi|<t} \frac{f(\xi, t-|x-\xi|)}{|x-\xi|}\, d\xi, \quad x \in R_3, \quad t > 0, \qquad (16)$$

in the case of two space variables $(n = 2)$ by Poisson's formula

$$u(x, t) = \frac{\partial}{\partial t}\left(\frac{1}{2\pi} \int\limits_{|x-\xi|<t} \frac{\varphi(\xi)\, d\xi}{\sqrt{t^2 - |x-\xi|^2}}\right) + \frac{1}{2\pi} \int\limits_{|x-\xi|<t} \frac{\psi(\xi)\, d\xi}{\sqrt{t^2 - |x-\xi|^2}}$$
$$+ \frac{1}{2\pi} \int\limits_0^t d\tau \int\limits_{|x-\xi|<\tau} \frac{f(\xi, t-\tau)\, d\xi}{\sqrt{\tau^2 - |x-\xi|^2}}, \quad x \in R_2, \quad t > 0, \qquad (17)$$

while in the case of a single space variable $(n = 1)$ by D'Alembert's formula

$$u(x, t) = \frac{\varphi(x+t) + \varphi(x-t)}{2} + \frac{1}{2} \int\limits_{x-t}^{x+t} \psi(\xi)\, d\xi$$
$$+ \frac{1}{2} \int\limits_0^t d\tau \int\limits_{x-\tau}^{x+\tau} f(\xi, \tau)\, d\xi, \quad x \in R_1, \quad t > 0. \qquad (18)$$

In view of this, the proof of the existence of the solution of the problem (14), (15) reduces to obtaining conditions under which the function u (x, t) given by the corresponding representation is a solution of this problem.

Consider first the case of three space variables $(n = 3)$. The following assertion holds.

If $\varphi \in C^3(R_3)$, $\psi \in C^2(R_3)$ *and the function* f *and all its derivatives with respect to* x_1, x_2, x_3 *up to second order are continuous in* $\{t \geqslant 0\}$, *then the function* u *defined by Kirchhoff's formula* (16) *is a solution of the* *problem* (14), (15); *moreover, for any point* $(X,\ T) \in \{t \geqslant 0\}$,

$$\|u\|_{C(\bar K_{X,\,T,\,0})} \leqslant \|\varphi\|_{C(\bar D_{X,\,T,\,0})} + T\,\|\,|\nabla\varphi|\,\|_{C(\bar D_{X,\,T,\,0})} + T\,\|\psi\|_{C(\bar D_{X,\,T,\,0})} + \frac{T^2}{2}\|f\|_{C(\bar K_{X,\,T,\,0})}. \qquad (19)$$

Remark. From formula (19) it follows that if the function f is bounded in $\{0 < t < T\}$, the function ψ is bounded in R_3, and the function φ is bounded in R_3 together with all its first derivatives, then the solution u of the problem (14), (15) is bounded in $\{0 < t < T\}$ and

$$\sup_{\{0<t<T\}} |u| \leqslant \sup_{R_3} |\varphi| + T \sup_{R_3} |\nabla\varphi| + T \sup_{R_3} |\psi| + \frac{T^2}{2} \sup_{\{0<t<T\}} |f|.$$

We shall first examine the function

$$u_g(x,\ t,\ \tau) = \frac{1}{4\pi t} \int\limits_{|x-\xi|=t} g(\xi,\ \tau)\, dS_\xi, \quad x \in R_3, \quad t > 0, \quad \tau > 0, \qquad (20)$$

where $g(x,\ \tau) \in C(\upsilon \geqslant 0)$. When the function g does not depend on the parameter τ, $g\,(x,\ \tau) = g(x)$, the function $u_g(x,\ t,\ \tau)$ will be denoted by $u_g(x,\ t)$.

Lemma 1. *If the function* $g(x,\ \tau)$ *and all its derivatives with respect to* $x_1,\ x_2,\ x_3$ *up to order* k, $k = 0,\ 1,\ \ldots$, *belong to* $C(\tau \geqslant 0)$, *then the function* $u_g(x,\ t,\ \tau)$ *and all its derivatives with respect to* $x_1,\ x_2,\ x_3,\ t$ *up to order* k *are continuous on the set* $\{x \in R_3,\ t \geqslant 0,\ \tau \geqslant 0\}$. *When* $k \geqslant 2$, *the function* $u_g\,(x,\ t,\ \tau)$ *for any* $\tau \geqslant 0$ *satisfies in* $\{t > 0\}$ *the equation* $\square\, u_g = 0$ *and the conditions* $u_g|_{t=0} = 0$, $\Delta u_g\,|_{t=0} = 0$, $u_{gt}\,|_{t=0} = g\,(x,\ \tau)$.

Proof. The first assertion of the lemma follows from the identity

$$u_g(x,\ t,\ \tau) = \frac{t}{4\pi} \int\limits_{|\eta|=1} g(x + t\eta,\ \tau)\, dS_\eta. \qquad (21)$$

It also follows from (21) that $u_g|_{t=0} = 0$. Since for $k \geqslant 2$

$$\Delta u_g\,(x,\ t,\ \tau) = \frac{t}{4\pi} \int\limits_{|\eta|=1} \Delta g\,(x + t\eta,\ \tau)\, dS_\eta, \qquad (22)$$

$\Delta u_g\,|_{t=0} = 0$.

Differentiation of (21) with respect to t yields

$$\frac{\partial u_g}{\partial t} = \frac{1}{4\pi} \int\limits_{|\eta|=1} g(x + t\eta,\ \tau)\, dS_\eta + \frac{t}{4\pi} \int\limits_{|\eta|=1} (\nabla g(x + t\eta,\ \tau),\ \eta)\, dS_\eta, \qquad (23)$$

whence it follows that

$$u_{gt}\,|_{t=0} = \lim_{t\to 0}\frac{\partial u_g}{\partial t} = \frac{1}{4\pi}\int\limits_{|\eta|=1} g(x,\tau)\,dS_\eta = g(x,\tau).$$

Since

$$\begin{aligned}\frac{t}{4\pi}\int\limits_{|\eta|=1}&(\nabla g(x+t\eta,\tau),\eta)\,dS_\eta\\ &= \frac{t}{4\pi}\int\limits_{|\eta|=1}\frac{\partial g(x+t\eta,\tau)}{\partial n_\eta}\,dS_\eta = \frac{1}{4\pi t}\int\limits_{|x-\xi|=t}\frac{\partial g(\xi,\tau)}{\partial n}\,dS_\xi\\ &= \frac{1}{4\pi t}\int\limits_{|x-\xi|<t}\Delta g(\xi,\tau)\,d\xi = \frac{I(x,t,\tau)}{4\pi t},\end{aligned}$$

where $I(x,t,\tau) = \int\limits_{|x-\xi|<t}\Delta g(\xi,\tau)\,d\xi$, (23) may be expressed in the form

$$\frac{\partial u_g}{\partial t} = \frac{1}{t}u_g + \frac{I}{4\pi t},$$

which yields

$$\begin{aligned}\frac{\partial^2 u_g}{\partial t^2} &= -\frac{1}{t^2}u_g + \frac{1}{t}\frac{\partial u_g}{\partial t} + \frac{1}{4\pi t}\frac{\partial I}{\partial t} - \frac{I}{4\pi t^2}\\ &= -\frac{u_g}{t^2} + \frac{1}{t}\left(\frac{u_g}{t} + \frac{1}{4\pi t}\right) + \frac{1}{4\pi t}\frac{\partial I}{\partial t} - \frac{I}{4\pi t^2} = \frac{1}{4\pi t}\frac{\partial I}{\partial t}\\ &= \frac{1}{4\pi t}\int\limits_{|x-\xi|=t}\Delta g(\xi,\tau)\,dS_\xi = \frac{t}{4\pi}\int\limits_{|\eta|=1}\Delta g(x+t\eta,\tau)\,dS_\eta. \qquad (24)\end{aligned}$$

It follows from (24) and (22) that $u_{gtt} = \Delta u_g$. ■

The second term on the right-hand side of (16) is $u_\psi(x,t)$, so by Lemma 1 ($\psi \in C^2(R_3)$) it belongs to $C^2(t \geqslant 0)$, is a solution of the homogeneous wave equation and satisfies the initial conditions

$$u_\psi\,|_{t=0} = 0, \qquad \left.\frac{\partial u_\psi}{\partial t}\right|_{t=0} = \psi.$$

The first term on the right-hand side of (16) is $\frac{\partial u_\varphi}{\partial t}$. Since $\varphi \in C^3(R_3)$, the function $\frac{\partial u_\varphi}{\partial t} \in C^2(t \geqslant 0)$ and is a solution of the homogeneous wave equation

$$\Box\left(\frac{\partial}{\partial t}u_\varphi\right) = \frac{\partial}{\partial t}\Box u_\varphi = 0$$

and satisfies the initial conditions

$$\left.\frac{\partial u_\varphi}{\partial t}\right|_{t=0} = \varphi, \quad \left.\frac{\partial}{\partial t}\left(\frac{\partial u_\varphi}{\partial t}\right)\right|_{t=0} = \Delta u_\varphi\,|_{t=0} = 0.$$

We denote the third term on the right-hand side of (16) by $F(x, t)$ and transform it as follows:

$$F(x, t) = \frac{1}{4\pi} \int\limits_{|x-\xi|<t} \frac{f(\xi, t-|x-\xi|)}{|x-\xi|}\, d\xi$$

$$= \frac{1}{4\pi} \int\limits_0^t \frac{d\rho}{\rho} \int\limits_{|x-\xi|=\rho} f(\xi, t-\rho)\, dS_\xi$$

$$= \int\limits_0^t \Bigg(\frac{1}{4\pi(t-\tau)} \int\limits_{|x-\xi|=t-\tau} f(\xi, \tau)\, dS_\xi \Bigg)\, d\tau = \int\limits_0^t G(x, t, \tau)\, d\tau,$$

where $G(x, t, \tau) = u_f(x, t-\tau, \tau)$. According to Lemma 1, the function $G(x, t, \tau)$ and all its derivatives with respect to x_1, x_2, x_3, t up to second order are continuous on the set $\{x \in R_3,\ t \geqslant 0,\ 0 \leqslant \tau \leqslant t\}$ and for any $\tau \geqslant 0$

$$G_{tt} - \Delta G = 0 \quad \text{for } t \geqslant \tau,$$

$$G|_{t=\tau} = 0, \quad G_t|_{t=\tau} = f(x, \tau).$$

Then the function $F(x, t) = \int\limits_0^t G(x, t, \tau)\, d\tau$ is continuous in $\{t \geqslant 0\}$ together with first derivative with respect to t and all the derivatives with respect to x_1, x_2, x_3 up to second order. And since

$$F_t = G|_{\tau=t} + \int\limits_0^t G_t(x, t, \tau)\, d\tau = \int\limits_0^t G_t(x, t, \tau)\, d\tau,$$

$F \in C^2(t \geqslant 0)$. Furthermore,

$$\Delta F(x, t) = \int\limits_0^t \Delta G(x, t, \tau)\, d\tau$$

and

$$F_{tt} = G_t|_{\tau=t} + \int\limits_0^t G_{tt}(x, t, \tau)\, d\tau = f + \int\limits_0^t \Delta G(x, t, \tau)\, d\tau.$$

Consequently, the function $F(x, t)$ satisfies the equation $\square F = f$ and homogeneous boundary conditions $F|_{t=0} = 0$, $F_t|_{t=0} = 0$.

Thus we have shown that the function

$$u(x, t) = \frac{\partial u_\varphi(x, t)}{\partial t} + u_\psi(x, t) + \int\limits_0^t u_f(x, t-\tau, \tau)\, d\tau$$

defined by formula (16) is a solution of the problem (14), (15).

We shall now establish the inequality (19). Suppose that (X, T) is any point of the half-space $\{t > 0\}$. By formula (20), for any point (x, t) of the cone $K_{X,T,0}$ and arbitrary $\tau > 0$

$$|u_g(x, t, \tau)| \leqslant t \max_{|x-\xi|=t} |g(\xi, \tau)| \leqslant T \max_{|x-\xi|=t} |g(\xi, \tau)|. \quad (25)$$

Consequently,

$$\|u_\psi\|_{C(\bar{K}_{X,T,0})} \leqslant T \max_{(x,t)\in K_{X,T,0}} \max_{|x-\xi|=t} |\psi(\xi)| = T \max_{|x-X|\leqslant T} |\psi| = T \|\psi\|_{C(\bar{D}_{X,T,0})} \quad (26)$$

and

$$\left\| \int_0^t u_f(x, t-\tau, \tau)\, d\tau \right\|_{C(\bar{K}_{X,T,0})} \leqslant \int_0^T (t-\tau) \max_{|x-\xi|=t-\tau} |f(\xi, \tau)|\, d\tau \leqslant \|f\|_{C(\bar{K}_{X,T,0})} \int_0^T (T-\tau)\, d\tau = \frac{T^2}{2} \|f\|_{C(\bar{K}_{X,T,0})}. \quad (27)$$

Similarly, by (23),

$$\left\| \frac{\partial u_\varphi}{\partial t} \right\|_{C(\bar{K}_{X,T,0})} \leqslant \|\varphi\|_{C(\bar{D}_{X,T,0})} + T \| |\nabla\varphi| \|_{C(\bar{D}_{X,T,0})}. \quad (28)$$

The inequality (19) follows immediately from (26)-(28).

Note that the assumptions regarding the functions φ, ψ and f under which the existence of a solution of the Cauchy problem has been established cannot be relaxed in some definite sense. The following example shows that the assumption $\varphi \in C^2(R_3)$ is not enough for the existence of a solution of the problem (14), (15).

Suppose that the function φ depending only on $|x|$, $\varphi(x) = \alpha(|x|)$, belongs to $C^2(R_3)$. Let there exist a solution $u(x, t)$ of the problem (14), (15) with this function $\varphi(x)$ and the functions $\psi \equiv 0$ and $f \equiv 0$. Then by (16)

$$u(x, t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int\limits_{|x-\xi|=t} \alpha(|\xi|)\, dS_\xi \right).$$

Let $|x| \neq 0$. Since for all points ξ on the sphere $\{|x-\xi| = t\}$, $|\xi^2| = |x|^2 + t^2 + 2|x| t \cos\theta$, where θ is the angle between the vectors x and $\xi - x$, we have

$$\int\limits_{x-\xi|=t} \alpha(|\xi|)\, dS_\xi = t^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \alpha\left(\sqrt{t^2 + |x|^2 + 2|x| t \cos\theta}\right) \sin\theta\, d\theta$$

$$= 2\pi t^2 \int_{-1}^{+1} \alpha\left(\sqrt{t^2 + |x|^2 + 2|x| t\lambda}\right) d\lambda = \frac{2\pi t}{x} \int_{|t-|x||}^{t+|x|} \rho\alpha(\rho)\, d\rho.$$

Therefore

$$u(x,\ t)=\frac{\partial}{\partial t}\left(\frac{1}{2|x|}\int\limits_{|t-|x||}^{t+|x|}\rho\alpha(\rho)\,d\rho\right)$$

$$=\frac{1}{2|x|}((t+|x|)\,\alpha(t+|x|)-(t-|x|)\,\alpha(|t-|x||))$$

$$=\frac{t}{2|x|}(\alpha(t+|x|)-\alpha(|t-|x||))+\frac{1}{2}(\alpha(t+|x|)+\alpha(|t-|x||)).$$

This implies that $u(0, t) = t\alpha'(t) + \alpha(t)$, because the solution is continuous. And since the function $u(0, t) \in C^2(t > 0)$, the function $\alpha(|x|)$ must belong to $C^3(|x| > 0)$ which, of course, does not follow from the assumption that φ belongs to $C^2(R_3)$.

Next we consider the case $n = 2$. We shall show that *if* $\varphi(x_1, x_2) \in C^3(R_2)$, $\psi(x_1, x_2) \in C^2(R_2)$ *and the function* $f(x_1, x_2, t)$ *is continuous in* $\{t \geqslant 0\}$ *together with all the derivatives up to second order with respect to* x_1, x_2, *then the function* $u(x_1, x_2, t)$ *given by Poisson's formula* (17) *is a solution of the problem* (14), (15). *Further, for any point* (X, T) *of the half-space* $\{t > 0\}$ *the inequality* (19) *holds.*

According to formula (10), for any x_3

$$u(x_1,\ x_2,\ t)=\frac{\partial}{\partial t}\left(\frac{1}{4\pi t}\int\limits_{S_t(x_1,x_2,x_3)}\varphi(\xi_1,\ \xi_2)\,dS_\xi\right)$$

$$+\frac{1}{4\pi t}\int\limits_{S_t(x_1,x_2,x_3)}\psi(\xi_1,\ \xi_2)\,dS_\xi+\frac{1}{4\pi}\int\limits_0^t\frac{d\tau}{\tau}\int\limits_{S_\tau(x_1,x_2,x_3)}f(\xi_1,\ \xi_2,\ t-\tau)\,dS_\xi, \tag{17'}$$

where $S_\rho(x_1, x_2, x_3)$ is a sphere of radius ρ centred at (x_1, x_2, x_3): $(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 + (x_3 - \xi_3)^2 = \rho^2$. As just now established, the function appearing in the right-hand side of the identity (17′) is a solution of the problem: $u_{tt} - u_{x_1x_1} - u_{x_2x_2} - u_{x_3x_3} = f(x_1, x_2, t)$ in $\{t > 0\}$, $u|_{t=0} = \varphi(x_1, x_2)$, $u_t|_{t=0} = \psi(x_1, x_2)$ and inequality (19) holds for it. And since the function u does not depend on x_3, it is a solution of the problem (14), (15) with $n = 2$.

For the case $n = 1$, it can be directly verified that the *function* $u(x, t)$ *defined by D'Alembert's formula* (18) *is a solution of the problem* (14), (15) *if* $\varphi \in C^2(R_1)$, $\psi \in C^1(R_1)$ *and the function* $f(x, t)$ *is continuous in* $\{t \geqslant 0\}$ *together with its first derivative with respect to* x. *Further, for all points* (X, T) *of the half-plane* $\{t > 0\}$ *the following inequality holds*:

$$\|u\|_{C(\bar{K}_{X,T,0})}\leqslant\|\varphi\|_{C(\bar{D}_{X,T,0})}+T\|\psi\|_{C(\bar{D}_{X,T,0})}+\frac{T^2}{2}\|f\|_{C(\bar{K}_{X,T,0})}$$

($K_{X,T,0}$ is the triangle $\{t + X - T < x < T + X - t,\ 0<t<T\}$ and $D_{X,T,0} = \{X - T < x < X + T,\ t = 0\}$ its base).

When the number of space variables is greater than three ($n > 3$), it is established, just like for the case $n = 3$, that if $\varphi \in C^{\left[\frac{n}{2}\right]+2}(R_n)$, $\psi \in C^{\left[\frac{n}{2}\right]+1}(R_n)$ and the function f is continuous on the set $\{t \geqslant 0\}$ together with its derivatives up to order $\left[\frac{n}{2}\right]+1$ with respect to $x_1, \ldots, x_n$, then the function u given by the corresponding representation is a solution of the problem (14), (15).

Theorem 3. *If* $\varphi(x) \in C^{m+3}(R_n)$, $\psi(x) \in C^{m+2}(R_n)$ *and the function* $f(x, t)$ *is continuous in* $\{t \geqslant 0\}$ *together with its derivatives up to order* $m+2$ *with respect to* $x_1, \ldots, x_n$, *where* $m = \max\left(\left[\frac{n}{2}\right]-1, 0\right)$, *then there exists a solution* $u(x, t)$ *of the problem* (14), (15). *Further, for any point* (X, T) *of the half-space* $\{t > 0\}$ *the inequality*

$$\|u\|_{C(\bar{K}_{X,T,0})} \leqslant C(\|\varphi\|_{C^{m+1}(\bar{D}_{X,T,0})} + \|\psi\|_{C^{m}(\bar{D}_{X,T,0})} + \sum_{|\alpha| \leqslant m} \|D^{\alpha} f\|_{C(\bar{K}_{X,T,0})}$$

holds with a constant C *depending only on* T.

As already noted in the preceding subsection, Poisson's formula in the case $n = 2$ or the corresponding representation in the case of even $n > 2$ implies that the value of the solution of the Cauchy problem (14), (15) at the point (x, t), $t > 0$, depends only on the values of the function f (in the case $n > 2$, also on the values of its derivatives with respect to the space variables) in the whole cone $\bar{K}_{x,t,0}$ as well as on the values of the initial functions φ and ψ (and the values of their derivatives) on the whole base $\bar{D}_{x,t,0}$ of this cone. In the case of any odd $n \geqslant 3$ (as in the case $n = 3$) the value of the solution at the point (x, t) depends on the values of function f, and in the case $n > 3$ also on the values of its derivatives with respect to space variables only on the lateral surface $\Gamma_{x,t,0}$ of the cone $K_{x,t,0}$ and on the values of the initial functions φ and ψ and their derivatives only on the boundary of the base, the sphere $S_{x,t,0}$, of the cone.

Due to this, the cone $K_{x,t,0}$ in the case of even number of space variables $n \geqslant 2$ and the conical surface $\Gamma_{x,t,0}$ in the case of odd $n \geqslant 3$ are referred to as the *region of dependence on the right-hand side of the equation* of the solution of the Cauchy problem (14), (15) at the point (x, t). Likewise, the ball $D_{x,t,0}$ lying on the initial plane in the case of even $n \geqslant 2$ and the boundary, the sphere $S_{x,t,0}$, of the ball in the case of odd $n \geqslant 3$ are known as the *region of dependence on the initial data* of the solution of the Cauchy problem at the point (x, t).

When $n = 1$, from D'Alembert's formula (18) it follows that the solution of the Cauchy problem at the point (x, t) depends only on the values of the function f in the triangle $K_{x, t, 0}$, on the values of the initial function ψ on the base of this triangle $D_{x, t, 0}$ and on the values of function φ on the boundary of the base, the points $(x + t, 0)$ and $(x - t, 0)$.

Suppose that for some $R > 0$ the initial functions φ and ψ vanish for $|x| \geqslant R$, while the function f vanishes for $|x| + t \geqslant R$. Then the solution u of the Cauchy problem vanishes at all points $(x, t) \in \{|x| \geqslant R + t, t \geqslant 0\}$, because the cone $K_{x, t, 0}$ for such (x, t) does not have common points with the set $\{|x| + t < R, t > 0\}$ and its base $\bar{D}_{x, t, 0}$ does not have common points with the ball $\{|x| < R, t = 0\}$. (Even when f vanishes only for $|x| \geqslant R + t$, this assertion remains valid.)

In the case of even number of space variables the set $\{|x| \geqslant R + t, t \geqslant 0\}$ is, generally speaking, maximal possible set where $u = 0$. For instance, with $n = 2$, if the function ψ is taken positive in the disc $\{|x| < R\}$ and the functions φ and f vanish, then it follows from Poisson's formula that $u(x, t) > 0$ for all $(x, t) \in \{|x| < R + t, t > 0\}$.

When the number of space variables is odd, $n \geqslant 3$, the function $u(x, t)$ vanishes not only on the set $\{|x| \geqslant R + t, t \geqslant 0\}$ but also on the set $\{|x| \leqslant t - R, t \geqslant R\}$, because for $(x, t) \in \{|x| \leqslant t - R, t \geqslant R\}$ the conical surface $\Gamma_{x, t, 0}$ does not have common points with the set $\{|x| + t < R, t > 0\}$ while the boundary of the base $S_{x, t, 0}$ does not have common points with the ball $\{|x| < R, t = 0\}$. The set $G = \{|x| \geqslant R + t, t \geqslant 0\} \cup \{|x| \leqslant t - R, t \geqslant R\}$ is, generally speaking, the maximal possible set where $u = 0$. For instance, with $n = 3$, if the function $\psi(x) > 0$ for $|x| < R$ and the functions φ and f vanish, then from Kirchhoff's formula it follows that $u(x, t) > 0$ on the region $\{||x| - t| < R, t > 0\}$, the complement of G.

If the right-hand side $f(x, t)$ of the equation (14) is defined not in the whole half-space $\{t > 0\}$ but only in the strip $\{0 < t < T\} = \Pi_T$ for some $T > 0$, then we consider the Cauchy problem for Eq. (14) in the strip Π_T.

A function $u(x, t)$ belonging to $C^2(0 < t < T) \cap C^1(0 \leqslant t < T)$ is called the *solution of the Cauchy problem* (14), (15) *in the strip* Π_T if for all points $(x, t) \in \Pi_T$ it satisfies Eq. (14) and for $t = 0$ the initial conditions (15). For the Cauchy problem in a strip there are, of course, existence and uniqueness theorems, analogous to corresponding theorems concerning the Cauchy problem in a half-space. The Cauchy problem in the strip Π_T cannot have more than one solution and in the case of $n = 3$, for instance, in order that the Cauchy problem in the strip Π_T may have a solution it is sufficient that $\varphi \in C^3(R_3)$, $\psi \in C^2(R_3)$ and the function f be continuous in

$\{0 \leqslant t < T\}$ together with all its derivatives up to second order with respect to the variables x_1, x_2, x_3. Moreover, the solution of the problem is represented in Π_T by Kirchhoff's formula.

Apart from considering the Cauchy problem in the half-space $\{t > 0\}$, we may also study the Cauchy problem in the half-spaces $\{t > t^0\}$ or $\{t < t^0\}$ for any t^0. A function $u\ (x, t)$ belonging to the space $C^2\ (t > t^0) \cap C^1\ (t \geqslant t^0)$ is called the *solution of the Cauchy problem in the half-space* $\{t > t^0\}$ *for the wave equation* if in $\{t > t^0\}$ it satisfies the equation $\Box u = f$ and for $t = t^0$ the initial conditions $u|_{t=t^0} = \varphi$, $u_t|_{t=t^0} = \psi$. Exactly similar is the definition of the solution of the Cauchy problem in the half-space $\{t < t^0\}$. The Cauchy problem in the half-space $\{t > t^0\}$ reduces to a Cauchy problem in the half-space $\{t > 0\}$ by changing t to $t - t^0$. Similarly, the Cauchy problem in the half-space $\{t < t^0\}$ transforms to a Cauchy problem in the half-space $\{t > 0\}$ by replacing t by $t^0 - t$. If one changes t to t/a (a is a positive constant), the Cauchy problem in the half-space $\{t > 0\}$ for the equation $\frac{1}{a^2} u_{tt} - \Delta u = f$ is transformed to the Cauchy problem (14), (15).

Let D be an n-dimensional region in the plane $\{t = 0\}$, and let the region Q lying in the half-space $\{t > 0\}$ be formed of the points (x, t) which are vertices of cones $K_{x,\,t,\,0}$ whose bases (balls $D_{x,\,t,\,0}$) lie in D. If, in particular, D is the ball $\{|\, x - x^0 \,| < R\}$, then the region Q is the cone $K_{x^0,\,R,\,0}$; if D is the cube $\{|\, x_i - x_i^0 \,| < a,\ i = 1, \ldots, n\}$, then Q is a pyramid whose base is this cube and vertex is at the point (x^0, a); if D is the entire plane $\{t = 0\}$, Q is the half-space $\{t > 0\}$.

A function $u(x, t)$ belonging to $C^2(Q) \cap C^1(Q \cup D)$ is called the *solution of the Cauchy problem in* Q *for the wave equation* if it satisfies in Q the equation $\Box u = f$ and for $t = 0$, $x \in D$, the initial conditions $u|_{t=0} = \varphi$, $u_t|_{t=0} = \psi$.

Theorem 1 of the preceding subsection immediately yields a uniqueness theorem for the Cauchy problem in Q: the Cauchy problem in Q cannot have more than one solution.

It is easily seen that the existence theorem, Theorem 3, is also valid for the case under consideration. For instance, with $n = 3$, a solution of the Cauchy problem in Q exists provided $\varphi \in C^3(D)$, $\psi \in C^2\ (D)$ and the function f is continuous in $Q \cup D$ together with its derivatives up to second order with respect to space variables. The solution $u\ (x, t)$ is given by Kirchhoff's formula (16).

Note that the solution of the Cauchy problem (14), (15) in the half-space $\{t > 0\}$ in Q coincides with the solution of the Cauchy problem in Q for Eq. (14) with initial functions φ and ψ considered only on D.

§ 2. MIXED PROBLEMS

1. Uniqueness of the Solution. Let D be a bounded region of the n-dimensional space R_n ($x = (x_1, \ldots, x_n)$ is a point in this space). In the $(n+1)$-dimensional space $R_{n+1} = R_n \times \{-\infty < t < +\infty\}$ we consider a bounded cylinder $Q_T = \{x \in D,\ 0 < t < T\}$ of height $T > 0$. We denote by Γ_T the lateral surface $\{x \in \partial D,\ 0 < t < T\}$ of the cylinder Q_T and by D_τ its intersection $\{x \in D,\ t = \tau\}$ with the plane $t = \tau$; in particular, the top of Q_T is $D_T = \{x \in D,\ t = T\}$, while its base is $D_0 = \{x \in D,\ t = 0\}$.

In the cylinder Q_T with $T > 0$, we examine the hyperbolic equation

$$\mathscr{L}u \equiv u_{tt} - \operatorname{div}(k(x)\,\nabla u) + a(x)\,u = f(x, t), \tag{1}$$

where $k(x) \in C^1(\bar{D})$, $a(x) \in C(\bar{D})$, $k(x) \geqslant k_0 = \text{const} > 0$.

A function $u(x, t)$ belonging to the space $C^2(Q_T) \cap C^1(Q_T \cup \Gamma_T \cup \bar{D}_0)$ that satisfies in Q_T Eq. (1), on D_0 the initial conditions

$$u|_{t=0} = \varphi, \tag{2}$$

$$u_t|_{t=0} = \psi, \tag{3}$$

and on Γ_T one of the boundary conditions

$$u|_{\Gamma_T} = \chi$$

or

$$\left(\frac{\partial u}{\partial n} + \sigma u\right)\bigg|_{\Gamma_T} = \chi,$$

where σ, a function continuous on Γ_T, is called (*classical*) *solution of the first or, correspondingly, the third mixed problem for Eq.* (1).

If $\sigma \equiv 0$ on Γ_T, the third mixed problem is known as the *second mixed problem.*

Since the case of nonhomogeneous boundary conditions is easily reduced to that of homogeneous boundary conditions, further consideration will be confined to the homogeneous boundary conditions

$$u|_{\Gamma_T} = 0 \tag{4}$$

and

$$\left(\frac{\partial u}{\partial n} + \sigma u\right)\bigg|_{\Gamma_T} = 0. \tag{5}$$

We shall assume that the coefficient $a(x)$ in Eq. (1) is nonnegative in Q_T, while the function σ in the boundary condition (5) depends only on x, $\sigma = \sigma(x)$, and is nonnegative on Γ_T.

Suppose that the function $u(x, t)$ is a solution of either of the problems (1)-(4) or (1), (2), (3), (5), where the right-hand side $f(x, t)$ of Eq. (1) belongs to $L_2(Q_T)$. Take any δ, $0 < \delta < T$. Multiply (1) by a function $v(x, t)$ belonging to $C^1(\bar{Q}_{T-\delta})$ and satisfying the condition

$$v|_{D_{T-\delta}} = 0 \tag{6}$$

and integrate the resulting identity over the cylinder $Q_{T-\delta}$. Since $u_{tt}v = (u_t v)_t - u_t v_t$ and $v \operatorname{div}(k\nabla u) = \operatorname{div}(kv\nabla u) - k\nabla u \nabla v$, we obtain by Ostrogradskii's formula, taking into account the initial condition (3) and condition (6),

$$\begin{aligned}\int_{Q_{T-\delta}} fv\,dx\,dt &= \int_{Q_{T-\delta}} ((u_t v)_t - \operatorname{div}(kv\nabla u))\,dx\,dt \\ &+ \int_{Q_{T-\delta}} (k\nabla u\nabla v + auv - u_t v_t)\,dx\,dt \\ &= \int_{D_{T-\delta}} u_t v\,dx - \int_{D_0} u_t v\,dx - \int_{\Gamma_{T-\delta}} k\frac{\partial u}{\partial n} v\,dS\,dt \\ &+ \int_{Q_{T-\delta}} (k\nabla u\nabla v + auv - u_t v_t)\,dx\,dt = -\int_{D_0} \psi v\,dx \\ &- \int_{\Gamma_{T-\delta}} kv\frac{\partial u}{\partial n}\,dS\,dt + \int_{Q_{T-\delta}} (k\nabla u\nabla v + auv - u_t v_t)\,dx\,dt. \end{aligned} \tag{7}$$

If $u(x, t)$ is a solution of the third (or second) mixed problem, the last identity shows, in view of (5), that $u(x, t)$ satisfies the integral identity

$$\int_{Q_{T-\delta}} (k\nabla u\nabla v + auv - u_t v_t)\,dx\,dt + \int_{\Gamma_{T-\delta}} k\sigma uv\,dS\,dt = \int_{Q_{T-\delta}} fv\,dx\,dt + \int_{D_0} \psi v\,dx$$

for all $v(x, t) \in C^1(\bar{Q}_{T-\delta})$ for which (6) holds, and therefore also for all $v(x, t) \in H^1(Q_{T-\delta})$ satisfying the condition (6).

When the function $u(x, t)$ is a solution of the first mixed problem, we additionally assume that $v(x, t)$ satisfies the condition

$$v|_{\Gamma_{T-\delta}} = 0. \tag{8}$$

Then (7) shows that $u(x, t)$ satisfies the integral identity

$$\int_{Q_{T-\delta}} (k\nabla u\nabla v + auv - u_t v_t)\,dx\,dt = \int_{D_0} \psi v\,dx + \int_{Q_{T-\delta}} fv\,dx\,dt$$

for all $v \in H^1(Q_{T-\delta})$ satisfying conditions (6) and (8).

By means of the above identities we can introduce the notion of generalized solutions of the mixed problems in question. We shall assume that $f(x, t) \in L_2(Q_T)$ and $\psi(x) \in L_2(D)$.

A function u belonging to the space $H^1(Q_T)$ is called the *generalized solution in Q_T of the first mixed problem* (1)-(4) if it satisfies the initial condition (2), the boundary condition (4) and the identity

$$\int_{Q_T} (k\nabla u\nabla v + auv - u_t v_t)\,dx\,dt = \int_{D_0} \psi v\,dx + \int_{Q_T} fv\,dx\,dt \tag{9}$$

for all $v \in H^1(Q_T)$ which satisfy the condition (4) and the condition

$$v|_{D_T} = 0. \tag{10}$$

A function u belonging to $H^1(Q_T)$ is known as the *generalized solution in Q_T of the third* (*second* if $\sigma = 0$) *mixed problem* (1), (2), (3), (5) if it satisfies the initial condition (2) and the identity

$$\begin{aligned}\int_{Q_T} (k\nabla u\nabla v + auv - u_t v_t)\,dx\,dt + \int_{\Gamma_T} k\sigma uv\,dS\,dt\\ = \int_{D_0} \psi v\,dx + \int_{Q_T} fv\,dx\,dt\end{aligned} \tag{11}$$

for all $v \in H^1(Q_T)$ for which condition (10) is fulfilled.

Note that the generalized solutions, like the classical solutions, have the following property. *If u is a generalized solution of the problem* (1)-(4) *or of* (1), (2), (3), (5) *in the cylinder Q_T, then it is also a generalized solution of the corresponding problem in the cylinder $Q_{T'}$ for any $T' < T$.*

Indeed, if u is a generalized solution in Q_T of any of the above problems, then for any $T' < T$, $u \in H^1(Q_{T'})$, in the case of the first mixed problem $u|_{\Gamma_{T'}} = 0$, and for it the corresponding integral identity holds for all v belonging to $H^1(Q_T)$ and satisfying the condition $v|_{D_T} = 0$, and in the case of the first mixed problem also the condition $v|_{\Gamma_T} = 0$.

It can be directly verified that if the function v belongs to $H^1(Q_{T'})$, $v|_{D_{T'}} = 0$ and $v = 0$ in $Q_T \setminus Q_{T'}$, then $v \in H^1(Q_T)$ and $v|_{D_T} = 0$, and if, in addition, $v|_{\Gamma_{T'}} = 0$, then $v|_{\Gamma_T} = 0$ also. Thus the function u satisfies the integral identity defining the generalized solution of the corresponding mixed problem in $Q_{T'}$.

We also note that the notion of a generalized solution of the mixed problem has been introduced as a generalization of the notion of a classical solution (with $f \in L_2(Q_T)$), and the following assertion has been established: *the classical solution in Q_T of each of the problems* (1)-(4) *and* (1), (2), (3), (5) *with $f \in L_2(Q_T)$ is a generalized solution of this problem in $Q_{T-\delta}$ for any $\delta \in (0, T)$.*

Apart from the classical and generalized solutions of mixed problems, we can also introduce the idea of an almost everywhere solution (a.e. solution). A function u is designated as an *a.e. solution of* *the mixed problem* (1)-(4) or *the third* (*second* if $\sigma = 0$) *mixed problem* (1), (2), (3), (5) if it belongs to $H^2(Q_T)$, satisfies Eq. (1) in Q_T (for almost all $(x, t) \in Q_T$) and the initial conditions (2) and (3) as well as one of the boundary conditions (4) or (5), respectively.

From this definition it immediately follows that if the classical solution of the problem (1)-(4) or (1), (2), (3), (5) belongs to the space $H^2(Q_T)$, it is also an a.e. solution of the corresponding problem. What is more, if an a.e. solution of the problem (1)-(4) (or the problem (1), (2), (3), (5)) belongs to $C^2(Q_T) \cap C^1(Q_T \cup \Gamma_T \cup \bar{D}_0)$, it is also a classical solution of this problem (the function $u_{tt} - \operatorname{div}(k\nabla u) + au - f$ is continuous and equal to zero a.e. in Q_T; accordingly, it vanishes everywhere in Q_T).

As shown above, a classical solution of the first or third (second) mixed problem for Eq. (1) in Q_T with $f \in L_2(Q_T)$ is a generalized solution of the corresponding problem in $Q_{T-\delta}$ for any $\delta \in (0, T)$. It can be similarly proved that an a.e. solution of the first or third (second) mixed problem for Eq. (1) in Q_T is a generalized solution of the corresponding problem in Q_T.

The following assertion also holds which is, in a definite sense, a converse of the above result.

Lemma 1. *If a generalized solution of the problem* (1)-(4) *or the problem* (1), (2), (3), (5) *belongs to the space* $H^2(Q_T)$, *it is an a.e. solution of the corresponding problem. If the generalized solution of the problem* (1)-(4) *or* (1), (2), (3), (5) *belongs to* $C^2(Q_T) \cap C^1(Q_T \cup \Gamma_T \cup \bar{D}_0)$, *then it is a classical solution of the corresponding problem.*

Proof. Both the assertions of the lemma will be proved simultaneously.

Let the generalized solution of the problem (1)-(4) or (1), (2), (3), (5) belong to $H^2(Q_T)$ (or to $C^2(Q_T) \cap C^1(Q_T \cup \Gamma_T \cup \bar{D}_0)$). Then to prove the lemma, it suffices to establish that in Q_T the function u satisfies Eq. (1), on D_0 the initial condition (3), and, in case of the third (second) mixed problem, also the boundary condition (5) on Γ_T.

Taking an arbitrary function $v \in \dot{C}^1(\bar{Q}_T)$, we transform (9) or, correspondingly, (11) by means of Ostrogradskii's formula in the following manner

$$\int_{Q_T} (-\operatorname{div}(k\nabla u) + au + u_{tt} - f)\, v\, dx\, dt = 0.$$

If $u \in H^2(Q_T)$, then $-\operatorname{div}(k\nabla u) + au + u_{tt} - f \in L_2(Q_T)$ and since the set $\dot{C}^1(\bar{Q}_T)$ is everywhere dense in $L_2(Q_T)$, the function u satisfies Eq. (1) a.e. in Q_T.

When $u \in C^2(Q_T) \cap C^1(Q_T \cup \Gamma_T \cup \bar{D}_0)$, then $-\operatorname{div}(k \nabla u) + au + u_{tt} - f \in L_2(Q')$ for any subregion $Q' \Subset Q$; since the set of functions $\dot{C}^1(\bar{Q}_T)$ is everywhere dense in $L_2(Q')$ and Q' is arbitrary, we find that u satisfies Eq. (1) in Q_T. (Since the function $-\operatorname{div}(k\nabla u) + au + u_{tt}$ is continuous in Q_T, so is the function f, that is, the function u satisfies Eq. (1) everywhere.)

Consider a function v belonging to $C^1(\bar{Q}_{T-\delta})$ for certain $\delta \in (0, T)$ and satisfying conditions (6) and (8). If $u \in H^2(Q_T)$, we use Ostrogradskii's formula to obtain from (9) or, correspondingly, from (11)

$$\int_{D_0} (u_t - \psi)\, v\, dx = 0.$$

This identity is true also when $u \in C^2(Q_T) \cap C^1(Q_T \cup \Gamma_T \cup \bar{D}_0)$, because in this case $-\operatorname{div}(k\nabla u) + u_{tt} = f - au \in L_1(Q_T)$ and $u \in C^1(\bar{Q}_{T-\delta})$. Since corresponding to any g in $\dot{C}^1(\bar{D}_0)$ (the set of such functions is everywhere dense in $L_2(D_0)$) we can construct a function v belonging to $C^1(\bar{Q}_{T-\delta})$ and satisfying conditions (6), (8) as well as the condition $v|_{D_0} = g$, the function u satisfies the initial condition (3).

We now take any function $v \in C^1(\bar{Q}_{T-\delta})$, $\delta \in (0, T)$, satisfying the condition (6). Then (11) yields

$$\int_{\Gamma_{T-\delta}} kv \left(\frac{\partial u}{\partial n} + \sigma u\right) dS\, dt = 0.$$

But for any continuously differentiable function g with compact support on Γ_T (the set of such functions is everywhere dense in $L_2(\Gamma_T)$) we can find a $\delta \in (0, T)$ and a function $v \in C^1(\bar{Q}_{T-\delta})$ satisfying condition (6) as well as the condition $v|_{\Gamma_{T-\delta}} = g/k$. So $\left(\frac{\partial u}{\partial n} + \sigma u\right)\Big|_{\Gamma_T} = 0.$ ∎

We shall now prove the following uniqueness theorem.

Theorem 1. *Each of the problems* (1)-(4) *and* (1), (2), (3), (5) *cannot have more than one generalized solution.*

Proof. Let u be the generalized solution of the problem (1)-(4) or (1), (2), (3), (5) with $f = 0$ in Q_T and $\varphi = 0$, $\psi = 0$ on D_0. We shall demonstrate that $u = 0$ in Q_T.

Take an arbitrary $\tau \in (0, T)$ and consider the function

$$v(x, t) = \begin{cases} \displaystyle\int_t^\tau u(x, \theta)\, d\theta, & 0 < t < \tau, \\ 0, & \tau < t < T. \end{cases}$$

It can be directly checked that in Q_T the function v has generalized derivatives

$$v_t = \begin{cases} -u, & 0 < t < \tau, \\ 0, & \tau < t < T, \end{cases}$$

and

$$v_{x_i} = \begin{cases} \int\limits_t^\tau u_{x_i}(x,\ \theta)\, d\theta, & 0 < t < \tau, \\ 0, & \tau < t < T. \end{cases}$$

Consequently, $v(x,\ t) \in H^1(Q_T)$. Moreover, $v|_{D_T} = 0$, and when u is a generalized solution of the first mixed problem, $v\, |_{\Gamma_T} = 0$.

We substitute v in identity (9) if u is the generalized solution of the problem (1)-(4), or in (11) if u is the generalized solution of the problem (1), (2), (3), (5). Then

$$\int\limits_{Q_\tau} \Big(k\nabla u \int\limits_t^\tau \nabla u\, d\theta - avv_t + u_t u \Big)\, dx\, dt = 0$$

in the case of the first mixed problem, or

$$\int\limits_{Q_\tau} \Big(k\nabla u \int\limits_t^\tau \nabla u\, d\theta - avv_t + u_t u \Big)\, dx\, dt$$
$$+ \int\limits_{\Gamma_\tau} k\sigma u\,(x,\ t) \int\limits_t^\tau u\,(x,\ \theta)\, d\theta\, dS\, dt = 0$$

in the case of the third (second) mixed problem. (Recall that in Q_τ $v_t = -u \in H^1(Q_\tau)$, and consequently $v_t|_{\Gamma_\tau} \in L_2(\Gamma_\tau)$.) Since

$$\int\limits_{Q_\tau} k\,(x)\, \nabla u\,(x,\ t) \int\limits_t^\tau \nabla u\,(x,\ \theta)\, d\theta\, dt\, dx$$
$$= \int\limits_D k\,(x) \int\limits_0^\tau \nabla u\,(x,\ t) \Big[\int\limits_t^\tau \nabla u\,(x,\ \theta)\, d\theta \Big]\, dt\, dx$$
$$= \int\limits_D k\,(x) \int\limits_0^\tau \nabla u\,(x,\ \theta)\, d\theta \int\limits_0^\theta \nabla u\,(x,\ t)\, dt\, dx$$
$$= \int\limits_D k\,(x) \int\limits_0^\tau \nabla u\,(x,\ \theta)\, d\theta \int\limits_0^\tau \nabla u\,(x,\ t)\, dt\, dx$$

$$-\int_D k(x)\int_0^\tau \nabla u(x,\ \theta)\,d\theta\int_\theta^\tau \nabla u(x,\ t)\,dt\,dx$$

$$=\int_D k(x)\left|\int_0^\tau \nabla u(x,\ t)\,dt\right|^2 dx-\int_{Q_\tau} k(x)\,\nabla u(x,\ t)\int_t^\tau \nabla u(x,\ \theta)\,d\theta\,dt\,dx,$$

it follows that

$$\int_{Q_\tau} k(x)\,\nabla u(x,\ t)\int_t^\tau \nabla u(x,\ \theta)\,d\theta\,dt\,dx=\frac{1}{2}\int_D k(x)\left|\int_0^\tau \nabla u(x,\ t)\,dt\right|^2 dx.$$

Similarly,

$$\int_{\Gamma_\tau} k\sigma u(x,\ t)\int_t^\tau u(x,\ \theta)\,d\theta\,dS\,dt$$

$$=\int_{\partial D} k\sigma\left(\int_0^\tau u(x,\ t)\,dt\right)^2 dS-\int_{\Gamma_\tau} k\sigma u(x,\ t)\int_t^\tau u(x,\ \theta)\,d\theta\,dS\,dt$$

yields

$$\int_{\Gamma_\tau} k\sigma u(x,\ t)\int_t^\tau u(x,\ \theta)\,d\theta\,dS\,dt=\frac{1}{2}\int_{\partial D} k\sigma\left(\int_0^\tau u(x,\ t)\,dt\right)^2 dS.$$

Furthermore,

$$\int_{Q_\tau} avv_t\,dx\,dt=-\int_{D_0} av^2\,dx-\int_{Q_\tau} av_tv\,dx\,dt.$$

Then

$$\int_{Q_\tau} avv_t\,dx\,dt=-\frac{1}{2}\int_{D_0} av^2\,dx.$$

Analogously,

$$\int_{Q_\tau} uu_t\,dx\,dt=\frac{1}{2}\int_{D_\tau} u^2\,dx.$$

Consequently, if u is a solution of the first mixed problem, then

$$\int_D k(x)\left|\int_0^\tau \nabla u(x,\ t)\,dt\right|^2 dx+\int_{D_0} av^2\,dx+\int_{D_\tau} u^2\,dx=0$$

and, if u is a solution of the third (second) mixed problem, then

$$\int_D k(x)\left|\int_0^\tau \nabla u(x,t)\,dt\right|^2 dx+\int_{D_0} av^2\,dx+\int_{D_\tau} u^2\,dx$$
$$+\int_{\partial D} k\sigma\left(\int_0^\tau u(x,t)\,dt\right)^2 dS=0.$$

Since $k(x)>0$, $a(x)\geqslant 0$ in Q_T and $\sigma(x)\geqslant 0$ on Γ_T, these identities imply that $\int_{D_\tau} u^2\,dx=0$. Since τ is an arbitrary number in the interval $(0, T)$, $u=0$ in Q_T. ∎

As shown above, the classical solutions of the problems (1)-(4) and (1), (2), (3), (5) are also generalized solutions of these problems in $Q_{T-\delta}$ for any $\delta\in(0, T)$. Therefore Theorem 1 at once gives the following result.

Corollary 1. *Each of the problem* (1)-(4) *and* (1), (2), (3), (5) *cannot have more than one classical solution.*

Since a.e. solutions of the problems (1)-(4) and (1)-(3), (5) are also generalized solutions of these problems, from Theorem 1 we also have the following result.

Corollary 2. *Each of the problem* (1)-(4) *and* (1), (2), (3), (5) *cannot have more than one a.e. solution.*

2. Existence of a Generalized Solution. We shall now establish the existence of solutions of the problems (1)-(4) and (1), (2), (3), (5). For this we make use of the *Fourier method*, according to which the solution of the mixed problem is sought in the form of a series in terms of eigenfunctions of the corresponding elliptic boundary-value problem.

Let $v(x)$ be the generalized eigenfunction of the first boundary-value problem

$$\operatorname{div}(k\nabla v)-av=\lambda v,\qquad x\in D,$$
$$v|_{\partial D}=0 \tag{12}$$

or the third (second if $\sigma=0$) boundary-value problem

$$\operatorname{div}(k\nabla v)-av=\lambda v,\qquad x\in D,$$
$$\left(\frac{\partial v}{\partial n}+\sigma v\right)\Big|_{\partial D}=0 \tag{13}$$

(λ is the corresponding eigenvalue). This means that in the case of the first boundary-value problem $v\in \mathring{H}^1(D)$ and for all $\eta\in\mathring{H}^1(D)$

$$\int_D (k\nabla v\nabla\eta+av\eta)\,dx+\lambda\int_D v\eta\,dx=0, \tag{14}$$

while in the case of the third (second) boundary-value problem $v \in H^1(D)$ and for all $\eta \in H^1(D)$

$$\int_D (k\nabla v\nabla\eta + av\eta)\,dx + \int_{\partial D} k\sigma v\eta\,dS + \lambda \int_D v\eta\,dx = 0. \tag{15}$$

Consider the orthonormal system $v_1, v_2, \ldots$ in $L_2(D)$ of all the generalized eigenfunctions of the problem (12) or correspondingly the problem (13); $\lambda_1, \lambda_2, \ldots$ is the sequence of corresponding eigenvalues (as usual, the sequence of eigenvalues is assumed nonincreasing and each eigenvalue is repeated according to its multiplicity). As demonstrated in Sec. 1, Chap. IV, the system $v_1, v_2, \ldots$ constitutes an orthonormal basis for $L_2(D)$ and $\lambda_k \to -\infty$ as $k \to \infty$. For the first, third, with $\sigma \not\equiv 0$ on ∂D, and second, with $a \not\equiv 0$ in D, boundary-value problems (recall that $k(x) \geqslant k_0 > 0$, $a(x) \geqslant 0$ in D and $\sigma(x) \geqslant 0$ on ∂D) the first eigenvalue $\lambda_1 < 0$, that is, $0 > \lambda_1 \geqslant \lambda_2 \geqslant \ldots$. If $a(x) \equiv 0$ in D, then for the second boundary-value problem $0 = \lambda_1 > \lambda_2 \geqslant \ldots$.

Assume that the initial functions $\varphi(x)$ and $\psi(x)$ in (2) and (3) belong to $L_2(D)$ and the function $f(x, t) \in L_2(Q_T)$. According to Fubini's theorem, $f(x, t) \in L_2(D)$ for almost all $t \in (0, T)$. We expand the functions $\varphi(x)$ and $\psi(x)$ and for almost all $t \in (0, T)$ the function $f(x, t)$ in Fourier series in terms of the system $v_1(x)$, $v_2(x), \ldots$ of generalized eigenfunctions of the problem (12) if the problem (1)-(4) is under consideration, or of the problem (13) if we consider the problem (1), (2), (3), (5):

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k v_k(x), \quad \psi(x) = \sum_{k=1}^{\infty} \psi_k v_k(x), \quad f(x, t) = \sum_{k=1}^{\infty} f_k(t)\, v_k(x), \tag{16}$$

where $\varphi_k = (\varphi, v_k)_{L_2(D)}$, $\psi_k = (\psi, v_k)_{L_2(D)}$ and $f_k(t) = \int_D f(x, t)\, v_k(x)dx$,

$k = 1, 2, \ldots$. Since $|f_k(t)|^2 \leqslant \int_D f^2(x, t)\,dx \cdot \int_D v_k^2\,dx = \int_D f^2(x, t)\,dx$,

we see that $f_k(t) \in L_2(0, T)$, $k = 1, 2, \ldots$. According to the Parseval-Steklov equality,

$$\sum_{k=1}^{\infty} \varphi_k^2 = \|\varphi\|^2_{L_2(D)}, \qquad \sum_{k=1}^{\infty} \psi_k^2 = \|\psi\|^2_{L_2(D)} \tag{17}$$

and for almost all $t \in (0, T)$

$$\sum_{k=1}^{\infty} f_k^2(t) = \int_D f^2(x, t)\,dx.$$

Hence

$$\sum_{k=1}^{\infty} \int_0^T f_k^2(t)\,dt = \int_{Q_T} f^2\,dx\,dt. \tag{17'}$$

For the initial functions in (2) and (3) we first take the functions $\varphi_k v_k(x)$ and $\psi_k v_k(x)$, the kth "harmonics" of the series (16), while for the function appearing on the right side of Eq. (1) we take the function $f_k(t)\,v_k(x)$, $k \geqslant 1$. Consider the function

$$u_k(x, t) = U_k(t)\,v_k(x), \tag{18}$$

where

$$U_k(t) = \varphi_k \cos \sqrt{-\lambda_k}\,t + \frac{\psi_k}{\sqrt{-\lambda_k}} \sin \sqrt{-\lambda_k}\,t + \frac{1}{\sqrt{-\lambda_k}} \int_0^t f_k(\tau) \sin \sqrt{-\lambda_k}\,(t-\tau)\,d\tau; \tag{19}$$

when $\lambda_1 = 0$,

$$\begin{aligned} U_1(t) &= \varphi_1 + \psi_1\,(t) + \int_0^t f_1(\tau)\,(t-\tau)\,d\tau \\ &= \lim_{\lambda_1 \to 0} \Big(\varphi_1 \cos \sqrt{-\lambda_1}\,t + \frac{\psi_1}{\sqrt{-\lambda_1}} \sin \sqrt{-\lambda_1}\,t \\ &\qquad + \frac{1}{\sqrt{-\lambda_1}} \int_0^t f_1(\tau) \sin \sqrt{-\lambda_1}\,(t-\tau)\,d\tau\Big). \end{aligned}$$

Clearly, the function $U_k(t)$ belongs to $H^2(0, T)$, satisfies for $t = 0$ the initial conditions $U_k(0) = \varphi_k$, $U_k'(0) = \psi_k$, and for almost all $t \in (0, T)$ is a solution of the equation

$$U_k'' - \lambda_k U_k = f_k, \qquad k = 1, 2, \ldots. \tag{20}$$

We shall show that if $v_k\,(x)$ and λ_k are the generalized eigenfunction and corresponding eigenvalue of the problem (12) (or of (13)), then the function $u_k(x, t)$ is a generalized solution of the first (respectively, of the third or second) mixed problem for the equation

$$u_{tt} - \operatorname{div}\,(k(x)\,\nabla u) + au = f_k(t)\,v_k(x)$$

with the initial conditions

$$u|_{t=0} = \varphi_k v_k(x), \qquad u_t|_{t=0} = \varphi_k v_k(x).$$

Indeed, the function $u_k(x, t) \in H^1(Q_T)$, satisfies on D_0 the initial condition (2) and, in the case of the first mixed problem, the boundary condition (4). Let us show that in the case of the first mixed problem the function $u_k(x, t)$ satisfies the integral identity

$$\int\limits_{Q_T} (k\nabla u_k \nabla v + a u_k v - u_{kt} v_t)\, dx\, dt = \psi_k \int\limits_{D_0} v_k(x)\, v\, dx + \int\limits_{Q_T} f_k(t)\, v_k(x)\, v\, dx\, dt \qquad (9_k)$$

for all functions v belonging to the space $H^1(Q_T)$ and satisfying conditions (4) and (10), and, in the case of the second and third mixed problems, the identity

$$\int\limits_{Q_T} (k\nabla u_k \nabla v + a u_k v - u_{kt} v_t)\, dx\, dt + \int\limits_{\Gamma_T} k\sigma u_k v\, dS\, dt = \psi_k \int\limits_{D_0} v_k(x)\, v\, dx + \int\limits_{Q_T} f_k(t)\, v_k(x)\, v\, dx\, dt \qquad (11_k)$$

for all $v \in H^1(Q_T)$ and satisfying condition (10). Obviously, it is enough to establish the validity of identities (9_k) and (11_k) for all the functions v continuously differentiable in $\overline{Q}_T$ and satisfying conditions (4) and (10) and condition (10), respectively.

In view of (10), (18) and (19),

$$\begin{aligned}\int\limits_{Q_T} u_{kt} v_t\, dx\, dt &= \int\limits_{D} v_k(x) \left[\int\limits_0^T U_k'(t)\, v_t\, dt\right] dx \\ &= \int\limits_{D} v_k(x) \left[-\psi_k v(x, 0) - \int\limits_0^T U_k''(t)\, v\, dt\right] dx \\ &= -\psi_k \int\limits_{D} v_k(x)\, v(x, 0)\, dx - \lambda_k \int\limits_{Q_T} u_k v\, dx\, dt - \int\limits_{Q_T} f_k(t)\, v_k(x)\, v\, dx\, dt.\end{aligned}$$

Therefore for the first mixed problem the identity (9_k) follows from (14):

$$\begin{aligned}\int\limits_{Q_T} (k\nabla u_k \nabla v + a u_k v - u_{kt} v_t)\, dx\, dt &= \int\limits_0^T U_k(t)\, dt \int\limits_{D} (k(x)\, \nabla v_k \nabla v + a v_k v + \lambda_k v_k v)\, dx \\ &\quad + \psi_k \int\limits_{D} v_k(x)\, v(x, 0)\, dx + \int\limits_{Q_T} f_k(t)\, v_k(x)\, v\, dx\, dt \\ &= \psi_k \int\limits_{D} v_k(x)\, v(x, 0)\, dx + \int\limits_{Q_T} f_k(t)\, v_k(x)\, v\, dx\, dt.\end{aligned}$$

Analogously, in the case of the third (second) mixed problem the identity (11_k) follows from (15):

$$\int\limits_{Q_T} (k(x)\,\nabla u_k \nabla v + a u_k v - u_{kt} v_t)\,dx\,dt + \int\limits_{\Gamma_T} k(x)\,\sigma u_k v\,dS\,dt$$

$$= \int\limits_0^T U_k(t)\,dt \left[\int\limits_D (k(x)\,\nabla v_k \nabla v + a v_k v + \lambda_k v_k v)\,dx + \int\limits_{\partial D} k(x)\,\sigma v_k v\,dS \right]$$

$$+ \psi_k \int\limits_D v_k(x)\,v(x,\ 0)\,dx + \int\limits_{Q_T} f_k(t)\,v_k(x)\,v\,dx\,dt$$

$$= \psi_k \int\limits_D v_k(x)\,v(x,\ 0)\,dx + \int\limits_{Q_T} f_k(t)\,v_k(x)\,v\,dx\,dt.$$

If we take for initial functions in (2) and (3) the partial sums $\sum\limits_{k=1}^{N} \varphi_k v_k(x)$ and $\sum\limits_{k=1}^{N} \psi_k v_k(x)$ of series (16) for some N and for f in (1) the partial sum $\sum\limits_{k=1}^{N} f_k(t)\,v_k(x)$ of its Fourier series, then a generalized solution of the problem (1)-(4) ((1), (2), (3), (5)) will be the function

$$S_N(x,\ t) = \sum_{k=1}^{N} u_k(x,\ t) = \sum_{k=1}^{N} U_k(t)\,v_k(x)$$

In particular, in the case of the first mixed problem this function satisfies the identity

$$\int\limits_{Q_T} (k\nabla S_N \nabla v + a S_N v - S_{Nt} v_t)\,dx\,dt$$

$$= \int\limits_D \sum_{k=1}^{N} \psi_k v_k(x)\,v(x,\ 0)\,dx + \int\limits_{Q_T} \sum_{k=1}^{N} f_k(t)\,v_k(x)\,v(x,\ t)\,dx\,dt \tag{21}$$

for all $v \in H^1(Q_T)$ satisfying conditions (4) and (10), while in the case of the third (second) mixed problem the identity

$$\int\limits_{Q_T} (k\nabla S_N \nabla v + a S_N v - S_{Nt} v_t)\,dx\,dt + \int\limits_{\Gamma_T} k\sigma S_N v\,dS\,dt$$

$$= \int\limits_D \sum_{k=1}^{N} \psi_k v_k(x)\,v(x,\ 0)\,dx + \int\limits_{Q_T} \sum_{k=1}^{N} f_k(t)\,v_k(x)\,v\,dx\,dt \tag{22}$$

for all $v \in H^1(Q_T)$ satisfying condition (10).

Thus it is natural to expect that under certain assumptions regarding φ, ψ and f the solution of the problem (1)-(4) ((1), (2), (3), (5)) may be represented by a series of the form

$$u(x,\ t)=\sum_{k=1}^{\infty} U_k(t)\, v_k(x), \tag{23'}$$

where v_1, v_2, . . . are generalized eigenfunctions of the problem (12) (respectively, of (13)).

Theorem 2. *Let* $f \in L_2(Q_T)$, $\psi \in L_2(D)$ *and* $\varphi \in \overset{\circ}{H}{}^1(D)$ *in the case of the first mixed problem* (1)-(4) *while* $\varphi \in H^1(D)$ *in the case of the third* (*second*) *mixed problem* (1), (2), (3), (5). *Then the generalized solution u of the respective problem exists and is represented by series* (23) *which converges in* $H^1(Q_T)$. *Moreover, the following inequality holds*:

$$\|u\|_{H^1(Q_T)} \leqslant C\,(\|\varphi\|_{H^1(D)}+\|\psi\|_{L_2(D)}+\|f\|_{L_2(Q_T)}), \tag{24}$$

where the positive constant C does not depend on φ, ψ *or* f.

Proof. From formula (19) it follows that for all $t \in [0,\ T]$

$$|U_k\ (t)| \leqslant |\varphi_k|+|\psi_k|\,|\lambda_k|^{-1/2}+|\lambda_k|^{-1/2}\int_0^T |f_k(t)|\,dt \ \text{ for } k>1$$

and

$$|U_1(t)| \leqslant |\varphi_1|+C_1|\psi_1|+C_1\int_0^T |f_1(t)|\,dt$$

(for the second mixed problem with $a\equiv 0$ $C_1=T$ and in the remaining cases $C_1 \doteq 1/\sqrt{|\lambda_1|}$). Therefore for all $t \in [0,\ T]$

$$U_k^2(t) \leqslant 3\varphi_k^2+3\psi_k^2|\lambda_k|^{-1}+3|\lambda_k|^{-1}\Big(\int_0^T |f_k|\,dt\Big)^2$$

$$\leqslant C\,(T)\Big(\varphi_k^2+\psi_k^2|\lambda_k|^{-1}+|\lambda_k|^{-1}\int_0^T f_k^2\,dt\Big) \ \text{ for } k>1, \tag{25}$$

$$U_1^2(t) \leqslant C\,(T)\Big(\varphi_1^2+\psi_1^2+\int_0^T f_1^2\,dt\Big). \tag{25'}$$

For any k, $k=1,\ 2,\ \ldots,$ $\left|\frac{dU_k}{dt}\right| \leqslant |\varphi_k|\,|\lambda_k|^{1/2}+\psi_k+\int_0^T |f_k|\,dt$, so for all $t \in [0, T]$

$$\left|\frac{dU_k}{dt}\right|^2 \leqslant C(T)\left(\varphi_k^2|\lambda_k| + |\psi_k|^2 + \int_0^T f_k^2\,dt\right). \tag{26}$$

Since the function φ belongs to $\overset{\circ}{H}{}^1(D)$ in the case of the first mixed problem (to $H^1(D)$ in the case of the third mixed problem), it follows from Theorem 3, Sec. 1.3, Chap. IV, that its Fourier series (16) in terms of the eigenfunctions of the problem (12) (or of (13), respectively) converges to it in the norm of the space $H^1(D)$. Further, there exists a constant $C > 0$ such that for all φ in $\overset{\circ}{H}{}^1(D)$ (or in $H^1(D)$, respectively)

$$\sum_{k=1}^{\infty} \varphi_k^2 |\lambda_k| \leqslant C_1 \|\varphi\|^2_{H^1(D)}. \tag{27}$$

Consider the partial sum $S_N(x, t) = \sum\limits_{k=1}^{N} U_k(t)\, v_k(x)$ of the series (23). For each $t \in [0, T]$ this series and its derivative with respect to t (according to Theorem 3, Sec. 6.2, Chap. III, the functions $U_k(t)$ and $U_k'(t)$, $k = 1, 2, \ldots$, are continuous on $[0, T]$) belong to $\overset{\circ}{H}{}^1(D_t)$ (or to $H^1(D_t)$).

In investigating the problem (1)-(4) it is convenient to define in the space $\overset{\circ}{H}{}^1(D_t)$ the scalar product

$$\int_{D_t} (k\nabla u \nabla v + auv)\,dx.$$

In investigating the problem (1), (2), (3), (5) we introduce in the space $H^1(D_t)$ the scalar product

$$\int_{D_t} (k\nabla u \nabla v + auv)\,dx + \int_{\partial D_t} k\sigma uv\,dS$$

if either $a \not\equiv 0$ in D or $\sigma \not\equiv 0$ on ∂D, and the scalar product

$$\int_{D_t} (k\nabla u \nabla v + uv)\,dx$$

if $a \equiv 0$ in D and $\sigma \equiv 0$ on ∂D. Since in the case of the first and third, with $\sigma \not\equiv 0$, mixed problems and in the case of the second mixed problem, with $a \not\equiv 0$, the system of functions $v_1/\sqrt{-\lambda_1}$, $v_2/\sqrt{-\lambda_2}$, ... is orthonormal in corresponding scalar products, and in the case of the second mixed problem, with $a \equiv 0$, the system of functions $v_1/\sqrt{1 - \lambda_1}$, $v_2/\sqrt{1 - \lambda_2}$, ... is orthonormal, then for all $t \in [0, T]$ and arbitrary M and N, $1 \leqslant M < N$, we have by (25),

$$\| S_N(x, t) - S_M(x, t) \|^2_{H^1(D_t)} = \Big\| \sum_{k=M+1}^{N} U_k(t)\, v_k(x) \Big\|^2_{H^1(D_t)}$$

$$= \sum_{k=M+1}^{N} U_k^2(t)\, |\lambda_k| \leqslant C(T) \sum_{k=M+1}^{N} \left(\varphi_k^2 |\lambda_k| + \psi_k^2 + \int_0^T f_k^2\, dt \right)$$

if either $a \not\equiv 0$ in D or $\sigma \not\equiv 0$ on ∂D, and

$$\| S_N(x, t) - S_M(x, t) \|^2_{H^1(D_t)} = \sum_{k=M+1}^{N} U_k^2(t)\, (1 - \lambda_k)$$

$$\leqslant \frac{1 + |\lambda_2|}{|\lambda_2|} C(T) \sum_{k=M+1}^{N} \left(\varphi_k^2 (1 + |\lambda_k|) + \psi_k^2 + \int_0^T f_k^2\, dt \right)$$

if $a \equiv 0$ in D and $\sigma \equiv 0$ on ∂D. Thus in all cases

$$\| S_N(x, t) - S_M(x, t) \|^2_{H^1(D_t)}$$

$$\leqslant C_2 \sum_{k=M+1}^{N} \left(\varphi_k^2 (1 + |\lambda_k|) + \psi_k^2 + \int_0^T f_k^2\, dt \right) \qquad (28)$$

for all $t \in [0, T]$. Similarly, by (26) for all $t \in [0, T]$

$$\left\| \frac{\partial S_N}{\partial t} - \frac{\partial S_M}{\partial t} \right\|^2_{L_2(D_t)} = \Big\| \sum_{k=M+1}^{N} U_k'(t)\, v_k(x) \Big\|^2_{L_2(D_t)} = \sum_{k=M+1}^{N} U_k'^2(t)$$

$$\leqslant C_3 \sum_{k=M+1}^{N} \left(\varphi_k^2 |\lambda_k| + \psi_k^2 + \int_0^T f_k^2\, dt \right). \qquad (28')$$

Apart from these inequalities, there are also the inequalities

$$\| S_N(x, t) \|^2_{H^1(D_t)} = \Big\| \sum_{k=1}^{N} U_k(t)\, v_k(x) \Big\|^2_{H^1(D_t)}$$

$$\leqslant C_4 \sum_{k=1}^{N} \left(\varphi_k^2 (1 + |\lambda_k|) + \psi_k^2 + \int_0^T f_k^2\, dt \right), \qquad (29)$$

$$\left\| \frac{\partial S_N}{\partial t} \right\|^2_{L_2(D_t)} = \Big\| \sum_{k=1}^{N} U_k'(t)\, v_k(x) \Big\|^2_{L_2(D_t)} = \sum_{k=1}^{N} U_k'^2(t)$$

$$\leqslant C_5 \sum_{k=1}^{N} \left(\varphi_k^2 (|\lambda_k| + 1) + \psi_k^2 + \int_0^T f_k^2\, dt \right), \qquad (29')$$

valid for all $t \in [0, T]$ and any $N \geqslant 1$.

Integrating inequalities (28) and (28′) with respect to $t \in (0, T)$ and adding them, we obtain the inequality

$$\|S_N(x, t)-S_M(x, t)\|^2_{H^1(Q_T)} \leqslant C_6 \sum_{k=M+1}^{N} \left(\varphi_k^2(1+|\lambda_k|)+\psi_k^2+\int_0^T f_k^2\,dt\right). \qquad (30)$$

By (17), (17′) and (27), the series $\sum_{k=1}^{\infty} \varphi_k^2(1+|\lambda_k|)$, $\sum_{k=1}^{\infty} \psi_k^2$ and $\sum_{k=1}^{\infty} \int_0^T f_k^2\,dt$ converge. Therefore it follows from (30) that the series (23) converges in $H^1(Q_T)$ and accordingly its sum $u \in H^1(Q_T)$. The function $u(x, t)$ obviously satisfies the initial condition (2) and in the case of the first mixed problem the boundary condition (4). Passing to the limit, as $N \to \infty$, in the identity (21) for the problem (1)-(4) or in (22) for the problem (1), (2), (3), (5), we find that u satisfies the identity (9) or (11), respectively. Thus u is a generalized solution of the first or third mixed problem, respectively. Integrating the inequalities (29) and (29′) with respect to $t \in (0, T)$, adding them and noting (17), (17′) and (27), we obtain the inequality (24). ▌

3. **The Galerkin Method.** It is possible to give alternative proofs of the existence of generalized solutions of mixed problems that are independent of the proofs of Subsec. 2 and do not use the properties of eigenfunctions. The present subsection is devoted to one of such methods, the Galerkin method, which is simultaneously an approximate method for solving mixed problems. Note that in contrast to the Fourier method, the Galerkin method enables us to investigate mixed problems in the case when the coefficients depend not only on space variables x but also on time t. For the sake of definiteness, we examine the first mixed problem (1)-(4). As above, it is assumed that $\varphi \in \overset{\circ}{H}{}^1(D)$, $\psi \in L_2(D)$, $f \in L_2(Q_T)$.

The Galerkin method is now set forth.

Let $v_1(x)$, $v_2(x)$, . . . be an arbitrary system of functions in $C^2(\bar{D})$ satisfying the boundary condition $v_k|_{\partial D} = 0$, $k = 1, 2, \ldots$, and let this system be linearly independent and complete in $\overset{\circ}{H}{}^1(D)$, that is, the linear manifold spanned by this system is everywhere dense in $\overset{\circ}{H}{}^1(D)$. For an arbitrary integer m in the finite-dimensional subspace V_m of the space $L_2(D)$ spanned by the functions v_k, $k = 1, \ldots, m$ we seek a solution of the problem obtained from the problem (1)-(4) by orthogonal projection onto this subspace, that is, we look for a function $w_m(x, t)$ (in $H^2(Q_T)$) that belongs to V_m for every $t \in [0, T]$, satisfies conditions (2), (3) with initial functions

$\varphi^m(x) = \sum_{k=1}^{m} \varphi_k v_k(x)$, $\psi^m(x) = \sum_{k=1}^{m} \psi_k v_k(x)$ which are the orthogonal projections onto V_m of the functions $\varphi(x)$ and $\psi(x)$, respectively, and such that for almost all $t \in (0, T)$ the orthogonal projections onto V_m (in $L_2(D)$ with a scalar product) of functions $f(x, t)$ and $w_{mtt} - \operatorname{div}(k\nabla w_m) + aw_m$ coincide. This means that we try to find functions $c_1(t), \ldots, c_m(t)$ (belonging to $H^2(0, T)$) satisfying the conditions $c_k(0) = \varphi_k$, $c_k'(0) = \psi_k$, $k = 1, \ldots, m$, such that the function $w_{mtt} - \operatorname{div}(k\nabla w_m) + aw_m - f$, where

$$w_m(x, t) = \sum_{k=1}^{m} c_k(t)\, v_k(x), \tag{31}$$

is for almost all $t \in (0, T)$ (for which $f \in L_2(D_t)$) orthogonal in $L_2(D)$ to the subspace V_m, that is,

$$\int_D (w_{mtt} - \operatorname{div}(k\nabla w_m) + aw_m)\, v_k\, dx = \int_D f v_k\, dx \tag{32}$$

for $k = 1, \ldots, m$.

According to the Galerkin method, the solution u of the problem (1)-(4) is approximated by the solutions w_m of the "projected" problems. To substantiate this, we must show that the solution w_m of each of these problems exists (and is unique) and the sequence w_m, $m = 1, 2, \ldots$, converges in some sense (weakly in $H^1(Q_T)$) to u.

For simplicity, we examine the case of homogeneous initial conditions ($\varphi = 0$, $\psi = 0$). Then $\varphi_k = \psi_k = 0$, $k = 1, \ldots$, that is,

$$c_k(0) = c_k'(0) = 0, \qquad k = 1, \ldots, m. \tag{33}$$

The equations (32) constitute a linear system, regarding functions $c_1(t), \ldots, c_m(t)$, of ordinary differential equations of second order with constant coefficients

$$\sum_{s=1}^{m} \left(c_s''(t)\,(v_k, v_s)_{L_2(D)} + c_s(t)\,(v_k, v_s)_{\overset{\circ}{H}{}^1(D)}\right) = f_k(t), \qquad k = 1, \ldots, m, \tag{34}$$

where

$$f_k(t) = \int_D f(x, t)\, v_k(x)\, dx \in L_2(0, T)$$

$$\left((h, g)_{\overset{\circ}{H}{}^1(D)} = \int_D (k\nabla h \nabla g + ahg)\, dx\right).$$

Let us show that the system (34) has a unique solution that belongs to $H^2(0, T)$ (all its components belong to $H^2(0, T)$) and satisfies the initial conditions (33).

Since the system of functions $v_1, v_2, \ldots$ is linearly independent, for any $m \geqslant 1$ the determinant of the matrix with elements $(v_k, v_s)_{L_2(D)}$, $k, s = 1, \ldots, m$, is different from zero (analogous assertion was established in Sec. 1.9, Chap. IV). Therefore the linear system of ordinary differential equations (34) can be solved for higher derivatives. Consequently, the problem (34), (33) is equivalent to the problem

$$c'(t) = Ac(t) + F(t), \qquad c(0) = 0, \tag{35}$$

where $c(t) = (c_1'(t), \ldots, c_m'(t), \ c_1(t), \ldots, c_m(t))$, $F(t) = (F_1(t), \ldots, F_{2m}(t))$, $(F_1(t), \ldots, F_m(t)) = \|(\varphi_k, \varphi_s)_{L_2(D)}\|^{-1} (f_1(t), \ldots, f_m(t))$, $F_{m+1}(t) \equiv \ldots \equiv F_{2m}(t) \equiv 0$ and

$$A = -\left\| \begin{matrix} 0, & \|(\varphi_k, \varphi_s)_{L_2(D)}\|^{-1} \cdot \|(\varphi_k, \varphi_s)_{\mathring{H}^1(D)}\| \\ I, & 0 \end{matrix} \right\|$$

is a matrix of order $2m$ (I is the identity matrix of order m). Clearly, the vector $F(t) \in L_2(0, T)$ ($F_i(t) \in L_2(0, T)$, $i = 1, \ldots, 2m$).

To prove the assertion, it suffices to show that the problem (35) has a unique solution belonging to $H^1(0, T)$. As usual, we replace the problem (35) by an equivalent system of integral equations

$$c(t) = \int_0^t Ac(\tau)\, d\tau + \int_0^t F(\tau)\, d\tau, \tag{36}$$

where the free term $\int_0^t F(\tau)\, d\tau$ belongs to $H^1(0, T)$ and is therefore continuous on $[0, T]$: if $c(t)$ is a solution of the problem (35) which belongs to $H^1(0, T)$, then it is continuous on $[0, T]$ and satisfies the system (36) by Theorem 3, Sec. 6.2, Chap. III; if $c(t)$ is a solution of the system (36) that is continuous on $[0, T]$, then it obviously belongs to $H^1(0, T)$ and is a solution of the problem (35). And the existence (as well as uniqueness) of the solution (continuous on $[0, T]$) of the system of integral equations (36) is established in courses of ordinary differential equations while proving the existence theorems regarding solutions of the Cauchy problem for linear normal system of ordinary differential equations (see, e.g., L.S. Pontryagin, *Ordinary Differential Equations*, Addison-Wesley, Reading, Mass., 1962).

Thus we have established the existence and uniqueness, for any $m = 1, 2, \ldots$, of functions $w_m(x, t)$ of the form (31) that satisfy Eqs. (32) and the initial conditions $w_m|_{t=0} = \left.\frac{\partial w_m}{\partial t}\right|_{t=0} = 0$.

Multiplying (32) by $c_k'(t)$ and integrating over $(0, \tau)$, where τ is any number in $[0, T]$, and then summing over k from 1 to m, we obtain the identity

$$\int_{Q_\tau} (w_{mtt}-\operatorname{div}(k\nabla w_m)+aw_m)\, w_{mt}\, dx\, dt = \int_{Q_\tau} f w_{mt}\, dx\, dt. \qquad (37)$$

Since $w_{mtt}w_{mt}=\frac{\partial}{\partial t}\left(\frac{1}{2}w_{mt}^2\right)$, $\operatorname{div}(k\nabla w_m)\cdot w_{mt}=\operatorname{div}(kw_{mt}\nabla w_m)-$ $-\frac{\partial}{\partial t}\left(\frac{1}{2}|\nabla w_m|^2\right)$ and $aw_m w_{mt}=\frac{\partial}{\partial t}\left(\frac{1}{2}aw_m^2\right)$, we have

$$\int_{Q_\tau} (w_{mtt}-\operatorname{div}(k\nabla w_m)+aw_m)\, w_{mt}\, dx\, dt$$

$$=\frac{1}{2}\int_{D_\tau} (w_{mt}^2+k|\nabla w_m|^2+aw_m^2)\, dx.$$

Noting that in the subspace $\tilde{H}^1(Q_T)$ of $H^1(Q_T)$ that consists of functions vanishing on $\Gamma_T \cup D_0$ we may introduce the norm

$$\|w\|_{\tilde{H}^1(Q_T)}=\left(\int_{Q_T} (w_t^2+k|\nabla w|^2+aw^2)\, dx\, dt\right)^{1/2}$$

equivalent to the usual one, we obtain

$$2\int_0^T d\tau \int_{Q_\tau} (w_{mtt}-\operatorname{div}(k\nabla w_m)+aw_m)\, w_{mt}\, dx = \|w_m\|^2_{\tilde{H}^1(Q_T)}.$$

Therefore by (37)

$$\|w_m\|^2_{\tilde{H}^1(Q_T)}=2\int_0^T d\tau \int_0^\tau dt \int_D f(x, t)\, w_{mt}(x, t)\, dx$$

$$=2\int_{Q_T} (T-t)\, f(x, t)\, w_{mt}(x, t)\, dx\, dt \leqslant 2T\, \|f\|_{L_2(Q_T)}\, \|w_{mt}\|_{L_2(Q_T)}$$

$$\leqslant 2T\, \|f\|_{L_2(Q_T)}\, \|w_m\|_{\tilde{H}^1(Q_T)},$$

whence

$$\|w_m\|_{\tilde{H}^1(Q_T)} \leqslant 2T\, \|f\|_{L_2(Q_T)}.$$

Thus the set of functions w_m, $m = 1, 2, \ldots$, is bounded in $\tilde{H}^1(Q_T)$. From Theorem 3, Sec. 3.8, Chap. II, it follows that this set is weakly compact in $\tilde{H}^1(Q_T)$, that is, from it we can choose a subsequence (denoted again by w_m) that converges weakly in $\tilde{H}^1(Q_T)$ to a function $u \in \tilde{H}^1(Q_T)$.

The function u is the desired generalized solution of the mixed problem. In order to establish this, it is enough to demonstrate that for any $v \in \tilde{\tilde{H}}^1(Q_T)$, where $\tilde{\tilde{H}}^1(Q_T)$ is the subspace of $H^1(Q_T)$ composed of functions vanishing on $D_T \cup \Gamma_T$, the integral identity (9) holds (with $\psi = 0$):

$$\int_{Q_T} (k\nabla u \nabla v + auv - u_t v_t)\, dx\, dt = \int_{Q_T} fv\, dx\, dt. \tag{38}$$

For this it suffices, in turn, to establish (38) for a set of functions $\mathcal{M}$ everywhere dense in $\tilde{\tilde{H}}^1(Q_T)$.

For $\mathcal{M}$ we take the set of all linear combinations of functions $v_k(x)\, \theta\, (t)$, where $k = 1, 2, \ldots$ and $\theta(t)$ is any function in $C^1([0, T])$ such that $\theta(T) = 0$. We shall first show that (38) is valid for any function $v(x, t) = v_k\, (x)\theta\, (t)$ and therefore for any v in $\mathcal{M}$, and then show that $\mathcal{M}$ is everywhere dense in $\tilde{\tilde{H}}^1(Q_T)$.

Multiplying (32) by $\theta(t)$ and then integrating over $(0,T)$, we obtain, for $m \geqslant k$,

$$\int_{Q_T} [(k\nabla w_m \nabla v_k + aw_m v_k)\, \theta - w_{mt} v_k \theta']\, dx\, dt = \int_{Q_T} fv_k \theta\, dx\, dt.$$

Hence follows (38) since, as $m \to \infty$, w_m converges weakly in $H^1(Q_T)$ to u.

Let us show that $\mathcal{M}$ is everywhere dense in $\tilde{\tilde{H}}^1(Q_T)$. For this, it suffices to establish that any function $\eta\, (x, t)$ belonging to $C^2(\bar{Q}_T)$ and satisfying the condition

$$\eta\,|_{\Gamma_T \cup D_T} = 0 \tag{39}$$

(the set of such functions is everywhere dense in $\tilde{\tilde{H}}^1(Q_T)$) can be approximated by functions belonging to $\mathcal{M}$ in the metric of the space $H^1(Q_T)$. The norm in $\tilde{\tilde{H}}^1(Q_T)$ is defined by the formula

$$\| f \|_{\tilde{\tilde{H}}^1(Q_T)} = \Big(\int_{Q_T} (f_t^2 + |\nabla f|^2)\, dx\, dt \Big)^{1/2}.$$

Note that the set $\mathcal{M}$ can be regarded as a set of all linear combinations of functions $v_k^*(x)\, \theta\, (t)$, where $\theta(t)$ is an arbitrary function in $C^1([0, T])$ that vanishes for $t = T$ and $v_1^*, v_2^*, \ldots$ is an orthonormal basis for the space $\overset{\circ}{H}{}^1(D)$ $\Big($in the scalar product $(f, g)_{\overset{\circ}{H}{}^1(D)} = \int_D \nabla f\, \nabla g\, dx\Big)$ obtained by orthonormalizing the system $v_1, v_2, \ldots$ according to the Gram-Schmidt method (see Sec. 2.5, Chap. II).

Let $\eta(x, t)$ be an arbitrary function in $C^2(\bar{Q}_T)$ satisfying the condition (39). Since for any $t \in [0, T]$ the functions $\eta(x, t)$ and

$\eta_t(x, t)$ belong to $\overset{\circ}{H}{}^1(D)$, they can be expanded in Fourier series which converge in the metric of $\overset{\circ}{H}{}^1(D)$:

$$\begin{aligned} \eta(x, t) &= \sum_{k=1}^{\infty} \eta_k(t)\, v_k^*(x), \\ \eta_t(x, t) &= \sum_{k=1}^{\infty} \eta_k'(t)\, v_k^*(x), \end{aligned} \tag{40}$$

where

$$\eta_k(t) = \int_D \nabla\eta(x, t)\, \nabla u_k^*(x)\, dx. \tag{41}$$

Furthermore,

$$\sum_{k=1}^{\infty} (\eta_k^2(t) + \eta_k'^2(t)) = \int_D (|\nabla\eta(x, t)|^2 + |\nabla\eta_t(x, t)|^2)\, dx, \qquad t \in [0, T]. \tag{42}$$

Let $\eta_N(x, t)$ denote the partial sum of the series in (40):

$$\eta_N(x, t) = \sum_{k=1}^{N} \eta_k(t)\, v_k^*(x). \tag{43}$$

It follows from (41) and (43) that for any $N \geqslant 1$ and all $t \in [0, T]$ the function $\eta_t - \eta_{Nt} \in \overset{\circ}{H}{}^1(D_t)$. Therefore by Steklov's inequality (Sec. 5.6, Chap. III),

$$\| \eta_t - \eta_{Nt} \|_{L_2(D_t)} \leqslant C \| \eta_t - \eta_{Nt} \|_{\overset{\circ}{H}{}^1(D_t)},$$

where the constant $C > 0$ depends only on the region D. Accordingly, for any $N \geqslant 1$ and all $t \in [0, T]$

$$\begin{aligned} \| \eta_t - \eta_{Nt} \|^2_{L_2(D_t)} + \| \eta - \eta_N \|^2_{\overset{\circ}{H}{}^1(D_t)} &\leqslant C^2 \| \eta_t - \eta_{Nt} \|^2_{\overset{\circ}{H}{}^1(D_t)} \\ &+ \| \eta - \eta_N \|^2_{\overset{\circ}{H}{}^1(D_t)} = \sum_{k=N+1}^{\infty} (\eta_k^2(t) + C^2 \eta_k'^2(t)). \end{aligned}$$

In view of (42), for any $t \in [0, T]$, $\sum\limits_{k=N+1}^{\infty} (\eta_k^2(t) + C^2\eta_k'^2(t)) \downarrow 0$ as $N \to \infty$. Hence Levi's theorem (Theorem 3, Sec. 1.6, Chap. II) implies, as $N \to \infty$, that

$$\| \eta - \eta_N \|^2_{\tilde{\tilde{H}}{}^1(Q_T)} = \int_0^T (\| \eta_t - \eta_{Nt} \|^2_{L_2(D_t)} + \| \eta - \eta_N \|^2_{\overset{\circ}{H}{}^1(D_t)})\, dt \to 0. \qquad \blacksquare$$

Note that since the generalized solution u of the problem (1)-(4) is unique (Theorem 1), it follows from what has been proved that not only a subsequence of the sequence w_m, $m = 1, 2, \ldots$, but also this sequence itself converges weakly in $H^1(Q_T)$ to u.

4. Smoothness of Generalized Solutions. Existence of A. E. Solution and Classical Solution. The investigation of smoothness of generalized solutions will be confined to the case of first and second mixed problems (in the boundary condition (5) $\sigma \equiv 0$) for a particular case of Eq. (1), the wave equation (in (1) $k \equiv 1$, $a \equiv 0$), although, if the coefficients of the equation and the function σ are sufficiently smooth, the same method can be applied to obtain analogous results in the general case.

Suppose that $u(x, t)$ is a generalized solution of the first or second mixed problem for the wave equation

$$u_{tt} - \Delta u = f(x, t), \tag{44}$$

$$u|_{t=0} = \varphi, \qquad u_t|_{t=0} = \psi \tag{45}$$

and either

$$u|_{\Gamma_T} = 0 \tag{46}$$

in the case of the first mixed problem, or

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_T} = 0 \tag{47}$$

in the case of the second mixed problem.

It was shown in earlier subsections that the problems (44)-(46) and (44), (45), (47) have (unique) generalized solutions provided $\psi \in L_2(D)$, $f \in L_2(Q_T)$ and the function φ belongs to the space $\mathring{H}^1(D)$ in the case of the first mixed problem or to the space $H^1(D)$ in the case of the second mixed problem. Furthermore, each of these generalized solutions $u(x, t)$ is represented by a convergent series in $H^1(Q_T)$:

$$u(x, t) = \sum_{k=1}^{\infty} U_k(t) v_k(x), \tag{48}$$

where

$$U_k(t) = \varphi_k \cos \sqrt{-\lambda_k}\, t + \frac{\psi_k}{\sqrt{-\lambda_k}} \sin \sqrt{-\lambda_k}\, t$$

$$+ \frac{1}{\sqrt{-\lambda_k}} \int_0^t f_k(\tau) \sin \sqrt{-\lambda_k}\,(t-\tau)\, d\tau, \qquad k = 1, 2, \ldots \tag{49}$$

(in the case of the second mixed problem

$$U_1(t) = \varphi_1 + t\psi_1 + \int\limits_0^t (t-\tau)\, f_1(\tau)\, d\tau$$

$$= \lim_{\lambda \to 0} \Big(\varphi_1 \cos \sqrt{-\lambda}\, t + \frac{\psi_1}{\sqrt{-\lambda}} \sin \sqrt{-\lambda}\, t$$

$$+ \frac{1}{\sqrt{-\lambda}} \int\limits_0^t f_1(\tau) \sin \sqrt{-\lambda}\,(t-\tau)\, d\tau \Big) \Big),$$

$$\varphi_k = (\varphi, v_k)_{L_2(D)}, \qquad \psi_k = (\psi, v_k)_{L_2(D)},$$

$$f_k(t) = \int\limits_{D_t} f(x, t)\, v_k(x)\, dx, \qquad k = 1, 2, \ldots, \tag{50}$$

and $v_1, v_2, \ldots$ and $\lambda_1, \lambda_2, \ldots$ are sequences of generalized eigenfunctions and corresponding eigenvalues of the first, if the problem (44)-(46) is considered, or the second, if the problem (44), (45), (47) is under consideration, boundary-value problem for the Laplace operator in D (recall that in the case of the first boundary-value problem $\lambda_k < 0$ for all $k = 1, 2, \ldots$, and in the case of the second boundary-value problem $\lambda_k < 0$ for $k = 2, 3, \ldots$ and $\lambda_1 = 0$ while $v_1 = \text{const} = 1/\sqrt{|D|}$).

Assume that the boundary ∂D of D belongs to the class C^s for some $s \geqslant 1$. Then, by Theorem 7, Sec. 2.4, Chap. IV, the eigenfunctions $v_k(x)$, $k = 1, 2, \ldots$, of the first and second boundary-value problems for the Laplace operator belong to the spaces $H^s_{\mathscr{D}}(D)$ and $H^s_{\mathscr{N}}(D)$, respectively, that is, belong to $H^s(D)$ and on ∂D satisfy the boundary conditions

$$v_k \big|_{\partial D} = \ldots = \Delta^{\left[\frac{s-1}{2}\right]} v_k \big|_{\partial D} = 0, \qquad k = 1, 2, \ldots,$$

in the case of the first boundary-value problem and the boundary conditions

$$\frac{\partial v_k}{\partial n}\bigg|_{\partial D} = \ldots = \frac{\partial}{\partial n} \Delta^{\left[\frac{s}{2}\right]-1} v_k \bigg|_{\partial D} = 0, \qquad k = 1, 2, \ldots,$$

for $s > 1$ in the case of the second boundary-value problem. Recall that $H^1_{\mathscr{N}}(D) = H^1(D)$.

Assume also that in the case of the first mixed problem (44)-(46) $\varphi \in H^s_{\mathscr{D}}(D)$, $\psi \in H^{s-1}_{\mathscr{D}}(D)$ and f belongs to the subspace $\tilde{H}^{s-1}_{\mathscr{D}}(Q_T)$ of $H^{s-1}(Q_T)$ which is composed of all functions $f \in H^{s-1}(Q_T)$, $s > 1$, such that

$$f|_{\Gamma_T}=\ldots=\Delta^{\left[\frac{s}{2}\right]-1}f|_{\Gamma_T}=0.$$

When $s=1$, $\widetilde{H}^{s-1}_{\mathscr{D}}(Q_T)=\widetilde{H}^{0}_{\mathscr{D}}(Q_T)=L_2(Q_T)$.

In the case of the second mixed problem (44), (45), (47) we assume that $\varphi\in H^{s}_{\mathscr{N}}(D)$, $\psi\in H^{s-1}_{\mathscr{N}}(D)$ and f belongs to the subspace $\widetilde{H}^{s-1}_{\mathscr{N}}(Q_T)$ of the space $H^{s-1}(Q_T)$ which is composed of all functions $f\in H^{s-1}(Q_T)$, $s>2$, such that

$$\frac{\partial f}{\partial n}\bigg|_{\Gamma_T}=\ldots=\frac{\partial}{\partial n}\,\Delta^{\left[\frac{s-1}{2}\right]-1}f|_{\Gamma_T}=0.$$

When $s=2$, $\widetilde{H}^{s-1}_{\mathscr{N}}(Q_T)=\widetilde{H}^{1}_{\mathscr{N}}(Q_T)=H^1(Q_T)$ and when $s=1$ $\widetilde{H}^{s-1}_{\mathscr{N}}(Q_T)=\widetilde{H}^{0}_{\mathscr{N}}(Q_T)=L_2(Q_T)$.

In this subsection we shall prove that under aforementioned assumptions the generalized solutions of the mixed problems belong to the space $H^s(Q_T)$ and are classical solutions for sufficiently large s.

Theorem 3. *Suppose that for some $s\geqslant 1$ $\partial D\in C^s$ and in the case of the first mixed problem* (44)-(46) $\varphi\in H^{s}_{\mathscr{D}}(D)$, $\psi\in H^{s-1}_{\mathscr{D}}(D)$, $f\in\widetilde{H}^{s-1}_{\mathscr{D}}(Q_T)$, *while in the case of the second mixed problem* (44), (45), (47) $\varphi\in H^{s}_{\mathscr{N}}(D)$, $\psi\in H^{s-1}_{\mathscr{N}}(D)$, $f\in\widetilde{H}^{s-1}_{\mathscr{N}}(Q_T)$. *Then the series* (48) *converges to the generalized solution $u(x,t)$ in $H^s(D_t)$ uniformly in $t\in[0,T]$. Furthermore, for any $p=1,\ldots,s$ the series obtained from* (48) *by p-times termwise differentiation with respect to t converges in $H^{s-p}(D_t)$ uniformly in $t\in[0,T]$, and for all $t\in[0,T]$ the following inequalities hold*

$$\sum_{p=0}^{s}\left\|\sum_{k=1}^{\infty}\frac{\partial^p}{\partial t^p}(U_k(t)\,v_k(x))\right\|^2_{H^{s-p}(D_t)}\leqslant C\left(\|\varphi\|^2_{H^s(D)}+\|\psi\|^2_{H^{s-1}(D)}+\|f\|_{H^{s-1}(Q_T)}\right). \tag{51}$$

The assertion of the theorem regarding convergence in $H^{s-p}(D_t)$, $p=0,\ldots,s$, which is uniform in $t\in[0,T]$, of the series obtained from (48) by p-times termwise differentiation with respect to t means that for any $t\in[0,T]$ the sequence of traces $\sum\limits_{k=1}^{N}\frac{\partial^p}{\partial t^p}\times$ $\times\,(U_k(t)\,v_k(x))\,|_{D_t}$ on D_t of pth derivatives with respect to t of the partial sums of the series (48) (each of these partial sums belongs to $H^1(Q_T)$) converges in $H^{s-p}(D_t)$ and this convergence is uniform in $t \in [0, T]$, that is,

$$\sup_{0\leqslant t\leqslant T} \left\| \sum_{k=M+1}^{N} \frac{\partial^p}{\partial t^p} (U_k(t)\, v_k(x)) \right\|_{H^{s-p}(D_t)} \to 0 \text{ as } M, N \to \infty.$$

Then the sequence of partial sums of the series (48) converges also in $H^s(Q_T)$ and the estimate (51) implies the inequality

$$\| u \|_{H^s(Q_T)} \leqslant C' (\| \varphi \|_{H^s(D)} + \| \psi \|_{H^{s-1}(D)} + \| f \|_{H^{s-1}(Q_T)}). \qquad (52)$$

Thus the following assertion holds.

Corollary 1. *For some $s \geqslant 1$, let $\partial D \in C^s$ and in the case of the first mixed problem* (44)-(46) $\varphi \in H^s_{\mathscr{D}}(D)$, $\psi \in H^{s-1}_{\mathscr{D}}(D)$, $f \in \tilde{H}^{s-1}_{\mathscr{D}}(Q_T)$ *and in the case of the second mixed problem* (44), (45), (47) $\varphi \in H^s_{\mathscr{N}}(D)$, $\psi \in H^{s-1}_{\mathscr{N}}(D)$, $f \in \tilde{H}^{s-1}_{\mathscr{N}}(Q_T)$. *Then the generalized solution of each of these problems belongs to $H^s(Q_T)$ and the series* (48) *converges to it in $H^s(Q_T)$. Moreover, the inequality* (52) *holds.*

For any $p = 0, \ldots, s-1$ the function $\frac{\partial^p u}{\partial t^p}$ has a trace on D_t for every $t \in [0, T]$ and the series obtained from (48) by p-times termwise differentiation with respect to t converges to $\left.\frac{\partial^p u}{\partial t^p}\right|_{D_t}$ in $H^{s-p}(D_t)$ uniformly in $t \in [0, T]$. Since also for $p = s$ the sequence of partial sums of the series $\sum\limits_{k=1}^{\infty} \frac{\partial^s}{\partial t^s} (U_k(t)\, v_k(x))\,|_{D_t}$ composed of the traces on D_t of the functions $\frac{\partial^s (U_k v_k)}{\partial t^s}$ belonging to $H^1(Q_T)$ converges in $L_2(D_t)$ (uniformly in $t \in [0, T]$), its limit for every $t \in [0, T]$ can be called the trace on D_t of the sth derivative with respect to t of the generalized solution $u(x, t)$.

Before proving Theorem 3, let us establish the following auxiliary result.

Lemma 2. *If $f \in H^q(Q_T)$, $q \geqslant 0$, and $g \in L_2(D)$, then the function*

$$h(t) = \int_{D_t} f(x, t)\, g(x)\, dx$$

belongs to $H^q(0, T)$ and the relations

$$\frac{d^p h(t)}{dt^p} = \int_{D_t} \frac{\partial^p f(x, t)}{\partial t^p}\, g(x)\, dx, \qquad 0 \leqslant p \leqslant q,$$

hold.

Proof. Since for $p = 0, 1, \ldots, q$ $\frac{\partial^p f}{\partial t^p} \in L_2(Q_T)$, by Fubini's theorem for almost all $t \in (0, T)$ the functions $g(x) \frac{\partial^p f(x, t)}{\partial t^p}$ are integrable over D_t and the functions

$$h^{(p)}(t)=\int\limits_{D_t}\frac{\partial^p f(x,t)}{\partial t^p}\,g(x)\,dx,\qquad p=0,1,\dots,q,$$

$(h^{(0)}(t)=h(t))$ are integrable over $(0, T)$. Further, because

$$\Big(\int\limits_{D_t}\frac{\partial^p f(x,t)}{\partial t^p}\,g(x)\,dx\Big)^2\leqslant\int\limits_{D_t}\Big(\frac{\partial^p f(x,t)}{\partial t^p}\Big)^2dx\cdot\|g\|^2_{L_2(D)},$$

$h^{(p)}(t)\in L_2(0,T)$, $p=0,\dots,q$.

For any function $\eta(x,t)\in\dot{C}^q(\overline{Q}_T)$

$$\int\limits_{Q_T}\frac{\partial^p f(x,t)}{\partial t^p}\,\eta(x,t)\,dx\,dt=(-1)^p\int\limits_{Q_T}f(x,t)\frac{\partial^p\eta(x,t)}{\partial t^p}\,dx\,dt,$$

so for any $\eta_1(t)\in\dot{C}^q([0,T])$ and $\eta_2(x)\in\dot{C}^q(\overline{D})$

$$\int\limits_0^T\eta_1(t)\Big(\int\limits_D\frac{\partial^p f}{\partial t^p}\,\eta_2(x)\,dx\Big)dt=(-1)^p\int\limits_0^T\frac{d^p\eta_1(t)}{dt^p}\Big(\int\limits_D f\eta_2(x)\,dx\Big)dt.$$

The set $\dot{C}^q(\overline{D})$ is everywhere dense in $L_2(D)$, therefore the last equality is also valid for arbitrary $\eta_2\in L_2(D)$, and, in particular, for $\eta_2=g$. Thus for all $\eta_1(t)\in\dot{C}^q([0,T])$

$$\int\limits_0^T\eta_1h^{(p)}dt=(-1)^p\int\limits_0^T\frac{d^p\eta_1}{dt^p}\,h\,dt,\qquad p=1,\dots,q.$$

This means that for $p=1,\dots,q$ the function $h^{(p)}(t)$ is the generalized derivative of pth order of the function $h(t)$, that is, $\frac{d^ph}{dt^p}=h^{(p)}\in L_2(0,T)$. ■

Proof of Theorem 3. It follows from Lemma 2 that the functions $f_k(t)$ $k=1,2,\dots$, defined by (50) belong to the space $H^{s-1}(0,T)$, and therefore (see Theorem 3, Sec. 6.2, Chap. III) to the space $C^{s-2}([0,T])$ for $s\geqslant 2$. Consequently, the functions U_k, $k=1,2,\dots$, defined by formula (49) and satisfying on $(0,T)$ the equations $U_k''-\lambda_kU_k=f_k$ belong to the space $H^{s+1}(0,T)$, thereby also to $C^s([0,T])$.

Then, by the properties of eigenfunctions $v_k(x)$, the partial sums $S_N(x,t)=\sum\limits_{k=1}^{N}U_k(t)\,v_k(x)$ of the series (48) belong to the space $H^s(Q_T)$ and for all $t\in[0,T]$ to the space $H^s_{\mathscr{D}}(D_t)$ in the case of problem (44)-(46) (or to the space $H^s_{\mathscr{N}}(D_t)$ in the case of problem (44), (45), (47)).

Further, for $p = 1, \ldots, s$ the function $\frac{\partial^p S_N}{\partial t^p}$ belongs to the space $H^{s-p+1}(Q_T)$ and for all $t \in [0, T]$ to the space $H^s_{\mathscr{D}}(D_t)$ $(H^s_{\mathscr{N}}(D_t))$. Therefore, in view of Lemma 3, Sec. 2.5, Chap. IV, and orthogonality of eigenfunctions $v_k(x)$ in $L_2(D)$ and $H^1(D)$, we have for all $t \in [0, T]$, arbitrary $p = 0, \ldots, s$ and any M and N, $1 \leqslant M < N$, the inequalities

$$\left\| \frac{\partial^p S_N}{\partial t^p} - \frac{\partial^p S_M}{\partial t^p} \right\|^2_{H^{s-p}(D_t)} \leqslant C_1 \left\| \Delta^{\frac{s-p}{2}} \frac{\partial^p}{\partial t^p} (S_N - S_M) \right\|^2_{L_2(D_t)}$$

$$= C_1 \left\| \sum_{k=M+1}^{N} |\lambda_k|^{\frac{s-p}{2}} \frac{d^p U_k(t)}{dt^p} v_k(x) \right\|^2_{L_2(D_t)}$$

$$= C_1 \sum_{k=M+1}^{N} |\lambda_k|^{s-p} \left(\frac{d^p U_k}{dt^p} \right)^2$$

if $s - p$ is even and

$$\left\| \frac{\partial^p S_N}{\partial t^p} - \frac{\partial^p S_M}{\partial t^p} \right\|^2_{H^{s-p}(D_t)} \leqslant C_1' \left\| \Delta^{\frac{s-p-1}{2}} \frac{\partial^p}{\partial t^p} (S_N - S_M) \right\|^2_{H^1(D_t)}$$

$$= C_1' \left\| \sum_{k=M+1}^{N} |\lambda_k|^{\frac{s-p-1}{2}} \frac{d^p U_k(t)}{dt^p} v_k(x) \right\|^2_{H^1(D_t)}$$

$$\leqslant C_1 \sum_{k=M+1}^{N} |\lambda_k|^{s-p} \left(\frac{d^p U_k(t)}{dt^p} \right)^2$$

if $s - p$ is odd. That is, for all $t \in [0, T]$, arbitrary $p = 0, \ldots,$ and arbitrary M and N, $1 \leqslant M < N$

$$\left\| \frac{\partial^p (S_N - S_M)}{\partial t^p} \right\|^2_{H^{s-p}(D_t)} \leqslant C_1 \sum_{k=M+1}^{N} |\lambda_k|^{s-p} \left(\frac{d^p U_k(t)}{\partial t^p} \right)^2. \tag{53}$$

Similarly, for all $t \in [0, T]$, arbitrary $p = 0, \ldots, s$ and any $N \geqslant 1$

$$\left\| \frac{\partial^p S_N}{\partial t^p} \right\|^2_{H^{s-p}(D_t)} \leqslant C_1 \sum_{k=1}^{N} |\lambda_k|^{s-p} \left(\frac{d^p U_k(t)}{dt^p} \right)^2$$

in the case of the first mixed problem ($\lambda_1 \neq 0$) and

$$\left\| \frac{\partial^p S_N}{\partial t^p} \right\|^2_{H^{s-p}(D_t)} = \left\| \frac{\partial^p (U_1 v_1)}{\partial t^p} + \frac{\partial^p (S_N - S_1)}{\partial t^p} \right\|^2_{H^{s-p}(D_t)}$$

$$\leqslant 2 \left(\frac{d^p U_1}{dt^p} \right)^2 \left\| \frac{1}{\sqrt{|D|}} \right\|^2_{H^{s-p}(D)_t} + 2 \left\| \frac{\partial^p (S_N - S_1)}{\partial t^p} \right\|^2_{H^{s-p}(D_t)}$$

$$\leqslant C_2 \left(\left(\frac{d^p U_1}{dt^p} \right)^2 + \sum_{k=2}^{N} |\lambda_k|^{s-p} \left(\frac{d^p U_k}{dt^p} \right)^2 \right)$$

in the case of the second mixed problem ($\lambda_1 = 0$). Thus for all $t \in [0, T]$, $p = 0, \ldots, s$, $N \geqslant 1$

$$\left\| \frac{\partial^p S_N}{\partial t^p} \right\|^2_{H^{s-p}(D_t)} \leqslant C_3 \left(\left(\frac{d^p U_1}{dt^p} \right)^2 + \sum_{k=1}^{N} |\lambda_k|^{s-p} \left(\frac{d^p U_k}{dt^p} \right)^2 \right).$$

Summing these inequalities over p from zero to s, we obtain

$$\sum_{p=0}^{s} \left\| \frac{d^p S_N}{\partial t^p} \right\|^2_{H^{s-p}(D_t)} \leqslant C_3 \sum_{p=0}^{s} \left[\left(\frac{d^p U_1}{\partial t^p} \right)^2 + \sum_{k=1}^{N} |\lambda_k|^{s-p} \left(\frac{d^p U_k}{dt^p} \right)^2 \right]. \quad (54)$$

We now apply the following lemma which will be proved later.

Lemma 3. *If for some* $s \geqslant 1$ $\partial D \in C^s$ *and* $\varphi \in H^s_{\mathscr{D}}(D)$, $\psi \in H^{s-1}_{\mathscr{D}}(D)$, $f \in \widetilde{H}^{s-1}_{\mathscr{D}}(Q_T)$ *in the case of the first mixed problem* (44)-(46) *or* $\varphi \in H^s_{\mathscr{N}}(D)$, $\psi \in H^{s-1}_{\mathscr{N}}(D)$, $f \in \widetilde{H}^{s-1}_{\mathscr{N}}(Q_T)$ *in the case of the second mixed problem* (44), (45), (47), *then for any* $p \leqslant s$ *the series* $\sum_{k=1}^{\infty} \left(\frac{d^p U_k(t)}{dt^p} \right)^2 \times$ $\times |\lambda_k|^{s-p}$ *converges uniformly in* $t \in [0, T]$ *and*

$$\sum_{k=1}^{\infty} \left(\frac{d^p U_k}{dt^p} \right)^2 |\lambda_k|^{s-p} \leqslant C (\|\varphi\|^2_{H^s(D)} + \|\psi\|^2_{H^{s-1}(D)} + \|f\|^2_{H^{s-1}(Q_T)}), \quad (55)$$

where the constant $C > 0$ *depends only on* Q_T.

In view of this lemma, the inequalities (53) imply that for every $p = 0, 1, \ldots, s$ the sequence $\left. \frac{\partial^p S_N}{\partial t^p} \right|_{D_t}$ converges in $H^{s-p}(D_t)$ uniformly with respect to $t \in [0, T]$, and the inequality (54) together with the obvious estimate $\left(\frac{d^p U_1}{dt^p} \right)^2 \leqslant \text{const} \, (\|\varphi\|^2_{L_2(D)} + \|\psi\|^2_{L_2(D)} + \|f\|^2_{H^{s-1}(Q_T)})$ implies inequality (51). ∎

When $s = 2$, it follows from Corollary 1 that the generalized solution of each of the mixed problems under consideration belongs to $H^2(Q_T)$, and is consequently an a.e. solution.

Note that in the hypothesis of Theorem 3 the following conditions, apart from the condition on smoothness of given functions, are assumed satisfied:

$$\varphi|_{\partial D}=\dots=\Delta^{\left[\frac{s-1}{2}\right]}\varphi|_{\partial D}=0,\quad \psi|_{\partial D}=\dots=\Delta^{\left[\frac{s}{2}\right]-1}\psi|_{\partial D}=0 \tag{56}$$

and

$$f|_{\Gamma_T}=\dots=\Delta^{\left[\frac{s}{2}\right]-1}f|_{\Gamma_T}=0 \tag{57}$$

in the case of the first mixed problem, and

$$\frac{\partial\varphi}{\partial n}\bigg|_{\partial D}=\dots=\frac{\partial}{\partial n}\Delta^{\left[\frac{s}{2}\right]-1}\varphi|_{\partial D}=0,$$

$$\frac{\partial\psi}{\partial n}\bigg|_{\partial D}=\dots=\frac{\partial}{\partial n}\Delta^{\left[\frac{s-3}{2}\right]}\psi|_{\partial D}=0 \tag{58}$$

and

$$\frac{\partial f}{\partial n}\bigg|_{\Gamma_T}=\dots=\frac{\partial}{\partial n}\Delta^{\left[\frac{s-3}{2}\right]}f|_{\Gamma_T}=0 \tag{59}$$

in the case of the second mixed problem. Note that for the validity of Theorem 3 some conditions of this kind are necessary.

Indeed, in the case of the first mixed problem, for example, with $s\geqslant 2$ the fact that $\varphi(x)=u(x, t)|_{t=0}$ is represented by a convergent series (48) in $H^s(D_0)$ and $\psi(x)=\frac{\partial u(x, t)}{\partial t}\Big|_{t=0}$ (u is an a.e. solution) by a convergent series $\sum\limits_{k=1}^{\infty}\frac{dU_k(t)}{dt}\Big|_{t=0} v_k(x)$ in $H^{s-1}(D_0)$ implies that conditions (56) hold. Since the series (48) converges to a.e. solution $u(x, t)$ in $H^s(Q_T)$, and thus the series $\sum\limits_{k=1}^{\infty} U_k(t)\,\Delta v_k(x)$ and $\sum\limits_{k=1}^{\infty}\frac{d^2U_k(t)}{dt^2}v_k(x)$ to Δu and u_{tt}, respectively, in $H^{s-2}(Q_T)$, it follows that for $s\geqslant 3$ $f=u_{tt}-\Delta u$ satisfies the conditions

$$f|_{\Gamma_T}=\dots=\Delta^{\left[\frac{s-3}{2}\right]}f|_{\Gamma_T}=0.$$

When s is even, Theorem 3 additionally assumes the condition $\Delta^{\left[\frac{s}{2}\right]-1}f|_{\Gamma_T}=0$. This condition is in fact unnecessary. It will be demonstrated for $s=2$.

Corollary 2. *Let* $\partial D\in C^2$ *and* $f\in H^1(Q_T)$, *and let* $\varphi\in H^2_{\mathscr{D}}(D)$, $\psi\in H^1_{\mathscr{D}}(D)$ *in the case of problem* (44)-(46) *or* $\varphi\in H^2_{\mathscr{N}}(D)$, $\psi\in H^1_{\mathscr{N}}(D)$ *in the case of problem* (44), (45), (47). *Then for* $p=0, 1, 2$

the series obtained from (48) *by p-times termwise differentiation with respect to t converges in* $H^{2-p}(D_t)$ *uniformly with respect to* $t \in [0, T]$ *and the sum* $u(x, t)$ *of the series* (48) *is an a.e. solution of the problem* (44)-(46) *or the problem* (44), (45), (47), *respectively. Moreover, for all* $t \in [0, T]$ *inequalities* (51) *hold for* $s = 2$.

Proof. In view of Theorem 3, it is enough to establish this assertion for homogeneous initial conditions: $\varphi = 0$, $\psi = 0$.

Since for $k > 1$

$$U_k(t) = \frac{1}{\sqrt{-\lambda_k}} \int_0^t f_k(\tau) \sin \sqrt{-\lambda_k}\,(t-\tau)\, d\tau$$

$$= \frac{1}{|\lambda_k|}\left(f_k(t) - f_k(0) \cos \sqrt{-\lambda_k}\, t\right)$$

$$- \frac{1}{|\lambda_k|} \int_k^t f_k'(\tau) \cos \sqrt{-\lambda_k}\,(t-\tau)\, d\tau,$$

$$U_k'(t) = \int_0^t f_k(\tau) \cos \sqrt{-\lambda_k}\,(t-\tau)\, d\tau$$

$$= \frac{1}{\sqrt{-\lambda_k}} f_k(0) \sin \sqrt{-\lambda_k}\, t + \frac{1}{\sqrt{-\lambda_k}} \int_0^t f_k'(\tau) \sin \sqrt{-\lambda_k}\,(t-\tau)\, d\tau,$$

$$U_k''(t) = f_k(t) + \lambda_k U_k(t)$$

$$= f_k(0) \cos \sqrt{-\lambda_k}\, t + \int_0^t f_k'(\tau) \cos \sqrt{-\lambda_k}\,(t-\tau)\, d\tau,$$

we have

$$\lambda_k^2 U_k^2(t) \leqslant \mathrm{const}\left(f_k^2(t) + f_k^2(0) + T \int_0^T (f_k'(\tau))^2\, d\tau\right),$$

$$|\lambda_k|\,(U_k'(t))^2 \leqslant \mathrm{const}\left(f_k^2(0) + T \int_0^T (f_k'(\tau))^2\, d\tau\right),$$

$$(U_k''(t))^2 \leqslant \mathrm{const}\left(f_k^2(0) + T \int_0^T (f_k'(\tau))^2\, d\tau\right).$$

Since Lemma 2 together with the fact that f belongs to $H^1(Q_T)$ implies that the series $\sum_{k=1}^{\infty} \int_0^T (f_k'(\tau))^2\, d\tau$ and $\sum_{k=1}^{\infty} f_k^2(t)$ converge uniformly in $t \in [0, T]$, the desired assertion follows from inequalities (53) and (54). ∎

Note that if $f=0$ the relation

$$\left\|\frac{\partial u}{\partial t}\right\|^2_{L_2(D_t)}+\left\|\,|\nabla u|\,\right\|^2_{L_2(D_t)}=\sum_{k=1}^{\infty}\left(\left(\frac{dU_k(t)}{dt}\right)^2+|\lambda_k|\,U_k^2(t)\right)$$

$$=\sum_{k=1}^{\infty}[(\psi_k\cos\sqrt{|\lambda_k|}\,t-\varphi_k\sqrt{|\lambda_k|}\sin\sqrt{|\lambda_k|}\,t)^2$$

$$+(\psi_k\sin\sqrt{|\lambda_k|}\,t+\varphi_k\sqrt{|\lambda_k|}\cos\sqrt{|\lambda_k|}\,t)^2]$$

$$=\sum_{k=1}^{\infty}(\psi_k^2+|\lambda_k|\,\varphi_k^2)=\|\psi\|^2_{L_2(D)}+\|\,|\nabla\varphi|\,\|_{L_2(D)}$$

implies that for all $t\in[0, T]$ the solutions satisfy the identity

$$\int\limits_{D_t}\left(\left(\frac{\partial u(x, t)}{\partial t}\right)^2+|\nabla u(x, t)|^2\right)dx=\int\limits_{D}(\psi^2+|\nabla\varphi|^2)\,dx,$$

which is referred to as "*the law of conservation of energy*".

Theorem 4. *Let* $\partial D\in C^{[\frac{n}{2}]+3}$. *Let* $\varphi\in H_{\mathscr{D}}^{[\frac{n}{2}]+3}(D)$, $\psi\in H_{\mathscr{D}}^{[\frac{n}{2}]+2}(D)$, $f\in\tilde{H}_{\mathscr{D}}^{[\frac{n}{2}]+2}(Q_T)$ *in the case of problem* (44)-(46) *and* $\varphi\in H_{\mathscr{N}}^{[\frac{n}{2}]+3}(D)$, $\psi\in H_{\mathscr{N}}^{[\frac{n}{2}]+2}(D)$, $f\in\tilde{H}_{\mathscr{N}}^{[\frac{n}{2}]+2}(Q_T)$ *in the case of problem* (44), (45), (47). *Then the series* (48) *converges in* $C^2(\bar{Q}_T)$ *and its sum* $u(x, t)$ *is a classical solution of the corresponding problem. Solution* u *satisfies the inequalities*

$$\|u\|_{C^p(\bar{Q}_T)}\leqslant C\left(\|\varphi\|_{H^{[\frac{n}{2}]+p+1}}+\|\psi\|_{H^{[\frac{n}{2}]+p}(D)}+\|f\|_{H^{[\frac{n}{2}]+p}(Q_T)}\right),\qquad p=0, 1, 2. \tag{60}$$

Proof. Since $\partial D\in C^{[\frac{n}{2}]+3}$, the generalized eigenfunctions $v_1(x)$, $v_2(x)$, . . . of the first and second boundary-value problems for the Laplace operator in D belong to the space $H^{[\frac{n}{2}]+3}(D)$ and hence, by Theorem 3, Sec. 6.2, Chap. III, to $C^2(\bar{D})$. Therefore the partial sums $S_N(x, t)$, $N = 1, 2, \ldots$, of the series (48) belong to $C^2(\bar{Q}_T)$.

According to Theorem 3, Sec. 6.2, Chap. III, and inequality (53), for all $t \in [0, T]$ and $1 \leqslant M < N$ we have

$$\|S_N - S_M\|^2_{C^2(\overline{D}_t)} + \left\|\frac{\partial}{\partial t}(S_N - S_M)\right\|^2_{C^1(\overline{D}_t)} + \left\|\frac{\partial^2}{\partial t^2}(S_N - S_M)\right\|^2_{C(\overline{D}_t)}$$

$$\leqslant C\left(\|S_N - S_M\|^2_{H^{\left[\frac{n}{2}\right]+3}(D_t)} + \left\|\frac{\partial}{\partial t}(S_N - S_M)\right\|^2_{H^{\left[\frac{n}{2}\right]+2}(D_t)} + \left\|\frac{\partial^2}{\partial t^2}(S_N - S_M)\right\|^2_{H^{\left[\frac{n}{2}\right]+1}(D_t)}\right)$$

$$\leqslant C_4 \sum_{p=0}^{2} \sum_{k=M+1}^{N} |\lambda_k|^{\left[\frac{n}{2}\right]+3-p}\left(\frac{d^pU_k}{dt^p}\right)^2,$$

and, hence,

$$\|S_N - S_M\|^2_{C^2(\overline{Q}_T)} \leqslant C_4 \max_{0 \leqslant t \leqslant T} \sum_{p=0}^{2} \sum_{k=M+1}^{N} |\lambda_k|^{\left[\frac{n}{2}\right]+3-p}\left(\frac{d^pU_k}{dt^p}\right)^2.$$

By Lemma 3, series with general terms $\left(\frac{d^pU_k}{dt^p}\right)^2 |\lambda_k|^{\left[\frac{n}{2}\right]+3-p}$, $p = 0, 1, 2$, converge uniformly on $[0, T]$, therefore series (48) converges in $C^2(\overline{Q}_T)$. Hence $u \in C^2(\overline{Q}_T)$. Since, by Theorem 3, Sec. 6.2, Chap. III, for $p = 0, 1, 2$

$$\|u\|_{C^p(\overline{Q}_T)} = \max_{0 \leqslant t \leqslant T} \sum_{q=0}^{p} \left\|\frac{\partial^q u}{\partial t^q}\right\|_{C^{p-q}(\overline{D}_t)} \leqslant C \max_{0 \leqslant t \leqslant T} \sum_{q=0}^{p} \left\|\frac{\partial^q u}{\partial t^q}\right\|_{H^{\frac{n}{2}+1+p-q}(D_t)},$$

inequalities (60) are a consequence of inequalities (51) with $s = \left[\frac{n}{2}\right] + 1 + p$. ∎

Proof of Lemma 3. The proof is conveniently carried out in two stages: first it will be established for $f = 0$ and then for $\varphi = \psi = 0$.

Let $f = 0$. It follows from (49) that for all $t \in [0, T]$ and $k = 1, 2, \ldots$ in the case of first mixed problem and $k = 2, 3, \ldots$ in the case of second mixed problem

$$|U_k(t)| \leqslant |\varphi_k| + \frac{|\psi_k|}{\sqrt{|\lambda_k|}},$$

and in the case of second mixed problem

$$|U_1(t)| \leqslant |\varphi_1| + T\,|\psi_1|.$$

Furthermore, for all $t \in [0, T]$ and any $p = 1, 2, \ldots$

$$\left|\frac{d^pU_k}{dt^p}\right| \leqslant |\varphi_k|\,|\lambda_k|^{p/2} + |\psi_k|\,|\lambda_k|^{(p-1)/2}, \qquad k = 1, 2, \ldots,$$

(if $\lambda_1 = 0$, then $(\lambda_1)^0 = 1$). Consequently, for all $t \in [0, T]$ with any $k \geqslant 1$ and p, $0 \leqslant p \leqslant s$,

$$\left(\frac{d^pU_k}{dt^p}\right)^2 |\lambda_k|^{s-p} \leqslant 2\left(\varphi_k^2 |\lambda_k|^s + \psi_k^2 |\lambda_k|^{s-1}\right).$$

Therefore the assertion of Lemma 3 (for $f = 0$) follows from the convergence of the number series $\sum\limits_{k=1}^{\infty} \varphi_k^2 |\lambda_k|^s$ and $\sum\limits_{k=1}^{\infty} \psi_k^2 |\lambda_k|^{s-1}$ and the inequalities

$$\sum_{k=1}^{\infty} \varphi_k^2 |\lambda_k|^s \leqslant C \|\varphi\|^2_{H^s(D)}, \qquad \sum_{k=1}^{\infty} \psi_k^2 |\lambda_k|^{s-1} \leqslant C \|\psi\|_{H^{s-1}(D)},$$

where the constant $C > 0$ does not depend on φ or ψ (Theorem 8, Sec. 2.5, Chap. IV).

In order to establish Lemma 3 for $\varphi = \psi = 0$, we require some auxiliary results.

Lemma 4. *Let $\partial D \in C^2$. Then*

(1) *if the function $f(x, t)$ belongs to $\tilde{H}^q_{\mathscr{D}}(Q_T)$, $q \geqslant 2$, then for any p, $p = 1, \ldots, q$, $\frac{\partial^p f}{\partial t^p}$ belongs to $\tilde{H}^{q-p}_{\mathscr{D}}(Q_T)$;*

(2) *if the function $f(x, t)$ belongs to $\tilde{H}^q_{\mathscr{N}}(Q_T)$, $q \geqslant 2$, then for any p, $p = 1, \ldots, q$, $\frac{\partial^p f}{\partial t^p}$ belongs to $\tilde{H}^{q-p}_{\mathscr{N}}(Q_T)$.*

Proof. In order to prove the first assertion of this lemma, it suffices, obviously, to establish that if $G \in H^2(Q_T)$ and $G|_{\Gamma_T} = 0$, then $G_t|_{\Gamma_T} = 0$.

For the proof of the second assertion, it is enough to show that if $G \in H^3(Q_T)$ and $\left.\frac{\partial G}{\partial n}\right|_{\Gamma_T} = 0$, then $\left.\frac{\partial}{\partial n} G_t\right|_{\Gamma_T} = 0$.

Let us prove the first assertion. Since $G|_{\Gamma_T} = 0$, we have for any i, $1 \leqslant i \leqslant n$,

$$\int_{Q_T} G_{x_i t} \eta \, dx\, dt = -\int_{Q_T} G_{x_i} \eta_t \, dx\, dt = \int_{Q_T} G \eta_{x_i t} \, dx\, dt = -\int_{Q_T} G_t \eta_{x_i} \, dx\, dt,$$

where η is any function belonging to $C^2(\overline{Q}_T)$ and satisfying the conditions $\eta|_{D_0} = \eta|_{D_T} = 0$. On the other hand, for any i, $1 \leqslant i \leqslant n$,

$$\int_{Q_T} G_{x_i t} \eta \, dx\, dt = \int_{\Gamma_T} G_t \eta n_i \, dS\, dt - \int_{Q_T} G_t \eta_{x_i} \, dx\, dt,$$

where n_i is cosine of the angle between the (outward) normal to Γ_T and the Ox_i-axis.

Thus for any function $\eta \in C^2(\overline{\Gamma}_T)$ such that $\eta|_{\partial D_0} = \eta|_{\partial D_T} = 0$, we have

$$\int_{\Gamma_T} G_t \eta n_i \, dS \, dt = 0, \qquad i = 1, \ldots, n. \tag{61}$$

We cover the closed surface $\overline{\Gamma}_T$ by a finite number of ($(n+1)$-dimensional open) balls $V_1, \ldots, V_m$ such that for each $j = 1, \ldots, m$ it is possible to find a number $i = i(j)$, $1 \leqslant i \leqslant n$, so that on $\overline{\Gamma}_{Tj}$, where $\Gamma_{Tj} = \Gamma_T \cap V_j$, the function $|n_{i(j)}(x)| > 0$. Take any j, $1 \leqslant j \leqslant m$, and any function $\eta(x, t) \in \dot{C}^2(\overline{\Gamma}_{Tj})$, which is extended beyond Γ_{Tj} (on Γ_T) by zero. By (61),

$$\int_{\Gamma_{Tj}} G_t n_{i(j)}(x) \eta(x, t) \, dS \, dt = 0.$$

Since the set of functions $n_{i(j)}(x) \eta(x, t)$ for any $\eta(x, t) \in \dot{C}^2(\overline{\Gamma}_{Tj})$ is everywhere dense in $L_2(\Gamma_{Tj})$, $G_t|_{\Gamma_{Tj}} = 0$. Accordingly, $G_t|_{\Gamma_T} = 0$. This proves the first assertion.

The proof of second assertion is exactly the same. Indeed, since $\left.\frac{\partial G}{\partial n}\right|_{\Gamma_T} = 0$, for any $\eta \in C^2(\overline{Q}_T)$, $\eta|_{D_0} = \eta|_{D_T} = 0$, we have

$$\int_{Q_T} \Delta G_t \cdot \eta \, dx \, dt = -\int_{Q_T} \Delta G \cdot \eta_t \, dx \, dt$$
$$= \int_{Q_T} \nabla G \cdot \nabla \eta_t \, dx \, dt = -\int_{Q_T} \nabla G_t \cdot \nabla \eta \, dx \, dt.$$

On the other hand,

$$\int_{Q_T} \Delta G_t \cdot \eta \, dx \, dt = \int_{\Gamma_T} \frac{\partial G_t}{\partial n} \cdot \eta \, dS \, dt - \int_{Q_T} \nabla G_t \cdot \nabla \eta \, dx \, dt,$$

from which

$$\int_{\Gamma_T} \frac{\partial}{\partial n} G_t \cdot \eta \, dS \, dt = 0$$

for any $\eta \in C^2(\overline{\Gamma}_T)$. Hence $\left.\frac{\partial G_t}{\partial n}\right|_{\Gamma_T} = 0$. ∎

Lemma 5. *If $\partial D \in C^2$ and the function $f(x, t)$ belongs either to the space $\widetilde{H}^q_{\mathscr{D}}(Q_T)$ or to $\widetilde{H}^q_{\mathscr{N}}(Q_T)$ for $q \geqslant 2$, then for any $t \in [0, T]$ and $p = 1, \ldots, q-1$ the trace of the function $\frac{\partial^p f}{\partial t^p}$ on D_t belongs to $H^{q-p-1}_{\mathscr{D}}(D_t)$ or to $H^{q-p-1}_{\mathscr{N}}(D_t)$, respectively.*

Proof. In view of Lemma 4, for the proof of Lemma 5 it is enough to establish the following assertion. If the function $G(x, t) \in H^2(Q_T)$, then for any $t \in [0, T]$ $G|_{D_t} \in H^1(D_t)$; if $G \in H^2(Q_T)$ and $G|_{\Gamma_T} = 0$, then $G|_{D_T} \in \overset{\circ}{H}{}^1(D_t)$ for any $t \in [0, T]$.

According to the theorem on traces (Theorem 1, Sec. 5.1, Chap. III), $G|_{D_t} \in L_2(D_t)$ and $G_{x_i}|_{D_t} \in L_2(D_t)$, $i = 1, \ldots, n$, for any $t \in$ $\in [0, T]$. Take any function $\eta_1(t)$ in $C^1([0, T])$ and any $\eta_2(x)$ in $\dot{C}^1(\bar{D})$. By Ostrogradskii's formula,

$$\int_{Q_T} G_{x_i} \eta_1 \eta_2 \, dx \, dt = -\int_{Q_T} G \eta_1 \eta_{2x_i} \, dx \, dt \tag{62}$$

for any $i = 1, \ldots, n$, that is,

$$\int_0^T \eta_1(t) \left[\int_{D_t} (G_{x_i} \eta_2 + G \eta_{2x_i}) \, dx \right] dt = 0.$$

Since the set $C^1([0, T])$ is everywhere dense in $L_2(0, T)$ and by Lemma 2 the function $\int_{D_t} (G_{x_i} \eta_2 + G\eta_{2x_i}) \, dx$ belongs to the space $H^1(0, T)$ and is thereby continuous on $[0, T]$,

$$\int_{D_t} G_{x_i} \eta_2 \, dx = -\int_{D_t} G \eta_{2x_i} \, dx \tag{63}$$

for any function $\eta_2(x) \in \dot{C}^1(\bar{D})$ and all $t \in [0, T]$.

Consequently, for any $t \in [0, T]$ $G|_{D_t} \in H^1(D_t)$ and its generalized derivative with respect to x_i is the trace on D_t of the function $G_{x_i}(x, t)$, $i = 1, \ldots, n$.

Suppose now $G \in H^2(Q_T)$ and $G|_{\Gamma_T} = 0$. Then relations (62) hold also for any function $\eta_2(x)$ belonging to $C^1(\bar{D})$. Therefore identities (63) also hold for any $\eta_2(x) \in C^1(\bar{D})$ for all $t \in [0, T]$.

By what has been proved, for any $t \in [0, T]$ $G|_{D_t} \in H^1(D_t)$, therefore $\eta_2(x) \in C^1(\bar{D})$, apart from satisfying the identities (63), also satisfies the identity

$$\int_{D_t} G_{x_i} \eta_2 \, dx = \int_{\partial D_t} G \eta_2 n_i \, dS - \int_{D_t} G \eta_{2x_i} \, dx.$$

Thus for any $\eta_2(x) \in C^1(\bar{D})$

$$\int_{\partial D_t} G \eta_2 n_i \, dS = 0, \qquad i = 1, \ldots, n.$$

From these identities it follows (compare the proof of Lemma 4) that the trace of function $G(x, t)|_{D_t}$ on the boundary ∂D_t of D_t is zero. ▮

Remark. The proof of Lemma 5 readily implies the following result. If $\partial D \in C^2$ and $f(x, t) \in H^q(Q_T)$, then $f|_{D_t} \in H^{q-1}(D_t)$ for any $t \in [0, T]$.

Lemma 6. *Let $v_1, v_2, \ldots$ be an orthonormal basis for the space $L_2(D)$. Then for any function $G(x, t) \in L_2(Q_T)$*

$$\sum_{k=1}^{\infty} \int_0^T \Big(\int_{D_t} G(x, t)\, v_k(x)\, dx \Big)^2 dt = \| G \|^2_{L_2(Q_T)}.$$

Proof. Since for almost all $t \in (0, T)$ the function $G(x, t) \in L_2(D_t)$,

$$\sum_{k=1}^{\infty} \Big(\int_{D_t} G(x, t)\, v_k(x)\, dx \Big)^2 = \| G(x, t) \|^2_{L_2(D_t)}$$

for these values of t. The desired result follows from integrating the last relation over $(0, T)$ and then applying Levi's theorem. ▮

Proof of Lemma 3 when $\varphi = \psi = 0$. By (49) and (50), we have

$$U_k(t) = \frac{1}{\sqrt{|\lambda_k|}} \int_{Q_t} f(x, \tau)\, v_k(x) \sin \sqrt{|\lambda_k|}\,(t-\tau)\, d\tau, \qquad k = 1, 2, \ldots \tag{49'}$$

(for the second mixed problem $U_1(t) = \frac{1}{\sqrt{|D|}} \int_{Q_t} (t - \tau)\, f(x, \tau) \times$ $\times\, dx\, d\tau$).

We first consider the case $p = 0$. Since the functions f and v_k belong to the space $\tilde{H}^{s-1}_{\mathscr{D}}(Q_T)$ (or to $\tilde{H}^{s-1}_{\mathscr{N}}(Q_T)$) and $\Delta^{\mu} v_k = \lambda_k^{\mu} v_k$ for any $\mu = 1, \ldots, [s/2]$, we have for all $t \in [0, T]$ and for even $s - 1$

$$|\lambda_k|^{s/2} U_k(t) = (-1)^{\frac{s-1}{2}} \int_{Q_t} f(x, \tau)\, \Delta^{\frac{s-1}{2}} v_k(x) \sin \sqrt{|\lambda_k|}\,(t-\tau)\, dx\, d\tau$$

$$= (-1)^{\frac{s-1}{2}} \int_{Q_t} \Delta^{\frac{s-1}{2}} f(x, \tau) \cdot v_k(x) \sin \sqrt{|\lambda_k|}\,(t-\tau)\, dx\, d\tau = \tilde{\alpha}_k(t),$$

where

$$\tilde{\alpha}_k^2(t) \leqslant \int_0^t \sin^2 \sqrt{|\lambda_k|}\,(t-\tau)\, d\tau \cdot \int_0^t d\tau \Big(\int_{D_\tau} \Delta^{\frac{s-1}{2}} f(x, \tau) \cdot v_k(x)\, dx \Big)^2$$

$$\leqslant T \int_0^T \Big(\int_{D_t} \Delta^{\frac{s-1}{2}} f(x, t) \cdot v_k(x)\, dx \Big)^2 dt.$$

Since $\Delta^{\frac{s-1}{2}} f \in L_2(Q_T)$, by Lemma 6 the number series $\sum\limits_{k=1}^{\infty} \int\limits_0^T \Big(\int\limits_{D_t} \Delta^{\frac{s-1}{2}} f(x, t)\, v_k(x)\, dx \Big)^2 dt$ converges and

$$\sum_{k=1}^{\infty} \int_0^T \Big(\int_{D_t} \Delta^{\frac{s-1}{2}} f(x, t)\, v_k(x)\, dx \Big)^2 dt$$

$$= \int_{Q_T} \Big(\Delta^{\frac{s-1}{2}} f \Big)^2 dx\, dt \leqslant C' \, \| f \|^2_{H^{s-1}(Q_T)}.$$

Accordingly, the series $\sum\limits_{k=1}^{\infty} \tilde{\alpha}_k^2(t)$ converges uniformly on $[0, T]$ and

$$\sum_{k=1}^{\infty} \tilde{\alpha}_k^2(t) \leqslant TC' \, \| f \|^2_{H^{s-1}(Q_T)} = C \, \| f \|^2_{H^{s-1}(Q_T)}.$$

When $s-1$ is odd,

$$|\lambda_k|^{s/2} U_k(t)$$

$$= (-1)^{\frac{s-2}{2}} |\lambda_k|^{1/2} \int_{Q_t} \Delta^{\frac{s-2}{2}} f(x, \tau)\, v_k(x) \sin \sqrt{|\lambda_k|}(t-\tau)\, dx\, d\tau$$

$$= (-1)^{\frac{s-2}{2}} \int_{Q_t} \Delta^{\frac{s-2}{2}} f(x, \tau)\, v_k(x) d \left(\cos \sqrt{|\lambda_k|}(t-\tau)\right) dx$$

$$= (-1)^{\frac{s-2}{2}} \Big\{ \int_{D_t} \Delta^{\frac{s-2}{2}} f(x, t)\, v_k(x)\, dx$$

$$- \cos \sqrt{|\lambda_k|}\, t \int_{D_0} \Delta^{\frac{s-2}{2}} f(x, 0)\, v_k(x)\, dx$$

$$- \int_0^t \cos \sqrt{|\lambda_k|}(t-\tau) \Big[\int_{D_\tau} \Delta^{\frac{s-2}{2}} f_\tau(x, \tau)\, v_k(x)\, dx \Big] d\tau \Big\}$$

$$= \tilde{\tilde{\alpha}}_k^{(1)}(t) + \tilde{\tilde{\alpha}}_k^{(2)}(t) + \tilde{\tilde{\alpha}}_k^{(3)}(t) = \tilde{\tilde{\alpha}}_k(t),$$

where

$$|\tilde{\tilde{\alpha}}_k^{(1)}(t)|^2 = \left| \int_{D_t} \Delta^{\frac{s-2}{2}} f(x, t)\, v_k(x)\, dx \right|^2,$$

$$|\tilde{\tilde{\alpha}}_k^{(2)}(t)|^2 \leqslant \left| \int_{D_0} \Delta^{\frac{s-2}{2}} f(x, 0)\, v_k(x)\, dx \right|^2,$$

$$|\tilde{\tilde{\alpha}}_k^{(3)}(t)|^2 \leqslant T \int_0^T \left(\int_{D_t} \Delta^{\frac{s-2}{2}} f_t(x, t)\, v_k(x)\, dx \right)^2 dt.$$

The function $\Delta^{\frac{s-2}{2}} f \in H^1(Q_T)$, so for all $t \in [0, T]$ $\Delta^{\frac{s-2}{2}} f(x, t) \in L_2(D_t)$ and

$$\| \Delta^{\frac{s-2}{2}} f(x, t) \|^2_{L_2(D_t)} \leqslant \text{const} \| \Delta^{\frac{s-2}{2}} f(x, t) \|^2_{H^1(Q_T)} \leqslant \text{const} \| f \|^2_{H^{s-1}(Q_T)}.$$

Consequently, the series $\sum_{k=1}^{\infty} |\tilde{\tilde{\alpha}}^{(2)}(t)|^2$ converges uniformly on $[0, T]$ and

$$\sum_{k=1}^{\infty} |\tilde{\tilde{\alpha}}_k^{(2)}(t)|^2 \leqslant C^{(2)} \| f \|^2_{H^{s-1}(Q_T)}.$$

Since for any function $G(x, t) \in H^1(Q_T)$ the integral

$$\| G \|^2_{L_2(D_{t'})} - \| G \|^2_{L_2(D_{t''})} = 2 \int_{t'}^{t''} \int_{D_\tau} G(x, \tau)\, G_\tau(x, \tau)\, dx\, d\tau = o(1)$$

converges absolutely as

$$| t' - t'' | \to 0, \qquad t' \in [0, T], \qquad t'' \in [0, T],$$

the function $\| \Delta^{\frac{s-2}{2}} f \|^2_{L_2(D_t)}$ is continuous on $[0, T]$. Therefore for all $t \in [0, T]$

$$\sum_{k=1}^{\infty} \left(\int_{D_t} \Delta^{\frac{s-2}{2}} f(x, t)\, v_k(x)\, dx \right)^2 = \| \Delta^{\frac{s-2}{2}} f \|^2_{L_2(D_t)} \leqslant C \| f \|^2_{H^{s-1}(Q_T)},$$

and, by Dini's theorem, the series on the left side of this relation (in view of Lemma 2, the terms of this series are continuous on $[0, T]$) converges uniformly on $[0, T]$. This means that the series

$\sum\limits_{k=1}^{\infty} |\tilde{\tilde{\alpha}}^{(1)}(t)|^2$ converges uniformly on $[0, T]$ and

$$\sum_{k=1}^{\infty} |\tilde{\tilde{\alpha}}^{(1)}(t)|^2 \leqslant C^{(1)} \|f\|^2_{H^{s-1}(Q_T)}.$$

Further, the function $\Delta^{\frac{s-2}{2}} f_t \in L_2(Q_T)$, therefore, by Lemma 6,

$$\sum_{k=1}^{\infty} \int_0^T \left(\int_{D_t} \Delta^{\frac{s-2}{2}} f_t(x, t)\, v_k(x)\, dx \right)^2 dt = \| \Delta^{\frac{s-2}{2}} f_t \|_{L_2(Q_T)} \leqslant C \|f\|^2_{H^{s-1}(Q_T)}.$$

Accordingly, the series $\sum\limits_{k=1}^{\infty} |\tilde{\tilde{\alpha}}^{(3)}(t)|^2$ converges uniformly on the segment $[0, T]$ and

$$\sum_{k=1}^{\infty} |\tilde{\tilde{\alpha}}^{(3)}(t)|^2 \leqslant C^{(3)} \|f\|^2_{H^{s-1}(Q_T)}.$$

Thus the series $\sum\limits_{k=1}^{\infty} \tilde{\tilde{\alpha}}_k^2(t)$ converges uniformly on $[0, T]$ and its sum

$$\sum_{k=1}^{\infty} \tilde{\tilde{\alpha}}_k^2(t) \leqslant C \|f\|^2_{H^{s-1}(Q_T)}.$$

This establishes Lemma 3 for $p = 0$.

When $p = 1$, the proof is analogous. According to (49′), (50), for even $s - 1$

$$|\lambda_k|^{\frac{s-1}{2}} \frac{dU_k}{dt} = (-1)^{\frac{s-1}{2}} \int_{Q_t} f(x, \tau)\, \Delta^{\frac{s-1}{2}} v_k(x) \cos \sqrt{|\lambda_k|}\,(t-\tau)\, dx\, d\tau$$

$$= (-1)^{\frac{s-1}{2}} \int_{Q_t} \Delta^{\frac{s-1}{2}} f(x, \tau) \cdot v_k(x) \cos \sqrt{|\lambda_k|}\,(t-\tau)\, dx\, d\tau = \tilde{\beta}_k(t).$$

Since

$$\tilde{\beta}_k^2(t) \leqslant T \int_0^T dt \left(\int_{D_t} \Delta^{\frac{s-1}{2}} f(x, t)\, v_k(x)\, dx \right)^2,$$

the series $\sum\limits_{k=1}^{\infty} \tilde{\beta}_k^2(t)$, like the series $\sum\limits_{k=1}^{\infty} \tilde{\alpha}_k^2(t)$, converges uniformly on $[0, T]$ and

$$\sum_{k=1}^{\infty} \tilde{\beta}_k^2(t) \leqslant C \|f\|^2_{H^{s-1}(Q_T)}.$$

When $s-1$ is odd,

$$|\lambda_k|^{\frac{s-1}{2}}\frac{dU_k}{dt}$$

$$=(-1)^{\frac{s-2}{2}}|\lambda_k|^{\frac{1}{2}}\int\limits_{Q_t}\Delta^{\frac{s-2}{2}}f(x,\tau)\,v_k(x)\cos\sqrt{|\lambda_k|}(t-\tau)\,dx\,d\tau$$

$$=(-1)^{\frac{s-2}{2}}\Big(\sin\sqrt{|\lambda_k|}t\cdot\int\limits_{D_0}\Delta^{\frac{s-2}{2}}f(x,0)\,v_k(x)\,dx$$

$$+\int\limits_{Q_t}\Delta^{\frac{s-2}{2}}f_\tau(x,\tau)\,v_k(x)\,\sin\sqrt{|\lambda_k|}(t-\tau)\,dx\,d\tau\Big)$$

$$=\tilde{\tilde{\beta}}_k^{(2)}(t)+\tilde{\tilde{\beta}}_k^{(3)}(t)=\tilde{\beta}_k(t).$$

Since

$$|\tilde{\tilde{\beta}}_k^{(2)}(t)|^2\leqslant\Big(\int\limits_{D_0}\Delta^{\frac{s-2}{2}}f(x,0)\;v_k(x)\,dx\Big)^2,$$

$$|\tilde{\tilde{\beta}}_k^{(3)}(t)|^2\leqslant T\int\limits_0^T\Big(\int\limits_{D_\tau}\Delta^{\frac{s-2}{2}}f_\tau(x,\tau)\;v_k(x)\,dx\Big)^2d\tau,$$

the series $\sum\limits_{k=1}^{\infty}(\tilde{\tilde{\beta}}_k^{(s)}(t))^2$ (like the series $\sum\limits_{k=1}^{\infty}(\tilde{\tilde{\alpha}}_k^{(s)}(t))^2$), $s=2,3,$ and thus also the series $\sum\limits_{k=1}^{\infty}\tilde{\tilde{\beta}}_k^2(t)$ converge uniformly on $[0,T]$ and

$$\sum_{k=1}^{\infty}\tilde{\tilde{\beta}}_k^2(t)\leqslant C\,\|f\|_{H^{s-1}(Q_T)}.$$

This completes the proof of Lemma 3 for $p=1$.

Suppose now that $p\geqslant 2$. Since the function $U_k(t)$ satisfies the differential equation $U_k''-\lambda_kU_k=f_k$, for even p, $2\leqslant p\leqslant s$, we have

$$\frac{d^pU_k}{dt^p}=\lambda_k^{\frac{p}{2}}U_k+\lambda_k^{\frac{p-2}{2}}f_k+\lambda_k^{\frac{p-4}{2}}\frac{d^2f_k}{dt^2}+\ldots+\frac{d^{p-2}f_k}{dt^{p-2}},$$

and for odd p, $2<p\leqslant s$,

$$\frac{d^pU_k}{dt^p}=\lambda_k^{\frac{p-1}{2}}\frac{dU_k}{dt}+\lambda_k^{\frac{p-3}{2}}\frac{df_k}{dt}+\ldots+\frac{d^{p-2}f_k}{dt^{p-2}}.$$

Therefore Lemma 3 will have been proved for any $p \leqslant s$ if we show that for any q, $0 \leqslant q \leqslant s-2$, the series $\sum\limits_{k=1}^{\infty} \lambda_k^{s-2-q} \left(\frac{d^q f_k}{dt^q}\right)^2$ converges uniformly on $[0, T]$ and the inequality

$$\sum_{k=1}^{\infty} |\lambda_k|^{s-2-q} \left(\frac{d^q f_k}{dt^q}\right)^2 \leqslant C \|f\|^2_{H^{s-1}(Q_T)}$$

holds, where the constant $C > 0$ depends only on Q_T.

When $s-q$ is even, by Lemmas 2 and 5 we have

$$|\lambda_k|^{\frac{s-q-2}{2}} \frac{d^q f_k}{dt^q} = |\lambda_k|^{\frac{s-q-2}{2}} \int\limits_{D_t} \frac{\partial^q f(x, t)}{\partial t^q} v_k(x)\, dx$$

$$= (-1)^{\frac{s-q-2}{2}} \int\limits_{D_t} \Delta^{\frac{s-q-2}{2}} \frac{\partial^q f(x, t)}{\partial t^q} v_k(x)\, dx = \tilde{\gamma}_k(t)$$

for all $t \in [0, T]$. Since $\Delta^{\frac{s-q-2}{2}} \frac{\partial^q f}{\partial t^q} \in H^1(Q_T)$, we find that for any $t \in [0, T]$, $\Delta^{\frac{s-q-2}{2}} \frac{\partial^q f}{\partial t^q} \in L_2(D_t)$ and the function $\left\| \Delta^{\frac{s-q-2}{2}} \frac{\partial^q f(x, t)}{\partial t^q} \right\|^2_{L_2(D_t)}$ is continuous on $[0, T]$. Hence the series $\sum\limits_{k=1}^{\infty} \tilde{\gamma}_k^2(t)$ converges uniformly on $[0, T]$ and

$$\sum_{k=1}^{\infty} \tilde{\gamma}_k^2(t) = \left\| \Delta^{\frac{s-q-2}{2}} \frac{\partial^q f(x, t)}{\partial t^q} \right\|^2_{L_2(D_t)} \leqslant C \|f\|^2_{H^{s-1}(Q_T)}.$$

Let $s-q$ be odd. Then for all $t \in [0, T]$

$$|\lambda_k|^{\frac{s-q-2}{2}} \frac{\partial^q f_k}{\partial t^q}$$

$$= (-1)^{\frac{s-q-3}{2}} |\lambda_k|^{\frac{1}{2}} \int\limits_{D_t} \Delta^{\frac{s-q-3}{2}} \frac{\partial^q f(x, t)}{\partial t^q} v_k(x)\, dx = \sqrt{|\lambda_k|}\, \tilde{\tilde{\gamma}}_k(t).$$

Since $\Delta^{\frac{s-q-3}{2}} \frac{\partial^q f}{\partial t^q} \in \tilde{H}^2_{\mathscr{D}}(Q_T)$ (or $\tilde{H}^2_{\mathscr{N}}(Q_T)$), by Lemma 5 $\Delta^{\frac{s-q-3}{2}} \frac{\partial^q f}{dt^q} \in \mathring{H}^1(D_t)$ (or $H^1(D_t)$) for any $t \in [0, T]$ and, moreover, the function $\left\| \Delta^{\frac{s-q-3}{2}} \frac{\partial^q f}{\partial t^q} \right\|^2_{H^1(D_t)}$ is continuous on $[0, T]$. Since for any $t \in [0, T]$

$\sum\limits_{k=1}^{\infty} \tilde{\tilde{\gamma}}_k^2(t)\,(|\lambda_k|+1) = \left\| \Delta^{\frac{s-q-3}{2}} \frac{\partial^q f}{\partial t^q} \right\|^2_{H^1(D_t)}$, the series $\sum\limits_{k=1}^{\infty} \tilde{\tilde{\gamma}}_k^2(t)\,(|\lambda_k|+1)$ and, more so, the series $\sum\limits_{k=1}^{\infty} \tilde{\tilde{\gamma}}_k^2(t)\,|\lambda_k|$ converge uniformly on $[0, T]$ and

$$\sum_{k=1}^{\infty} \tilde{\tilde{\gamma}}_k^2(t)\,|\lambda_k| \leqslant \left\| \Delta^{\frac{s-q-3}{2}} \frac{\partial^q f}{\partial t^q} \right\|^2_{H^1(D_t)} \leqslant C \| f \|^2_{H^{s-1}(Q_T)}. \blacksquare$$

As already noted, if the given functions do not satisfy conditions of the type (56) and (57) for the first mixed problem or (58) and (59) for the second mixed problem, then Theorems 3 and 4 are not valid. Nevertheless, if we wish to establish only the smoothness of generalized solutions and not the convergence of Fourier series in the corresponding space, the conditions (56), (57) and conditions (58), (59), respectively, can be very much relaxed. Let us consider, for example, the first mixed problem.

Theorem 3′. *Let* $\partial D \in C^s$, $\varphi \in H^s(D)$, $\psi \in H^{s-1}(D)$, $f \in H^{s-1}(Q_T)$, *for* $s \geqslant 1$ *and let the following compatibility conditions be fulfilled:*

$$\varphi|_{\partial D} = \ldots = \left[\Delta^{\left[\frac{s-1}{2}\right]} \varphi + \sum_{i=0}^{\left[\frac{s-3}{2}\right]} \Delta^{\left[\frac{s-3}{2}\right]-i} \frac{\partial^{2i} f}{\partial t^{2i}} \right]\Bigg|_{\partial D_0} = 0 \tag{64}$$

and for $s \geqslant 2$

$$\psi|_{\partial D} = \ldots = \left[\Delta^{\left[\frac{s}{2}\right]-1} \psi + \sum_{i=0}^{\left[\frac{s}{2}\right]-2} \Delta^{\left[\frac{s}{2}\right]-2-i} \frac{\partial^{2i+1} f}{\partial t^{2i+1}} \right]\Bigg|_{\partial D_0} = 0 \tag{65}$$

(*it is assumed that for* $s < 0$ $\sum\limits_{i=0}^{s} a_i = 0$). *Then the generalized solution of the first mixed problem* (44)-(46) *belongs to* $H^s(Q_T)$.

The compatibility conditions (64) and (65) in Theorem 3′ have the form

$$\varphi|_{\partial D} = 0$$

when $s = 1$,

$$\varphi|_{\partial D} = \psi|_{\partial D} = 0$$

when $s = 2$ and

$$\varphi|_{\partial D} = \psi|_{\partial D} = 0, \qquad (\Delta\varphi + f)|_{\partial D_0} = 0$$

when $s = 3$. Since $f \in H^{s-1}(Q_T)$, its trace $f|_{D_0}$ belongs to $H^{s-2}(D_0)$ by Remark to Lemma 5. Consequently, with $s \geqslant 3$ for any

$i=0, \ldots, \left[\frac{s-3}{2}\right]$ there exists the trace $\Delta^{\left[\frac{s-3}{2}\right]-i} \frac{\partial^{2i} f}{\partial t^{2i}}\Big|_{\partial D_0}$ or for any $i=0, \ldots, \left[\frac{s}{2}\right]-2$ with $s \geqslant 4$ the trace $\Delta^{\left[\frac{s}{2}\right]-2-i} \frac{\partial^{2i+1} f}{\partial t^{2i+1}}\Big|_{\partial D_0}$ belonging to $L_2(\partial D_0)$.

Proof. When $s = 1$, the conclusion of the theorem is evident. When $s = 2$, it follows from Corollary 2 to Theorem 3. Let us prove the theorem for $s = 3$; for $s > 3$ the proof is analogous.

Together with problem (44)-(46), we consider the problem

$$v_{tt} - \Delta v = f_t, \tag{66}$$

$$v|_{t=0} = \psi, \tag{67}$$

$$v_t|_{t=0} = \Delta\varphi + f|_{D_0} \tag{68}$$

The existence of generalized solution $v(x, t)$ of the problem (66)-(68) is guaranteed by the hypotheses of the theorem. By Corollary 2 to Theorem 3, the function $v(x, t)$ belongs to $H^2(Q_T)$ and is an a.e. solution of the problem (66)-(68). We shall show that $v = u_t$.

Evidently, the function

$$w(x, t) = \varphi(x) + \int_0^t v(x, \tau)\, d\tau$$

belongs to $H^2(Q_T)$, and

$$\nabla w = \nabla\varphi + \int_0^t \nabla v(x, \tau)\, d\tau,$$

$$w_t = v.$$

Since v is a generalized solution of the problem (66)-(68), the function w satisfies the integral identity

$$\int_{Q_T} (\nabla w_t \nabla \eta - w_{tt}\eta_t)\, dx\, dt = \int_{D_0} (f + \Delta\varphi)\, \eta\, dx + \int_{Q_T} f_t \eta\, dx\, dt \tag{69}$$

for all $\eta \in H^1(Q_T)$ such that

$$\eta|_{D_T} = 0, \qquad \eta|_{\Gamma_T} = 0. \tag{70}$$

Let $\eta \in C^2(\overline{Q}_T)$ and satisfy the conditions

$$\eta|_{D_T} = \eta_t|_{D_T} = 0, \qquad \eta|_{\Gamma_T} = 0. \tag{71}$$

Then

$$\int_{Q_T} (\nabla w_t \nabla \eta - w_{tt}\eta_t)\,dx\,dt$$

$$= -\int_{Q_T} \left(\nabla w \nabla \eta_t - w_t \eta_{tt}\right) dx\,dt - \int_{D_0} (\nabla \varphi \nabla \eta - \psi \eta_t)\,dx$$

$$= -\int_{Q_T} (\nabla w \nabla \eta_t - w_t \eta_{tt})\,dx\,dt + \int_{D_0} (\Delta \varphi \cdot \eta + \psi \eta_t)\,dx$$

and

$$\int_{Q_T} f_t \eta\,dx\,dt = -\int_{Q_T} f \eta_t\,dx\,dt - \int_{D_0} f \eta\,dx.$$

Substituting these inequalities in (69), we obtain

$$\int_{Q_T} (\nabla w \nabla \eta_t - w_t \eta_{tt})\,dx\,dt = \int_{D_0} \psi \eta_t\,dx + \int_{Q_T} f \eta_t\,dx\,dt.$$

Since, for any function $\zeta(x, t)$ belonging to $C^2(\bar{Q}_T)$ and satisfying conditions (70), we can find a function $\eta(x, t)$ in $C^2(\bar{Q}_T)$ which satisfies conditions (71), such that $\zeta = \eta_t \left(\eta(x, t) = -\int_t^T \zeta(x, \tau)\,d\tau\right)$, the function w satisfies the integral identity

$$\int_{Q_T} (\nabla w \nabla \zeta - w_t \zeta_t)\,dx\,dt = \int_{D_0} \psi \zeta\,dx + \int_{Q_T} f \zeta\,dx\,dt$$

for all $\zeta(x, t)$ belonging to $C^2(\bar{Q}_T)$ and satisfying conditions (70), and, consequently, also for all $\zeta \in H^1(Q_T)$ satisfying conditions (70). Since the generalized solution of the problem (44)-(46) is unique, $w = u$ and hence $v = u_t$.

Thus $u \in H^2(Q_T)$ and $u_t \in H^2(Q_T)$. Since u is an a.e. solution of the problem (44)-(46), for almost all $t \in [0, T]$, the function $u(x, t)$ is an a.e. solution of the first boundary-value problem for Poisson's equation

$$\Delta u = f_1, \qquad x \in D_t,$$

$$u|_{\partial D_t} = 0,$$

where $f_1 = (f + u_{tt})|_{D_t}$. By Corollary 2 to Theorem 3, $u_{tt}|_{D_t} = v_t|_{D_t} \in H^1(D_t)$, so $f_1 \in H^1(D_t)$ and according to Theorem 4, Sec. 2.3, Chap. IV, $u \in H^3(D_t)$ for almost all $t \in [0, T]$ and

$$\|u\|_{H^3(D_t)} \leqslant \text{const}\,\|f_1\|_{H^1(D_t)} \leqslant \text{const}\,[\|f\|_{H^1(D_t)} + \|u_{tt}\|_{H^1(D_t)}]$$
$$\leqslant \text{const}\,[\|f\|_{H^2(Q_T)} + \|\psi\|_{H^2(D)} + \|\Delta\varphi + f\|_{H^1(D_0)} + \|f_t\|_{H^1(Q_T)}]$$
$$\leqslant \text{const}\,[\|f\|_{H^2(Q_T)} + \|\psi\|_{H^2(D)} + \|\varphi\|_{H^3(D)}].$$

Hence $u \in H^3(Q_T)$. ▮

§ 3. GENERALIZED SOLUTION OF THE CAUCHY PROBLEM

In the strip $\Pi_T = \{x \in R_n,\ 0 < t < T\}$ with certain $T > 0$, we examine the hyperbolic equation

$$u_{tt} - \operatorname{div}(k(x)\,\nabla u) + a(x)\,u = f, \tag{1}$$

where $k(x) \in C^1(R_n)$, $a(x) \in C(R_n)$, $\inf\limits_{x \in R_n} k(x) = k_0 > 0$, $\sup\limits_{x \in R_n} k(x) = k_1 < \infty$; it is also assumed that $a(x) \geqslant 0$.

A function $u(x, t) \in C^2(\Pi_T) \cap C^1(\Pi_T \cup \{t = 0\})$ is called the *classical solution of the Cauchy problem* for Eq. (1) in the strip Π_T if in Π_T it satisfies Eq. (1) and for $t = 0$ the initial conditions

$$u|_{t=0} = \varphi(x), \tag{2}$$

$$u_t|_{t=0} = \psi(x). \tag{3}$$

For any $R > 0$, let $Q_{T,R}$ denote the cylinder $\{|x| < R,\ 0 < t < T\}$, $S_{T,R}$ its lateral surface $\{|x| = R,\ 0 < t < T\}$, $D_{\tau,R}$, $\tau \in [0, T]$, the set $\{|x| < R,\ t = \tau\}$; in particular, $D_{0,R}$ and $D_{T,R}$ denote the base and top, respectively, of the cylinder $Q_{T,R}$.

Suppose that $u(x, t)$ is the classical solution of the Cauchy problem (1)-(3) in the strip $\Pi_{T+\delta}$ with certain $\delta > 0$, where the function $f(x, t)$ belongs to the space $L_2(Q_{T,R})$ for any $R > 0$. Multiplying (1) by an arbitrary function $v(x, t)$, such that

$$\begin{aligned} &v(x, t) \in H^1(Q_{T,R_0}), \qquad v(x, t) = 0 \ \text{in}\ \Pi_T \setminus Q_{T,R_0}, \\ &v|_{D_{T,R_0}} = 0, \qquad v|_{S_{T,R_0}} = 0 \end{aligned} \tag{4}$$

with some $R_0 = R_0(v) > 0$, and integrating the resulting identity over the strip Π_T, we obtain by means of Ostrogradskii's formula

$$\int\limits_{\Pi_T} (k\nabla u \nabla v + auv - u_t v_t)\,dx\,dt = \int\limits_{\Pi_T} fv\,dx\,dt + \int\limits_{R_n} \psi(x)\,v(x, 0)\,dx \tag{5}$$

(in this identity the actual integration is taken over not the whole strip Π_T and the plane $\{x \in R_n,\ t = 0\}$ but only over the cylinder Q_{T,R_0} and its base D_{0,R_0}, respectively).

Let $f(x, t) \in L_2(Q_{T,\,R})$ and $\psi(x) \in L_2(|x| < R)$ for any $R > 0$. We introduce the following definition.

A function u is called the *generalized solution of the Cauchy problem* (1)-(3) *in the strip* Π_T if it belongs to $H^1(Q_{T,\,R})$ for all $R > 0$, satisfies the integral identity (5) for all v which obey condition (4) for some $R_0 = R_0(v) > 0$ and satisfies the initial condition (2) (that is, $u(x, t)\,|_{D_{0,R}} = \varphi(x)$ for any $R > 0$).

Apart from the notions of classical and generalized solutions of the Cauchy problem (1)-(3), we can also define an a.e. solution of this problem.

A function u is said to be an *a.e. solution in* Π_T *of the Cauchy problem* (1)-(3) if it belongs to $H^2(Q_{T,\,R})$ for all $R > 0$, satisfies Eq. (1) for almost all $(x, t) \in \Pi_T$ together with the initial conditions (2) and (3) (that is, $u|_{D_{0,\,R}} = \varphi$, $u_t|_{D_{0,\,R}} = \psi$ for any $R > 0$).

It was shown above that a classical solution in $\Pi_{T+\delta}$ (for any $\delta > 0$) of the problem (1)-(3), where f belongs to $L_2\,(Q_{T,\,R})$ for any $R > 0$, is a generalized solution in Π_T of the same problem. It can be similarly established that an a.e. solution of the problem (1)-(3) (in Π_T) is also a generalized solution (in Π_T) of the same problem.

Just as in the case of mixed problems, it is easy to show (compare with Lemma 1, Sec. 2.1) that if the generalized solution in Π_T of the problem (1)-(3) belongs to $H^2(Q_{T,\,R})$ for any $R > 0$, then it is also an a.e. solution, and if it belongs to $C^2(\Pi_T) \cap C^1(\Pi_T \cup \{t = 0\})$, it is a classical solution.

We shall now prove the existence and uniqueness theorems regarding the generalized solution of the problem (1)-(3). For this we require the following auxiliary proposition.

Take a number $\gamma > \sqrt{k_1}$ $(k_1 = \sup\limits_{x \in R_n} k(x) < \infty)$. Let t_1 be any number greater than T and x^0 a point in R_n. We denote by $K_{t_1,\,\tau}\,(x^0)$, $\tau \in (0, T]$, the truncated cone $\{|x - x^0| < \gamma\,(t_1 - t),\ 0 < t < \tau\}$ lying in Π_T, by $\Gamma_{t_1,\,\tau}\,(x^0)$ its lateral surface, $\Gamma_{t_1,\,\tau}\,(x^0) = \{|x - x^0| = \gamma\,(t_1 - t),\ 0 < t < \tau\}$, by $D_{\theta,\,\gamma(t_1-\theta)}\,(x^0)$ the set $\{|x - x^0| < \gamma\,(t_1 - \theta),\ t = \theta\}$, $0 \leqslant \theta \leqslant \tau$ (then $D_{0,\,\gamma t_1}\,(x^0)$ and $D_{\tau,\,\gamma(t_1-\tau)}\,(x^0)$ are its base and top, respectively). If x^0 is the origin in the space R_n, then the cone $K_{t_1,\,\tau}\,(x^0) = K_{t_1,\,\tau}\,(0)$ will be denoted by $K_{t_1,\,\tau}$ and the surface $\Gamma_{t_1,\,\tau}\,(0)$ by $\Gamma_{t_1,\,\tau}$. In this case $D_{\tau,\gamma\,(t_1-\tau)}\,(0) = D_{\tau,\,\gamma(t_1-\tau)}$ and, in particular, $D_{0,\,\gamma t_1}$ and $D_{T,\,\gamma(t_1-T)}$ are the base and top of the cone $K_{t_1,\,T}$.

Lemma 1. *For certain* $t_1 > T$ *and* $x^0 \in R_n$, *let the function* $u(x, t) \in H^1(K_{t_1,\,T}\,(x^0))$, $u|_{D_{0,\,\gamma t_1}(x^0)} = 0$ *and*

$$\int\limits_{K_{t_1,\,T}(x^0)} (k(x)\,\nabla u \nabla v + auv - u_t v_t)\,dx\,dt = 0 \tag{6}$$

for all v *satisfying the condition*

$$v \in H^1(K_{t_1,T}(x^0)), \qquad v = 0 \text{ in } \Pi_T \setminus K_{t_1,T}(x^0),$$
$$v|_{D_{T,\gamma(t_1-T)}(x^0)} = 0, \qquad v|_{\Gamma_{t_1,T}(x^0)} = 0.$$

Then $u = 0$ *in* $K_{t_1,T}(x^0)$.

Proof. Evidently, it is enough to establish the lemma for $x^0 = 0$. Take any $\tau \in (0, T]$, and in $K_{t_1,T}$ consider the function

$$v(x, t) = \begin{cases} \int\limits_t^{\theta(x)} u(x, z)\,dz & \text{in } K_{t_1,\tau}, \\ 0 & \text{in } K_{t_1,T} \setminus K_{t_1,\tau}, \end{cases}$$

where

$$\theta(x) = \begin{cases} t_1 - \dfrac{|x|}{\gamma} & \text{for } \gamma(t_1 - \tau) < |x| < \gamma t_1, \\ \tau & \text{for } |x| < \gamma(t_1 - \tau) \end{cases}$$

($t = \theta(x)$, $|x| < \gamma t_1$, is the equation of the surface $\Gamma_{t_1,\tau} \cup \bar{D}_{\tau,\gamma(t_1-\tau)}$). The function $v(x, t)$ belongs to $H^1(K_{t_1,T})$, $v|_{\Gamma_{t_1,\tau'}} = 0$, $v|_{D_{\tau',\gamma(t_1-\tau')}} = 0$ for all $\tau' \in [\tau, T]$, and generalized derivatives of v have the form

$$\nabla v = \begin{cases} \int\limits_t^{\theta(x)} \nabla u(x, z)\,dz + u(x, \theta(x))\,\nabla\theta & \text{in } K_{t_1,\tau}, \\ 0 & \text{in } K_{t_1,T} \setminus K_{t_1,\tau}, \end{cases} \tag{7}$$

$$v_t = \begin{cases} -u(x, t) & \text{in } K_{t_1,\tau}, \\ 0 & \text{in } K_{t_1,T} \setminus K_{t_1,\tau}. \end{cases} \tag{8}$$

One can verify this most easily as follows. Since $u \in H^1(K_{t_1,T})$, there is a sequence of functions u_s, $s = 1, 2, \ldots$, in $C^1(\bar{K}_{t_1,T})$ which converges to u in $H^1(K_{t_1,T})$. Consider a sequence of functions $v_1, v_2, \ldots$ belonging to $C^1(\bar{K}_{t_1,T})$:

$$v_m(x, t) = \begin{cases} \zeta_m(t) \int\limits_t^{\theta(x)} u_m(x, z)\,dz & \text{in } K_{t_1,\tau}, \\ 0 & \text{in } K_{t_1,T} \setminus K_{t_1,\tau}, \end{cases}$$

where $\zeta_m(t)$ is equal to 1 for $t < \tau(1 - 1/m)$, to 0 for $t > \tau$, $0 \leqslant \zeta_m(t) \leqslant 1$, $\zeta_m(t) \in C^1(-\infty, +\infty)$, $\left|\frac{d\zeta_m}{dt}\right| \leqslant C_0 m$. For any $\tau' \in [\tau, T]$ $v_m|_{\Gamma_{t_1,\tau'} \cup D_{\tau',\gamma(t_1-\tau')}} = 0$. We show that the sequence v_m, $m = 1, 2, \ldots$, converges to v in $H^1(K_{t_1,T})$. Indeed, the sequence v_m, $m = 1, 2, \ldots$, obviously converges to v in

$L_2(K_{t_1,T})$ and the sequence ∇v_m, $m = 1, 2, \ldots$,

$$\nabla v_m = \begin{cases} \zeta_m(t) \int\limits_t^{\theta(x)} \nabla u_m(x, z)\, dz + \zeta_m(t)\, u_m(x, \theta(x))\, \nabla\theta & \text{in } K_{t,\tau}, \\ & \text{in } K_{t_1,T} \diagdown K_{t_1,\tau} \end{cases}$$

converges in $L_2(K_{t_1,T})$ to the vector function ∇v defined by (7) (that is, $(v_m)_{x_i} \to v_{x_i}$, $i = 1, \ldots, n$, as $m \to \infty$) and since, as $m \to \infty$,

$$C_0^2 m^2 \int\limits_{K_{t_1,\tau} \diagdown K_{t_1,\tau(1-1/m)}} \left(\int\limits_t^{\theta(x)} u_m(x, z)\, dz \right)^2 dx\, dt$$

$$\leqslant C_0^2 T^2 \int\limits_{K_{t_1,\tau} \diagdown K_{t_1,\tau(1-1/m)}} u_m^2(x, z)\, dx\, dz$$

$$\leqslant 2C_0^2 T^2 \left[\int\limits_{K_{t_1,T}} (u - u_m)^2\, dx\, dt + \int\limits_{K_{t_1,\tau} \diagdown K_{t_1,\tau(1-1/m)}} u^2\, dx\, dt \right] \to 0,$$

the sequence v_{1t}, v_{2t}, ...,

$$v_{mt} = \begin{cases} -\zeta_m(t)\, u_m(x, t) + \zeta_m'(t) \int\limits_\tau^{\theta(x)} u_m(x, z)\, dz & \text{in } K_{t_1,\tau}, \\ 0 & \text{in } K_{t_1,T} \diagdown K_{t_1,\tau}, \end{cases}$$

converges in $L_2(K_{t_1,T})$ to the function v_t defined by (8).

Substituting v in identity (6), we obtain

$$\int\limits_{K_{t_1,\tau}} k(x)\, \nabla u(x, t) \int\limits_t^{\theta(x)} \nabla u(x, z)\, dz\, dx\, dt$$

$$+ \int\limits_{K_{t_1,\tau}} k(x)\, u(x, \theta(x))\, \nabla u(x, t)\, \nabla\theta\, dx\, dt$$

$$- \int\limits_{K_{t_1,\tau}} a(x)\, v_t v\, dx\, dt + \int\limits_{K_{t_1,\tau}} u_t u\, dx\, dt = 0. \qquad (9)$$

Since $v|_{\Gamma_{t_1,\tau} \cup D_{\tau,\gamma(t_1-\tau)}} = 0$, we have

$$\int\limits_{K_{t_1,\tau}} a v v_t\, dx\, dt = -\frac{1}{2} \int\limits_{D_{0,\gamma t_1}} a v^2\, dx \leqslant 0. \qquad (10)$$

Similarly, since $u|_{D_{0,\gamma t_1}} = 0$,

$$\int\limits_{K_{t_1,\tau}} u u_t\, dx\, dt = \frac{1}{2} \int\limits_{|x|<\gamma t_1} u^2(x, \theta(x))\, dx. \qquad (11)$$

Since

$$\int\limits_{K_{t_1,\tau}} k(x)\,\nabla u(x,t) \int\limits_{t}^{\theta(x)} \nabla u(x,z)\,dz\,dt\,dx$$

$$= \int\limits_{|x|<\gamma t_1} k(x) \int\limits_{0}^{\theta(x)} \nabla u(x,t) \int\limits_{t}^{\theta(x)} \nabla u(x,z)\,dz\,dt\,dx$$

$$= \int\limits_{|x|<\gamma t_1} k(x) \int\limits_{0}^{\theta(x)} \nabla u(x,t) \int\limits_{0}^{\theta(x)} \nabla u(x,z)\,dz\,dt\,dx$$

$$- \int\limits_{|x|<\gamma t_1} k(x) \int\limits_{0}^{\theta(x)} \nabla u(x,t) \int\limits_{0}^{t} \nabla u(x,z)\,dz\,dt\,dx$$

$$= \int\limits_{|x|<\gamma t_1} k(x) \left| \int\limits_{0}^{\theta(x)} \nabla u(x,t)\,dt \right|^2 dx$$

$$- \int\limits_{|x|<\gamma t_1} k(x) \int\limits_{0}^{\theta(x)} \nabla u(x,z) \int\limits_{z}^{\theta(x)} \nabla u(x,t)\,dt\,dz\,dx,$$

according to (7)

$$\int\limits_{K_{t_1,\tau}} k(x)\,\nabla u(x,t) \int\limits_{t}^{\theta(x)} \nabla u(x,z)\,dz\,dt\,dx$$

$$= \frac{1}{2} \int\limits_{|x|<\gamma t_1} k(x) \left| \int\limits_{0}^{\theta(x)} \nabla u(x,t)\,dt \right|^2 dx$$

$$= \frac{1}{2} \int\limits_{|x|<\gamma t_1} k(x) \Bigg[\left| \int\limits_{0}^{\theta(x)} \nabla u(x,t)\,dt + u(x,\theta(x))\,\nabla\theta \right|^2$$

$$- 2u(x,\theta(x))\nabla\theta \int\limits_{0}^{\theta(x)} \nabla u(x,t)\,dt - u^2(x,\theta(x))\,|\nabla\theta|^2 \Bigg] dx$$

$$\geqslant \int\limits_{K_{t_1,\tau}} k(x)\,u(x,\theta(x))\,\nabla\theta\nabla u(x,t)\,dx\,dt$$

$$- \frac{1}{2} \int\limits_{|x|<\gamma t_1} k(x)\,u^2(x,\theta(x))\,|\nabla\theta|^2\,dx.$$

Hence

$$\int\limits_{K_{t_1,\tau}} k(x)\,\nabla u(x,t)\int\limits_{t}^{\theta(x)}\nabla u(x,z)\,dz\,dt\,dx$$
$$+\int\limits_{K_{t_1,\tau}} k(x)\,u(x,\theta(x))\,\nabla u\cdot\nabla\theta\,dx\,dt$$
$$\geqslant -\frac{1}{2}\int\limits_{|x|<\gamma t_1} k(x)\,u^2(x,\theta(x))\,|\nabla\theta|^2\,dx$$
$$\geqslant -\frac{k_1}{2\gamma^2}\int\limits_{|x|<\gamma t_1} u^2(x,\theta(x))\,dx. \qquad (12)$$

If we insert relations (10)-(12) into (9), we find that $\left(1-\frac{k_1}{\gamma^2}\right)\int\limits_{|x|<\gamma t_1} u^2(x,\ \theta(x))\ \ dx\leqslant 0$ which implies that $\int\limits_{D\tau,\,\gamma(t_1-\tau)} u^2(x,\tau)\,dx=0$. And since $\tau\in(0,\ T)$ is arbitrary, this means that $u=0$ in $K_{t_1,T}$. ▮

From Lemma 1 there readily follows a uniqueness theorem regarding the generalized solution of the Cauchy problem (1)-(3), and therefore also regarding an a.e. solution and the classical solution.

Theorem 1. *The Cauchy problem* (1)-(3) *cannot have more than one generalized solution, one a.e. solution and one classical solution.*

Proof. Let u_1 and u_2 be generalized solutions (and, in particular, a.e. solutions) of the problem (1)-(3) ($f\in L_2(Q_{T,\,R})$, $\psi\in L_2(|x|<R)$ for all $R>0$). Then their difference u_1-u_2 satisfies the hypotheses of Lemma 1 for any $t_1>T$ and $x^0\in R_n$. Therefore $u_1=u_2$.

If u_1 and u_2 are classical solutions of the problem (1)-(3), then their difference u_1-u_2 is a classical solution of the same problem in which the functions f, φ and ψ are equal to zero; therefore u_1-u_2 is a generalized solution and consequently $u_1=u_2$. ▮

Let $\varphi\in H^1(|x|<R)$, $\psi\in L_2(|x|<R)$, $f\in L_2(Q_{T,\,R})$ for any $R>0$. For each m, $m=1, 2, \ldots$, take an infinitely differentiable function $\zeta_m(x,t)$ in $\overline{\Pi}_T$ such that it is equal to 1 in $K_{8mT,\,T}$ and to zero in $\Pi_T\diagdown K_{8(m+1/2)T,\,T}$, and let $u_m(x,t)$ denote the generalized solution in the cylinder $Q_{T,\,8(m+1)T\gamma}$ of the following mixed problem:

$$\begin{gathered}u_{mtt}-\operatorname{div}(k(x)\nabla u_m)+a(x)\,u_m=f_m(x,t),\\ u_m|_{D_{0,\,8(m+1)T\gamma}}=\varphi_m(x),\\ u_{mt}|_{D_{0,\,8(m+1)T\gamma}}=\psi_m(x),\\ u_m|_{S_{T,\,8(m+1)T\gamma}}=0,\end{gathered}\qquad(13)$$

where $\varphi_m(x) = \varphi(x)\,\zeta_m(x, 0)$, $\psi_m(x) = \psi(x)\,\zeta_m(x, 0)$, $f_m(x, t) = f(x, t)\,\zeta_m(x, t)$. This means that the function u_m belongs to $H^1(Q_{T,\,8(m+1)T\gamma})$, satisfies the initial condition $u_m|_{D_{0,\,8(m+1)T\gamma}} = 0$ and satisfies the integral identity

$$\int\limits_{Q_{T,\,8(m+1)T\gamma}} (k(x)\,\nabla u_m \nabla v + a u_m v - u_{mt} v_t)\,dx\,dt$$

$$= \int\limits_{Q_{T,\,8(m+1)T\gamma}} f_m v\,dx\,dt$$

$$+ \int\limits_{D_{0,\,8(m+1)T\gamma}} \psi_m(x)\,v(x, 0)\,dx, \qquad m = 1, 2, \ldots, \tag{14}$$

for all $v \in H^1(Q_{T,\,8(m+1)T\gamma})$ such that $v|_{D_{T,\,8(m+1)T\gamma}} = 0$, $v|_{S_{T,\,8(m+1)T\gamma}} = 0$.

Since $\varphi_m \in \overset{\circ}{H}{}^1(D_{0,\,8(m+1)T\gamma})$ ($\varphi \in H^1(D_{0,\,8(m+1)T\gamma})$ and is equal to zero for $8(m + 1/2)\,T\gamma < |x| < 8(m + 1)\,T\gamma$), in view of Theorem 1 of the preceding section the generalized solutions u_m exist.

Take any point $x^0 \in R_n$ such that $|x^0| = (8m + 6)\,T\gamma$ and in identity (14) substitute an arbitrary function v satisfying the following condition:

$$v \in H^1(K_{2T,\,T}(x^0)), \qquad v = 0 \text{ in } \Pi_T \setminus K_{2T,\,T}(x^0),$$

$$v|_{D_{T,\,T\gamma}(x^0)} = 0, \qquad v|_{\Gamma_{2T,\,T}(x^0)} = 0$$

(it is not difficult to see that such a v belongs to $H^1(Q_{T,\,8(m+1)T\gamma})$ and that its traces on $D_{T,\,8(m+1)T\gamma}$ and $S_{T,\,8(m+1)T\gamma}$ are zero). Applying Lemma 1, we find that $u_m = 0$ in $K_{2T,\,T}(x^0)$. Since x^0 is an arbitrary point on the n-dimensional sphere $\{|x| = (8m + 6)\,T\gamma,\ t = 0\}$, $u_m = 0$ in the cylindrical layer $\{(8m + 5)\,T\gamma < |x| < (8m + 7)\,T\gamma,\ 0 < t < T\}$.

Let $\tilde{u}_m(x, t)$ be a function equal to $u_m(x, t)$ in $Q_{T,\,(8m+6)T\gamma}$ and to zero in $\Pi_T \setminus Q_{T,\,(8m+6)T\gamma}$, $m = 1, 2, \ldots$. Evidently, the function $\tilde{u}_m$, $m = 1, 2, \ldots$, belongs to the space $H^1(Q_{T,\,R})$ and

$$\tilde{u}_m|_{D_{0,\,R}} = \varphi_m \tag{15}$$

for any $R > 0$.

Take an arbitrary function v satisfying condition (4) with some $R_0 = R_0(v) > 0$. If $R_0 < (8m + 7)\,T\gamma$, the function v may be substituted in (14). And since $u_m(x, t) = \tilde{u}_m(x, t)$ in $Q_{T,\,(8m+7)T\gamma}$, we have

$$\int_{\Pi_T} (k\nabla\tilde{u}_m\nabla v + a\tilde{u}_m v - \tilde{u}_{mt}v_t)\,dx\,dt$$

$$= \int_{\Pi_T} f_m v\,dx\,dt + \int_{R_n} \psi_m(x)\,v(x,\,0)\,dx. \qquad (16)$$

Suppose that $R_0 > (8m+7)T\gamma$. Since $\tilde{u}_m = u_m$ in $Q_{T,\,(8m+7)T\gamma}$ and $\tilde{u}_m = 0$ in $\Pi_T \setminus Q_{T,\,(8m+5)T\gamma}$ (in $Q_{T,\,(8m+7)T\gamma} \setminus Q_{T,\,(8m+5)T\gamma}$, $\tilde{u}_m = u_m = 0$), we have

$$\int_{\Pi_T} (k\nabla\tilde{u}_m\cdot\nabla v + a\tilde{u}_m v - \tilde{u}_{mt}v_t)\,dx\,dt$$

$$= \int_{Q_{T,\,(8m+5)T\gamma}} (k\nabla u_m\cdot\nabla v + au_m v - u_{mt}v_t)\,dx\,dt.$$

Taking an infinitely differentiable function $\tilde{\zeta}_m(x)$ in $\overline{\Pi}_T$ that is equal to 1 in $Q_{T,\,(8m+5)T\gamma}$ and to zero in $\Pi_T \setminus Q_{T,\,(8m+7)T\gamma}$ and substituting the function $v\zeta_m$ in (14), we obtain

$$\int_{Q_{T,\,(8m+5)T\gamma}} (k\nabla u_m\cdot\nabla v + au_m v - u_{mt}v_t)\,dx\,dt$$

$$= \int_{Q_{T,\,8(m+1)T\gamma}} (k\nabla u_m\cdot\nabla(v\tilde{\zeta}_m) + au_m(v\tilde{\zeta}_m) - u_{mt}(v\tilde{\zeta}_m)_t)\,dx\,dt$$

$$= \int_{\Pi_T} f_m v\,dx\,dt + \int_{R_n} \psi_m(x)\,v(x,\,0)\,dx$$

($u_m = 0$ in $Q_{T,\,(8m+7)T\gamma} \setminus Q_{T,\,(8m+5)T\gamma}$, $f_m = 0$ in $\Pi_T \setminus Q_{T,\,(8m+4)T\gamma}$ and $\psi_m = 0$ in $\{|x| > (8m+4)T\gamma\}$).

Thus the function $\tilde{u}_m$ satisfies the integral identity (16) for all v which satisfy condition (4) for some $R_0 = R_0(v) > 0$. Consequently, the function $\tilde{u}_m$ is a generalized solution in Π_T of the Cauchy problem (1)-(3) with functions $\varphi = \varphi_m$, $\psi = \psi_m$, $f = f_m$, $m = 1, 2, \ldots$. Since the function $\tilde{u}_{m'} - \tilde{u}_m$ (we assume $m' > m$) is a generalized solution in Π_T of the Cauchy problem (1)-(3) with functions $\varphi = \varphi_{m'} - \varphi_m = 0$ for $|x| < 8mT\gamma$, $\psi = \psi_{m'} - \psi_m = 0$ for $|x| < 8mT\gamma$ and $f = f_{m'} - f_m = 0$ in $K_{8mT,\,T}$, by Lemma 1 $\tilde{u}_{m'} - \tilde{u}_m = 0$ in $K_{8mT,\,T}$. That is, for all $m' \geqslant m$ $\tilde{u}_{m'} = \tilde{u}_m$ in $K_{8mT,\,T}$, and therefore also in $Q_{T,\,(8m-1)T\gamma}$. This means that the sequence of functions $\tilde{u}_1, \tilde{u}_2, \ldots$ converges almost everywhere in Π_T to some function u; moreover, for any $R > 0$ there is a number $N=N(R)$ $\left(N(R) = 1+\left[\frac{R+T\gamma}{8T\gamma}\right]\right)$ such that $u=\tilde{u}_m = u_m$ in $Q_{T,\,R}$ for all $m \geqslant N$. From (15) and (16) it follows that $u|_{t=0} = \varphi$ (for any $R > 0$ and $m \geqslant N(R)$ $\varphi_m = \varphi$ in $D_{0,\,R}$ and $\varphi_m = \tilde{u}_m|_{D_{0,\,R}} = u|_{D_{0,\,R}}$) and u satisfies the integral identity (5) for all v obeying condition (4) with a certain $R_0 = R_0(v) > 0$.

Consequently, u is a generalized solution of the Cauchy problem (1)-(3) in Π_T. Thus we have established the following result.

Theorem 2. *If* $\varphi(x) \in H^1(|x| < R)$, $\psi(x) \in L_2(|x| < R)$ and $f \in L_2(Q_{T,\,R})$ *for any* $R > 0$, *then the Cauchy problem* (1)-(3) *has a generalized solution in* Π_T.

Note that we have also proved the following result. For $R > 0$ there is a number $N = N(R)$ such that for all $m \geqslant N$ the generalized solution u of the Cauchy problem (1)-(3) in the cylinder $Q_{T,\,R}$ coincides with the solutions u_m of the mixed problems (13).

We shall now examine a particular case of Eq. (1), the wave equation ($k \equiv 1$, $a \equiv 0$ in (1))

$$u_{tt} - \Delta u = f. \tag{17}$$

Suppose that with a certain integer $s > 1$ $\varphi \in H^s(|x| < R)$, $\psi \in H^{s-1}(|x| < R)$, $f \in H^{s-1}(Q_{T,\,R})$ for any $R > 0$. Then by Theorem 3, Sec. 2.4, the generalized solution $u_m(x, t)$ of the mixed problem (13) (with $k \equiv 1$, $a \equiv 0$) belongs to $H^s(Q_{T,\,8(m+1)T\gamma})$

$$(\varphi_m = \zeta_m(x, 0)\,\varphi(x) \in H^s_{\mathscr{D}}(D_{0,\,8(m+1)T\gamma}),$$

$$\psi_m = \zeta_m(x, 0)\,\psi(x) \in H^{s-1}_{\mathscr{D}}(D_{0,\,8(m+1)T\gamma}),$$

$$f_m = \zeta_m f \in \tilde{H}^{s-1}_{\mathscr{D}}(Q_{T,\,8(m+1)T\gamma})).$$

Consequently, for any $R > 0$ the generalized solution u of the Cauchy problem (17), (2), (3) in Π_T belongs to $H^s(Q_{T,\,R})$.

Thus we have proved the following theorem.

Theorem 3. *If for any* $R > 0$ $\varphi \in H^s(|x| < R)$, $\psi \in H^{s-1}(|x| < R)$, $f \in H^{s-1}(Q_{T,\,R})$ *with certain integer* $s > 1$, *then the generalized solution of the Cauchy problem* (17), (2), (3) *belongs to* $H^s(Q_{T,\,R})$ *for any* $R > 0$.

Since for any $R > 0$ the generalized solution of the Cauchy problem belonging to $H^2(Q_{T,\,R})$ is an a.e. solution of the same problem, from Theorem 3 (with $s = 2$) there follows the following result.

Theorem 4. *If* $\varphi \in H^2(|x| < R)$, $\psi \in H^1(|x| < R)$, $f \in H^1(Q_{T,\,R})$ *for any* $R > 0$, *then the Cauchy problem* (17), (2), (3) *in* Π_T *has an a.e. solution.*

Note that the orders of smoothness of the initial functions and right-hand side of the equation that guarantee the existence of the generalized solution or an a.e. solution of the Cauchy problem do not depend (Theorems 2, 4) on the dimension of space.

Let $s=\left[\frac{n}{2}\right]+3$. Then, by Theorem 4, Sec. 2.4, the generalized solution $u_m(x, t)$ of the mixed problem (13) (with $k\equiv 1$, $a\equiv 0$) is a classical solution of this problem. Consequently, the generalized solution $u(x, t)$ of the problem (17), (2), (3) is a classical solution of the same problem.

Thus we have proved the following theorem.

Theorem 5. *If* $\varphi\in H^{\left[\frac{n}{2}\right]+3}(|x|<R)$, $\psi\in H^{\left[\frac{n}{2}\right]+2}(|x|<R)$, $f\in H^{\left[\frac{n}{2}\right]+2}(Q_{T,R})$ *for any* $R>0$, *then the Cauchy problem* (17), (2), (3) *has a classical solution.*

We supplement Theorems 4 and 5 regarding the existence of an a.e. solution and the classical solution of the Cauchy problem (17), (2), (3) by proving the following assertion.

Theorem 6. *Let* u *be an a.e. solution of the problem* (17), (2), (3) *in* Π_T *or a classical solution of this problem with* $f\in L_2(Q_{T,R})$ *for any* $R>0$. *Then for any* $R>0$ *and any* t, $0<t<\min(R, T)$ *the inequality*

$$E_R^{1/2}(t)\leqslant E_R^{1/2}(0)+2\sqrt{t}\,\|f\|_{L_2(K_{R,T})} \tag{18}$$

holds, where

$$E_R(t)\int\limits_{D_{t,R-t}}(u_t^2+|\nabla u|^2)\,dx, \tag{19}$$

$$D_{\tau,R-\tau}=\{|x|<R-\tau,\ t=\tau\},$$

$$K_{R,\tau}=\{|x|<R-t,\quad 0<t<\tau\},\qquad \tau\in[0,\ \min(R, T)].$$

Proof. Take any $\tau\in(0, \min(R, T))$, multiply identity (17) by u_t and then integrate it over the truncated cone $K_{R,\tau}$:

$$\int\limits_{K_{R,\tau}}(u_{tt}u_t-u_t\Delta u)\,dx\,dt=\int\limits_{K_{R,\tau}}fu_t\,dx\,dt.$$

According to Ostrogradskii's formula, we obtain

$$\int\limits_{D_{\tau,R-\tau}}(u_t^2+|\nabla u|^2)\,dx-\int\limits_{D_{0,R}}(\psi^2+|\nabla\varphi|^2)\,dx$$
$$+\int\limits_{\Gamma_{R,\tau}}\left[(u_t^2+|\nabla u|^2)\,n_0-2u_t\sum_{i=1}^{n}u_{x_i}n_i\right]dS=2\int\limits_{K_{R,\tau}}fu_t\,dx\,dt,$$

where $\Gamma_{R,\tau}$ is the lateral surface of the cone $K_{R,\tau}$, $\Gamma_{R,\tau}=$ $=\{|x|=R-t,\ 0<t<\tau\}$ and $(n_0, n_1, \ldots, n_n)=$ $=\left(\frac{1}{\sqrt{2}}, \frac{x_1}{\sqrt{2}(R-t)}, \ldots, \frac{x_n}{\sqrt{2}(R-t)}\right)$ is the outward normal vector to $\Gamma_{R,\tau}$. Since on $\Gamma_{R,\tau}$

$$(u_t^2+|\nabla u|^2)\,n_0-2u_t\sum_{i=1}^{n}u_{x_i}n_i$$

$$=\frac{1}{\sqrt{2}}\sum_{i=1}^{n}\left(\left(\frac{u_t x_i}{R-t}\right)^2-2\frac{u_t x_i}{R-t}u_{x_i}+u_{x_i}^2\right)$$

$$=\frac{1}{\sqrt{2}}\sum_{i=1}^{n}\left(\frac{u_t x_i}{R-t}-u_{x_i}\right)^2\geqslant 0,$$

we have

$$E_R(\tau)\leqslant E_R(0)+2\int\limits_{K_{R,\tau}}|f|\,|u_t|\,dx\,dt\leqslant E_R(0)$$

$$+2\|f\|_{L_2(K_{R,\tau})}\left(\int\limits_{K_{R,\tau}}(u_t^2+|\nabla u|^2)\,dx\,dt\right)^{1/2}. \qquad (20)$$

Since $2|ab|=2\left|a\sqrt{2\tau}\frac{b}{\sqrt{2\tau}}\right|\leqslant 2\tau a^2+\frac{b^2}{2\tau}$, it follows from (20) that

$$E_R(t)\leqslant E_R(0)+2\tau\|f\|_{L_2(K_{R,\tau})}^2+\frac{1}{2\tau}\int\limits_{K_{R,\tau}}(u_t^2+|\nabla u|^2)\,dx\,dt$$

for all $t\in(0,\tau)$. Integrating the last inequality with respect to $t\in(0,\tau)$, we have

$$\int\limits_{K_{R,\tau}}(u_t^2+|\nabla u|^2)\,dx\,dt\leqslant\tau E_R(0)+2\tau^2\|f\|_{L_2(K_{R,\tau})}^2$$

$$+\frac{1}{2}\int\limits_{K_{R,\tau}}(u_t^2+|\nabla u|^2)\,dx\,dt,$$

hence

$$\int\limits_{K_{R,\tau}}(u_t^2+|\nabla u|^2)\,dx\,dt\leqslant 2\tau E_R(0)+4\tau^2\|f\|_{L_2(K_{R,\tau})}^2$$

$$\leqslant(2\sqrt{\tau}E_R^{1/2}(0)+2\tau\|f\|_{L_2(K_{R,\tau})})^2. \qquad (21)$$

The inequality (18) follows from (20) and (21). ▮

PROBLEMS ON CHAPTER V

Let $u(x, t)$, $x = (x_1, x_2) \in R_2$, be a (classical) solution of the Cauchy problem

$$u_{tt} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}, \qquad u|_{t=0} = \varphi, \qquad u_t|_{t=0} = \psi, \tag{1}$$

where the initial functions φ and ψ have compact supports: $\varphi = \psi = 0$ for $|x|^2 = x_1^2 + x_2^2 > R_0^2$.

1. Show that the function $u(x, t)$ is analytic in the cone $\{|x| < t - R_0, t > R_0\}$.

2. Prove that there is a constant $C > 0$ such that for all $x \in R_2$ and $t \geqslant 0$ the solution $u(x, t)$ of problem (1) admits the estimate

$$|u(x, t)| \leqslant \frac{C}{\sqrt{t}\,(1 + \sqrt{|t - |x||})}.$$

Moreover, if $\varphi \equiv 0$ and $\psi \geqslant 0$, $\psi \not\equiv 0$, then there are positive constants C_0 and T such that

$$u(x, t) \geqslant \frac{C_0}{\sqrt{t}\,(1 + \sqrt{|t - |x||})}$$

for all $t \geqslant T$ and $|x| \leqslant t - R_0$.

3. Show that for any $R > 0$ there is a $T > 0$ such that for all $(x, t) \in \{|x| \leqslant R, t \geqslant T\}$ the solution $u(x, t)$ of problem (1) is the sum of the convergent series

$$u(x, t) = \sum_{m=0}^{\infty} \frac{c_m(x)}{t^{m+1}};$$

find c_0 and c_1.

Prove that if for some disc $K_0 = \{|x - x^0| < r_0\}$ (x^0 is a point of R_2 and r_0 is a positive number) and all natural numbers l $t^l u(x, t) \to 0$ uniformly in $x \in K_0$, then $u \equiv 0$.

4. Let $\varphi \in C^2(R_2)$ and $\psi \in C^1(R_2)$, and let all the second derivatives of the function φ and all the first derivatives of the function ψ belong to the class $C^{\alpha}(R_2)$ (see Probl. 17, Chap. III) for some $\alpha > 1/2$. Prove that a classical solution of the Cauchy problem (1) exists.

5. Let $u(x, t)$, $x = (x_1, x_2, x_3) \in R_3$, be the (classical) solution of the Cauchy problem

$$u_{tt} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2}, \qquad u|_{t=0} = \varphi, \qquad u_t|_{t=0} = \psi, \tag{2}$$

where the initial functions φ and ψ have compact supports.

Show that, for all $x \in R_3$ and $t > 0$,

$$|u(x, t)| \leqslant C/t,$$

where C is a positive constant.

6. Let $u(x, t)$, $x = (x_1, x_2, x_3) \in R_3$, belong to $C^3(Q \setminus \{x^0, t^0\})$, where Q is a region in the four-dimensional space R_4 and (x^0, t^0) is a point of Q, and satisfy the wave equation (2) in $Q \setminus \{x^0, t^0\}$. Show that u belongs to $C^2(Q)$ (that is, it can be redefined at the point (x^0, t^0) so that it becomes twice continuously differentiable in Q).

7. Suppose that the function $u(x, t) = v(x - nt)$, $x = (x_1, x_2, x_3)$, $n = (n_1, n_2, n_3)$ is a constant vector, belongs to $C^3(R_4 \setminus L)$, where L is a line defined by the equation $x - nt = 0$, and satisfies the wave equation (2) outside L. Show that if $u(x, t) = o\,(1/r)$, where r is the distance between the point (x, t) and the line L, then $u(x, t) \in C^2(R_4)$ (that is, it can be redefined on L so that it becomes twice continuously differentiable in R_4).

8. Let $u(x, t)$ be the classical or generalized solution in $\{t > 0\}$ of the Cauchy problem for the wave equation

$$u_{tt} = \Delta u + f(x, t), \tag{3}$$

$$u\,|_{t=0} = \varphi(x), \qquad u_t\,|_{t=0} = \psi(x), \tag{4}$$

and let $u_R(x, t)$ be the classical or generalized solution of the second mixed problem for the wave equation in the cylinder $\{|\,x\,| < R + 1,\ t > 0\}$, $R > 0$:

$$(u_R)_{tt} = \Delta u_R + f_R,$$

$$u_R\,|_{t=0} = \varphi_R, \qquad u_{Rt}\,|_{t=0} = \psi_R, \qquad \left.\frac{\partial u_R}{\partial n}\right|_{|x|=R+1} = 0,$$

where $\varphi_R = \varphi$, $\psi_R = \psi$ for $|\,x\,| < R$ and $f_R = f$ for $|\,x\,| < R$, $t < R$. Show that in any cylinder $Q_T = \{x \in D_0,\ 0 < t < T\}$, where D_0 is any n-dimensional region and T any positive number, the difference $u - u_R = 0$ for sufficiently large R.

9. A solution of the first mixed problem for the wave equation in the cylinder $Q_T = \{x \in D_0,\ 0 < t < T\}$ can be defined as follows: a function $u(x, t)$ is said to be the classical solution of the first mixed problem for the wave equation (3) if it belongs to $C^2(Q_T) \cap C^1(Q_T \cup D_0) \cap C(Q_T \cup \Gamma_T \cup \bar{D}_0)$, satisfies Eq. (3) in Q_T and the initial conditions (4) on D_0 together with the boundary condition

$$u|_{\Gamma_T} = \chi. \tag{5}$$

Show that this solution is unique.

10. Establish the existence and uniqueness of the generalized solution of the third mixed problem (see Sec. 2.1) for the wave equation in the cylinder $Q_T = \{x \in D_0,\ 0 < t < T\}$:

$$u_{tt} - \Delta u = f(x, t),$$

$$\left.\left\{\frac{\partial u}{\partial n} + \sigma(x)\,u\right\}\right|_{\Gamma_T} = 0,$$

$$u\,|_{t=0} = \varphi(x), \qquad u_t\,|_{t=0} = \psi(x)$$

($\varphi \in H^1(D_0)$, $\psi \in L_2(D_0)$, $f \in L_2(Q_T)$) when the arbitrary function $\sigma(x)$ is continuous on ∂D_0 (without assuming that it is nonnegative).

11. A function $u(x, t)$ belonging to $H^1(Q_T)$ is called the generalized solution of the problem

$$u_{tt} = \Delta u, \qquad (x, t) \in Q_T,$$

$$\left.\left\{\frac{\partial u}{\partial n} + \sigma(x)\,\frac{\partial u}{\partial t}\right\}\right|_{\Gamma_T} = 0,$$

$$u\,|_{t=0} = \varphi(x), \qquad u_t\,|_{t=0} = \psi(x),$$

where $\sigma(x) \in C(\partial D_0)$, $\sigma(x) \geqslant 0$, $\varphi \in H^1(D)$, $\psi \in L_2(D)$, if it satisfies the initial condition $u|_{t=0} = \varphi$ and the integral identity

$$\int\limits_{Q_T} (u_t v_t - \nabla u \nabla v)\, dx\, dt = \int\limits_{\Gamma_T} \sigma u v_t\, dS\, dt + \int\limits_{D_0} \psi v\, dx + \int\limits_{\partial D_0} \sigma \varphi v\, dS$$

for all $v \in C^1(\overline{Q}_T)$ such that $v_t \in C^1(\overline{Q}_T)$ and $v|_{t=T} = 0$.

Establish the existence and uniqueness of the generalized solution of this problem.

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CHAPTER VI

PARABOLIC EQUATIONS

In this chapter we shall study the Cauchy problem and mixed problems for a parabolic equation of the form

$$u_t - \operatorname{div}\,(k(x)\,\nabla u(x, t)) + a(x)\,u(x, t) = f(x, t).$$

Here $(x, t) = (x_1, \ldots, x_n, t)$ is a point of the $(n+1)$-dimensional space R_{n+1}, $x \in R_n$, $t \in R_1$; $\nabla v\,(x, t) = \left(\frac{\partial v}{\partial x_1}, \ldots, \frac{\partial v}{\partial x_n}\right)$ and $\operatorname{div}\,(w_1(x, t), \ldots, w_n\,(x, t)) = \frac{\partial w_1}{\partial x_1} + \ldots + \frac{\partial w_n}{\partial x_n}$; moreover, by $\Delta v(x, t)$ we shall mean $\operatorname{div}\,\nabla v(x, t) = \frac{\partial^2 v}{\partial x_1^2} + \ldots + \frac{\partial^2 v}{\partial x_n^2}$. The data of the problems will be assumed real-valued functions and we shall study only real-valued solutions of these problems. Therefore the function spaces $C^{p,\,q}$, $H^{p,\,q}$, ... to be used in what follows will be considered as real-valued*.

§ 1. PROPERTIES OF SOLUTIONS OF HEAT EQUATION. THE CAUCHY PROBLEM FOR HEAT EQUATION

1. Properties of Solutions of the Heat Equation. Let us consider the simplest parabolic equation, the heat equation

$$\mathscr{L}u \equiv u_t - \Delta u = f(x, t). \tag{1}$$

First we construct in the half-space $\{t > 0\} = \{x \in R_n,\ t > 0\}$ some special solutions of the homogeneous heat equation

$$\mathscr{L}u \equiv u_t - \Delta u = 0. \tag{1_0}$$

---

* The definitions of spaces $C^{p,q}$ and $H^{p,q}$ are given, respectively, in Secs. 7.1 and 7.2, Chap. III.

We first examine the case of one space variable, $n = 1$. The function $u(x, t) = w(x^2/t)$, depending only on x^2/t, being a solution of the equation $u_t - u_{xx} = 0$ in $\{t > 0\}$ satisfies the ordinary differential equation

$$4zw''(z) + (2 + z)\, w'(z) = 0.$$

The general solution of this equation on the semi-axis $(0, \infty)$ is given by the formula $c_1 \int\limits_0^z e^{-\zeta/4} \zeta^{-1/2}\, d\zeta + c_2$, where c_1 and c_2 are arbitrary constants. Then the function $c_1 \int\limits_0^{x^2/t} e^{-\zeta/4} \zeta^{-1/2}\, d\zeta + c_2$ is a solution of Eq. (1_0) (when $n = 1$) in the regions $\{x > 0,\ t > 0\}$ and $\{x < 0,\ t > 0\}$.

Set $c_2 = 0$ and $c_1 = \frac{1}{4\sqrt{\pi}}$ for $x > 0$, while $c_1 = -\frac{1}{4\sqrt{\pi}}$ for $x < 0$. The resulting function, as is easy to check, is continuously differentiable in the half-plane $\{x \in R_1,\ t > 0\}$ and, consequently, satisfies Eq. (1_0) in this half-plane. Then any derivative (with respect to x or t) of this function, in particular, the first derivative with respect to x, $U(x, t) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$, will also satisfy this equation.

Now let $n > 1$. To construct desired solutions of Eq. (1_0), note that if the functions $v_i(x, t)$, $i = 1, \ldots, n$, are solutions of Eq. (1_0) for $n = 1$ in the half-plane $\{x \in R_1,\ t > 0\}$, then the function $v(x, t) = v_1(x_1, t)\, v_2(x_2, t) \ldots v_n(x_n, t)$ is a solution of Eq. (1_0) in the half-space $\{x \in R_n,\ t > 0\}$. Therefore, in particular, the function

$$U(x, t) = \prod_{i=1}^{n} \frac{e^{-\frac{x_i^2}{4t}}}{2\sqrt{\pi t}} = \frac{e^{-\frac{|x|^2}{4t}}}{(2\sqrt{\pi t})^n}$$

is a solution of Eq. (1_0) in the half-space $\{t > 0\}$. Then it follows that if (x^0, t^0) is an arbitrary point in R_{n+1}, then the function

$$U(x - x^0,\ t - t^0) = \frac{e^{-\frac{|x - x^0|^2}{4(t - t^0)}}}{(2\sqrt{\pi (t - t^0)})^n}$$

is a solution of Eq. (1_0) in the half-space $\{t > t^0\} = \{x \in R_n,\ t > t^0\}$. This function is called the *fundamental solution of the heat equation* (with singularity at the point (x^0, t^0)).

Let us note the following properties of the fundamental solution.

If the function $U(x-x^0,\ t-t^0)$ is extended as being equal to zero into the half-space $\{t<t^0\}=\{x\in R_n,\ t<t^0\}$, then the resulting function will be infinitely differentiable in $R_{n+1}\diagdown\{x^0,\ t^0\}$.

For all $x^0\in R_n$, $t>t^0$

$$\int\limits_{R_n} U(x-x^0,\ t-t^0)\,dx=\frac{1}{\pi^{n/2}}\int\limits_{R_n} e^{-|\xi|^2}\,d\xi=1. \tag{2}$$

The function $U(x-x^0,\ t-t^0)$ regarded as a function of variables $(x^0,\ t^0)=(x_1^0,\ \ldots,\ x_n^0,\ t^0)$ is a solution of the equation

$$\mathcal{L}^*_{x^0,\,t^0}U(x-x^0,\ t-t^0)=-\frac{\partial U}{\partial t^0}-\sum_{i=1}^{n}\frac{\partial^2 U}{\partial x_i^{0\,2}} \tag{1_0^*}$$

in the half-space $\{t^0<t\}=\{x^0\in R_n,\ t^0<t\}$.

Consider a strip (bounded by the characteristics of Eq. (1)) $\{0<t<T\}=\{x\in R_n,\ 0<t<T\}$. As in the case of Laplace's equation and the wave equation, let us establish, using the constructed special solutions (fundamental solution), in this strip the representation of an arbitrary function $u(x,\ t)$ belonging to $C^{2,\,1}(0<t<T)\cap$ $\cap\ C(0\leqslant t<T)$ in terms of the functions $\mathcal{L}u=u_t-\Delta u$ and $u(x,\ 0)$, the value of the function $u(x,\ t)$ on the plane $\{t=0\}=$ $=\{x\in R_n,\ t=0\}$. We shall assume that the functions $u(x,\ t)$ and $\mathcal{L}u(x,\ t)$ are bounded in $\{0<t<T\}$.

We take the functions $\zeta_N(x)=\zeta_N(|\,x\,|)$, $N=1,\ 2,\ \ldots$, which are infinitely differentiable in R_n and satisfy the following conditions: $\zeta_N(x)\equiv 1$ for $|\,x\,|<N$, $\zeta_N(x)\equiv 0$ for $|\,x\,|>N+1$ and $|\,\zeta_N(x)\,|\leqslant C_0$, $|\,\nabla\zeta_N\,|\leqslant C_0$, $|\,\Delta\zeta_N\,|\leqslant C_0$, where the constant C_0 does not depend on N.

Let $(\xi,\ \tau)$ be any point in the strip $\{0<t<T\}$. Since the functions $\zeta_N(x)\ u(x,\ t)$, $N=1,\ 2,\ \ldots$, and $U(\xi-x,\ \tau-t)$ belong to $C^{2,\,1}(0<t<\tau)$, from (1_0^*) and the relation

$$\mathcal{L}(u(x,\ t)\,\zeta_N(x))=\zeta_N\mathcal{L}u-2\nabla\zeta_N\cdot\nabla u-u\Delta\zeta_N$$

it follows that

$$\begin{aligned}U(\xi-x,\ \tau-t)\,(\zeta_N(x)\,\mathcal{L}u(x,t)-2\nabla\zeta_N(x)\cdot\nabla u(x,t)-u(x,\ t)\,\Delta\zeta_N(x))&\\ =U(\xi-x,\ \tau-t)\ \mathcal{L}(u(x,\ t)\,\zeta_N(x))&\\ -u(x,\ t)\,\zeta_N(x)\ \mathcal{L}^*_{x,\,t}\,(U(\xi-x,\ \tau-t))&\\ =(u\zeta_N U)_t+\sum_{i=1}^{n}(u\zeta_N U_{x_i}-(u\zeta_N)_{x_i}U)_{x_i}&\end{aligned}$$

for all $(x,\ t)\in\{0<t<\tau\}$.

We integrate the last identity over the cylinder $\{|\,x\,|<N+1,\ \varepsilon<t<\tau-\varepsilon\}$ for some $\varepsilon\in(0,\ \tau/2)$. By means of Ostrogradskii's formula, we obtain

$$\begin{aligned}
\int_{\varepsilon}^{\tau-\varepsilon} dt \int_{|x|<N+1} \zeta_N(x)\, \mathscr{L}u(x,\, t)\cdot U(\xi - x,\, \tau - t)\, dx \\
= \int_{|x|<N+1} u(x,\, \tau-\varepsilon)\, \zeta_N(x)\, U(\xi - x,\, \varepsilon)\, dx \\
- \int_{|x|<N+1} u(x,\, \varepsilon)\, \zeta_N(x)\, U(\xi - x,\, \tau-\varepsilon)\, dx \\
+ \int_{\varepsilon}^{\tau-\varepsilon} dt \int_{N<|x|<N+1} u(x,\, t)\cdot \Delta\zeta_N(x)\cdot U(\xi - x,\, \tau - t)\, dx \\
+ 2\int_{\varepsilon}^{\tau-\varepsilon} dt \int_{N<|x|<N+1} \nabla u(x,\, t)\cdot \nabla\zeta_N(x)\cdot U(\xi - x,\, \tau - t)\, dx \\
= \int_{|x|<N+1} u(x,\, \tau-\varepsilon)\, \zeta_N(x)\, U(\xi - x,\, \varepsilon)\, dx \\
- \int_{|x|<N+1} u(x,\, \varepsilon)\zeta_N(x)\, U(\xi - x,\, \tau-\varepsilon)\, dx \\
- \int_{\varepsilon}^{\tau-\varepsilon} dt \int_{N<|x|<N+1} u(x,\, t)\, \Delta\zeta_N(x)\cdot U(\xi - x,\, \tau - t)\, dx \\
- 2\int_{\varepsilon}^{\tau-\varepsilon} dt \int_{N<|x|<N+1} u(x,\, t)\, \nabla\zeta_N(x)\cdot \nabla_x U(\xi - x,\, \tau - t)\, dx \\
= I_{1,\,\varepsilon,\,N} + I_{2,\,\varepsilon,\,N} + I_{3,\,\varepsilon,\,N} + I_{4,\,\varepsilon,\,N}\,. \qquad (3)
\end{aligned}$$

We pass to the limit in (3) first as $N \to \infty$ and then as $\varepsilon \to +0$. Since for all $N = 1, 2, \ldots$ and all $(x,\, t) \in \{0 < t < \tau\}$ $|\, \zeta_N(x)\, \mathscr{L}u(x,\, t)\,| \leqslant C_0 \cdot \sup\limits_{\{0<t<T\}} |\, \mathscr{L}u\,|$ ($\mathscr{L}u$ is bounded in $\{0 < t < T\}$), in view of (2)

$$\int_{R_n} |\, \zeta_N(x)\, \mathscr{L}u(x,\, t)\cdot U(\xi - x,\, \tau - t)\,|\, dx \leqslant C_0 \sup_{\{0<t<T\}} |\, \mathscr{L}u(x,\, t)\,|$$

for all $t \in (0,\, \tau)$. Consequently, according to Lebesgue's theorem, we have

$$\lim_{\varepsilon\to 0} \lim_{N\to\infty} \int_{\varepsilon}^{\tau-\varepsilon} dt \int_{|x|<N+1} \zeta_N(x)\, \mathscr{L}u(x,\, t)\cdot U(\xi - x,\, \tau - t)\, dx$$

$$= \int_0^{\tau} dt \int_{R_n} \mathscr{L}u(x,\, t)\cdot U(\xi - x,\, \tau - t)\, dx. \qquad (4)$$

Since for all $N = 1, 2, \dots$ and all $(x, t) \in \{0 < t < \tau\}$ $| \zeta_N(x)\, u(x, t)| \leqslant C_0 \cdot \sup\limits_{\{0<t<T\}} | u(x, t) |$ (u is bounded in $\{0 < t < T\}$), for all $t \in (0, \tau)$

$$\int_{R_n} | \zeta_N(x)\; u(x, t) \cdot U(\xi - x, \tau - t) |\, dx \leqslant C_0 \cdot \sup_{\{0<t<T\}} | u(x, t) |,$$

hence

$$\lim_{N\to\infty} \int_{|x|<N+1} \zeta_N(x)\, u(x, \tau - \varepsilon) \cdot U(\xi - x, \varepsilon)\, dx = \int_{R_n} u(x, \tau - \varepsilon)\, U(\xi - x, \varepsilon)\, dx$$

and

$$\lim_{N\to\infty} \int_{|x|<N+1} \zeta_N(x)\, u(x, \varepsilon)\, U(\xi - x, \tau - t)\, dx = \int_{R_n} u(x, \varepsilon)\, U(\xi - x, \tau - \varepsilon)\, dx.$$

But the function $u(x, t)$ is continuous and bounded in $\{0 \leqslant t \leqslant \tau\}$, therefore

$$\begin{aligned} \lim_{\varepsilon\to 0} \lim_{N\to\infty} I_{1,\varepsilon,N} &= \lim_{\varepsilon\to 0} \int_{R_n} u(x, \tau - \varepsilon)\, U(\xi - x, \varepsilon)\, dx \\ &= \frac{1}{(\pi)^{n/2}} \lim_{\varepsilon\to 0} \int_{R_n} u(\xi + 2\sqrt{\varepsilon}\,\eta, \tau - \varepsilon)\, e^{-|\eta|^2}\, d\eta \\ &= u(\xi, \tau) \frac{1}{\pi^{n/2}} \int_{R_n} e^{-|\eta|^2}\, d\eta = u(\xi, \tau) \qquad (5) \end{aligned}$$

and

$$\lim_{\varepsilon\to 0} \lim_{N\to\infty} I_{2,\varepsilon,N} = \int_{R_n} u(x, 0)\, U(\xi - x, \tau)\, dx. \qquad (6)$$

Since for all $N = 1, 2, \dots$ and all $(x, t) \in \{0 < t < \tau\}$ $| u(x, t)\, \Delta\zeta_N(x) | \leqslant C_0 \cdot \sup\limits_{\{0<t<T\}} | u(x, t) |$, for all $t \in (0, \tau)$

$$\int_{R_n} | u(x, t)\, \Delta\zeta_N(x) \cdot U(\xi - x, \tau - t) |\, dx \leqslant C_0 \sup_{\{0<t<T\}} | u |.$$

Consequently,

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty}|I_{3,\varepsilon,N}|$$

$$\leqslant \lim_{\varepsilon\to 0}\lim_{N\to\infty}\int_{\varepsilon}^{\tau-\varepsilon} dt \int_{N<|x|<N+1} |u(x, t)\,\Delta\zeta_N(x)\cdot U(\xi-x, \tau-t)|\,dx$$

$$\leqslant C_0 \lim_{\varepsilon\to 0}\int_{\varepsilon}^{\tau-\varepsilon} dt \lim_{N\to\infty}\int_{N<|x|<N+1} |u(x, t)|\,U(\xi-x, \tau-t)\,dx = 0. \qquad (7)$$

Finally, we consider the term $I_{4,\varepsilon,N}$ in (3). Since for all $(x, t) \in \{0 < t < \tau\}$ and all $N = 1, 2, \ldots$ $|\nabla\zeta_N\cdot u| \leqslant C_0 \times \sup_{\{0<t<T\}} |u(x, t)|$, it follows that for all $t \in (0, \tau)$

$$\int_{R_n} |u(x, t)\,\nabla\zeta_N(x)\cdot\nabla_x U(\xi-x, \tau-t)|\,dx \leqslant C_0 \int_{R_n} |u(x, t)|\,|\nabla_x U|\,dx$$

$$\leqslant C_0\cdot \sup_{\{0<t<T\}} |u| \int_{R_n} \frac{|x-\xi|\,e^{-\frac{|x-\xi|^2}{4(\tau-t)}}}{2(\tau-t)\,(2\sqrt{\pi(\tau-t)})^n}\,dx$$

$$= \frac{C_0 \sup|u|}{\pi^{n/2}\sqrt{\tau-t}} \int_{R_n} |\eta|\,e^{-|\eta|^2}\,d\eta = \frac{C_1}{\sqrt{\tau-t}},$$

where the constant C_1 does not depend on N. Therefore

$$\lim_{\varepsilon\to 0}\lim_{N\to\infty} I_{4,\varepsilon,N} = 0. \qquad (8)$$

The desired representation for the function u follows from relations (3)-(8).

Thus, the following statement holds.

If the function $u(x, t)$ *belongs to* $C^{2,1}(0 < t < T) \cap C(0 \leqslant t < T)$ *is bounded in* $\{0 < t < T\}$ *and the function* $\mathscr{L}u$ *is bounded in* $\{0 < t < T\}$, *then for any point* (x, t) *in* $\{0 < t < T\}$ *the representation*

$$u(x, t) = \int_{R_n} u(\xi, 0)\,U(x-\xi, t)\,d\xi + \int_0^t d\tau \int_{R_n} \mathscr{L}u(\xi, \tau)\,U(x-\xi, t-\tau)\,d\xi \qquad (9)$$

holds.

Using representation (9), we shall establish some properties of solutions of the heat equation.

Theorem 1. *If the function* $u(x, t)$ *belongs to* $C^{2,1}(Q)$ *and* $\mathscr{L}u = u_t - \Delta u = 0$ *in* Q, *where* Q *is a region in the* $(n+1)$*-dimensional* *space* R_{n+1}, *then* $u(x, t) \in C^\infty(Q)$ *and for any* t^0 *the function* $u(x, t^0)$ *(regarded as a function of variables* $x_1, \ldots, x_n$*) is analytic in* $Q \cap \{t = t^0\}$.

Proof. Let (x^0, t^0) be an arbitrary point in Q. We assume that $t^0 > 0$ (this can be always achieved by shifting the origin). Take a $\delta = \delta(x^0, t^0) > 0$ such that the cylinder $Q_{x^0, t^0, 2\delta} = \{|x - x^0| < 2\delta, \ |t - t^0| < 2\delta\} \Subset Q \cap \{t > 0\}$, and let $\zeta(x, t)$ be an infinitely differentiable function in R_{n+1} which is equal to 1 in $Q_{x^0, t^0, \delta} = \{|x - x^0| < \delta, \ |t - t^0| < \delta\}$ and vanishes outside $Q_{x^0, t^0, 2\delta}$. Then the function $\tilde{u}(x, t)$, which is equal to $u(x, t)\ \zeta(x, t)$ in $Q_{x^0, t^0, 2\delta}$ and to zero outside $Q_{x^0, t^0, 2\delta}$, belongs to $C^{2,1}(0 < t < T) \cap C(0 \leqslant t \leqslant T)$ with $T > t^0 + 2\delta$, is bounded in $\{0 < t < T\}$, coincides with the function $u(x, t)$ in $Q_{x^0, t^0, \delta}$ and $\tilde{u}(x, 0) = 0$; furthermore, the function $\mathscr{L}(\tilde{u}(x, t))$ is bounded in $\{0 < t < T\}$ and $\mathscr{L}(\tilde{u}) = 0$ when $(x, t) \in Q_{x^0, t^0, \delta}$ as well as when $(x, t) \in \{0 < t < T\} \setminus Q_{x^0, t^0, 2\delta}$. By (9), for all points $(x, t) \in Q_{x^0, t^0, \delta}$ we have

$$
\begin{aligned}
u(x, t) &= \int_0^t d\tau \int_{R_n} U(x - \xi, \ t - \tau)\ \mathscr{L}(\tilde{u}(\xi, \tau))\, d\xi \\
&= \int_{t^0 - 2\delta}^{t^0 - \delta} d\tau \int_{|x^0 - \xi| < 2\delta} \frac{g(\xi, \tau)}{(t - \tau)^{n/2}}\, e^{-\frac{|x - \xi|^2}{4(t - \tau)}}\, d\xi \\
&\quad + \int_{t^0 - \delta}^{t} d\tau \int_{\delta < |x^0 - \xi| < 2\delta} \frac{g(\xi, \tau)}{(t - \tau)^{n/2}}\, e^{-\frac{|x - \xi|^2}{4(t - \tau)}}\, d\xi,
\end{aligned}
$$

where $g(\xi, \tau) = \dfrac{1}{(2\sqrt{\pi})^n}\ \mathscr{L}(\tilde{u}(\xi, \tau))$.

From this representation it immediately follows that $u(x, t) \in C^\infty(Q_{x^0, t^0, \delta})$. Thus the first assertion of the theorem (the point (x^0, t^0) is an arbitrary point in Q) is proved.

We shall now show that the function

$$
\begin{aligned}
u(x, t^0) &= \int_{t^0 - 2\delta}^{t^0 - \delta} d\tau \int_{|x^0 - \xi| < 2\delta} \frac{g(\xi, \tau)}{(t^0 - \tau)^{n/2}}\, e^{-\frac{|x - \xi|^2}{4(t^0 - \tau)}}\, d\xi \\
&\quad + \int_{t^0 - \delta}^{t^0} d\tau \int_{\delta < |x^0 - \xi| < 2\delta} \frac{g(\xi, \tau)}{(t^0 - \tau)^{n/2}}\, e^{-\frac{|x - \xi|^2}{4(t^0 - \tau)}}\, d\xi \\
&= \int_D \frac{g(\xi, \tau)}{(t^0 - \tau)^{n/2}}\, e^{-\frac{|x - \xi|^2}{4(t^0 - \tau)}}\, d\xi\, d\tau, \qquad (10)
\end{aligned}
$$

where the region $D = \{|x^0 - \xi| < 2\delta,\ t^0 - 2\delta < \tau < t^0\} \setminus \{|\xi - x^0| \leqslant \delta,\ t^0 - \delta \leqslant \tau \leqslant t^0\}$, is analytic in some neighbourhood of the point x^0. For this, in the $(3n+1)$-dimensional (real) space R_{3n+1} of variables x, y, ξ, τ $(x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n),\ \xi = (\xi_1, \ldots, \xi_n))$ consider the region $D_1 = \{|x_1 - x^0| < \delta/4,\ |y| < \delta/4,\ (\xi, \tau) \in D\}$ and a complex-valued function defined in it:

$$G(x, y, \xi, \tau) = \frac{g(\xi, \tau)}{(t^0 - \tau)^{n/2}} e^{-\frac{1}{4(t^0-\tau)} \sum\limits_{k=1}^{n} (x_k + iy_k - \xi_k)^2}$$

$$= \frac{g(\xi, \tau)}{(t^0 - \tau)^{n/2}} e^{-\frac{|y|^2 - |x-\xi|^2}{4(t^0-\tau)}} e^{-i\frac{(x-\xi,\, y)}{4(t^0-\tau)}}.$$

Note that for $y = 0$ the function G coincides (for $|x - x^0| < \delta/4$, $(\xi, \tau) \in D$) with the integrand function in (10).

The function G and its derivatives G_{x_k} and G_{y_k}, $k = 1, \ldots, n$, evidently belong to $C(\bar{D}_1 \setminus \{\tau = t^0\})$ (here $\{\tau = t^0\} = \{x \in R_n,\ y \in R_n,\ \xi \in R_n,\ \tau = t^0\}$). Let us examine the functions G, G_{x_k}, G_{y_k}, $k = 1, \ldots, n$, in $D_1' = \{|x - x^0| < \delta/4,\ |y| < \delta/4,\ \delta < |\xi - x^0| < 2\delta,\ t^0 - \delta < \tau < t^0\}$, a subregion of D_1. Since in D_1' $|\xi - x| = |\xi - x^0 + x^0 - x| \leqslant |\xi - x^0| + |x^0 - x| \leqslant 9\delta/4$, $|\xi - x| \geqslant |\xi - x^0| - |x^0 - x| \geqslant \frac{3\delta}{4}$ and $|y| < \frac{\delta}{4}$, for all points (x, y, ξ, τ) in D_1' we have

$$|G| \leqslant \frac{g_0}{(t^0 - \tau)^{n/2}} e^{-\frac{\delta^2}{8(t^0-\tau)}},$$

$$|G_{x_k}| = |G_{y_k}| \leqslant g_0 \frac{|x - \xi| + |y|}{2(t^0 - \tau)^{\frac{n}{2}+1}} e^{\frac{|y|^2 - |x-\xi|^2}{4(t^0-\tau)}}$$

$$\leqslant \frac{5 g_0 \delta}{4(t^0 - \tau)^{\frac{n}{2}+1}} e^{-\frac{\delta^2}{8(t^0-\tau)}}, \quad k = 1, \ldots, n,$$

where $g_0 = \max\limits_{(\xi, \tau) \in \bar{D}} |g(\xi, \tau)|$.

Consequently, the functions G, G_{x_k}, G_{y_k}, $k = 1, 2, \ldots, n$, belong to $C(\bar{D}_1')$ $(G(x, y, \xi, t^0) = G_{x_k}(x, y, \xi, t^0) = G_{y_k}(x, y, \xi, t^0) = 0,\ k = 1, \ldots, n)$ and hence to $C(\bar{D}_1)$.

Furthermore, since the function G is analytic in each of the variables $x_1 + iy_1, \ldots, x_n + iy_n$ (for any $\tau < t^0$), for each k, $k = 1, \ldots, n$, it satisfies in D_1 the Cauchy-Riemann equations

$$(\operatorname{Re} G)_{x_k} = (\operatorname{Im} G)_{y_k}, \qquad (\operatorname{Re} G)_{y_k} = -(\operatorname{Im} G)_{x_k}.$$

Hence the complex-valued function

$$F(x, y) = \int\limits_D G(x, y, \xi, \tau)\, d\xi\, d\tau$$

$$= \int\limits_D \frac{g(\xi, \tau)}{(t^0-\tau)^{n/2}} e^{-\frac{1}{4(t^0-\tau)}\sum\limits_{k=1}^{n}(x_k+iy_k-\xi_k)^2} d\xi\, d\tau$$

is continuously differentiable in the region $V = \{|x - x^0| < \delta/4, |y| < \delta/4\}$ of the space R_{2n}, and for all $(x, y) \in V$ and for any k, $k = 1, \ldots, n$,

$$(\operatorname{Re} F)_{x_k} = (\operatorname{Im} F)_{y_k}, \qquad (\operatorname{Re} F)_{y_k} = -(\operatorname{Im} F)_{x_k}.$$

Therefore for any point (x^1, y^1) in V the function $F(x_1^1, \ldots, x_{k-1}^1, x_k, x_{k+1}^1, \ldots, x_n^1, y_1^1, \ldots, y_{k-1}^1, y_k, y_{k+1}^1, \ldots, y_n^1)$ of two (real) variables x_k and y_k is an analytic function of the complex variable $x_k + iy_k$ at the point $x_k^1 + iy_k^1$, $k = 1, \ldots, n$. It is easy to show[*] that then the function $F(x, y)$ is an analytic function of n complex variables $x_1 + iy_1, \ldots, x_n + iy_n$ in V. And since for $|x - x^0| < \delta$ the function $F(x, 0)$ coincides with the function $u(x, t^0)$ in question, the assertion of the theorem is established. ∎

Remark. The function $u(x, t)$ which satisfies the homogeneous heat equation in some region Q of the space R_{n+1} is not necessarily analytic in t.

For example, the function $u(x, t)$ which is equal to $t^{-n/2}e^{-\frac{|x|^2}{4t}}$ for $|x| > 1$, $t > 0$ and to zero for $|x| > 1$, $t \leqslant 0$, satisfies Eq. (1_0) in $\{|x| > 1, -\infty < t < \infty\}$ but is not analytic in t (of course, it belongs to $C^\infty(|x| > 1, -\infty < t < \infty)$).

2. The Cauchy Problem for the Heat Equation. A function $u(x, t)$ belonging to the space $C^{2,1}(0 < t < T) \cap C(0 \leqslant t < T)$ is called the *solution* (*classical solution*) *of the Cauchy problem for Eq.* (1) if it satisfies Eq. (1) in $\{0 < t < T\}$ and for $t = 0$ satisfies the initial condition

$$u|_{t=0} = \varphi(x), \tag{11}$$

where $f(x, t)$ and $\varphi(x)$ are given functions.

First we shall prove the following uniqueness theorem.

Theorem 2. *The Cauchy problem* (1), (11) *cannot have more than one classical solution bounded in* $\{0 < t < T\}$.

Proof. Let $u_1(x, t)$ and $u_2(x, t)$ be two classical solutions of the problem (1), (11) that are bounded in $\{0 < t < T\}$. Then the function $u = u_1 - u_2$ is a solution of the homogeneous heat equation (1_0)

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[*] See, for instance, Vladimirov, V. S. *Methods of Theory of Several Complex Variables*, Nauka, Moscow, 1964, p. 42 (in Russian) or Shabat, B. V. *Introduction to Complex Analysis*, Nauka, Moscow, 1969, p. 273 (in Russian).

which is bounded in $\{0 < t < T\}$ and satisfies the homogeneous initial condition

$$u|_{t=0} = 0. \qquad (11_0)$$

Consequently, for the function u the representation (9) of the preceding subsection holds in the strip $\{0 < t < T\}$ which immediately implies that $u \equiv 0$ in $\{0 < t < T\}$. ▮

By $M_\sigma = M_\sigma(T)$, $\sigma \geqslant 0$, we denote the set of all functions $u(x, t)$ defined in $\{0 \leqslant t < T\}$ for each of which there are positive constants A and a (depending on this function) such that

$$|u(x, t)| \leqslant Ae^{a|x|^\sigma} \text{ for all } (x, t) \in \{0 \leqslant t < T\}.$$

It is clear that the set M_σ is a linear space for any $\sigma \geqslant 0$; moreover, $M_\sigma \subset M_{\sigma'}$ for $\sigma \leqslant \sigma'$; M_0 is the set of all the functions bounded in $\{0 \leqslant t < T\}$, and M_2 is the set of all functions for each of which there are positive constants A and a such that

$$|u(x, t)| \leqslant Ae^{a|x|^2} \text{ for all } (x, t) \in \{0 \leqslant t < T\}. \qquad (12)$$

In Theorem 2 the uniqueness of the solution of the Cauchy problem (1), (11) was established in the set of bounded functions M_0. In fact, the solution is unique in M_2 too, and hence in any M_σ, $0 \leqslant \sigma \leqslant 2$. Namely, the following assertion, generalizing Theorem 2, holds.

Theorem 2′. *The Cauchy problem* (1), (11) *cannot have more than one solution belonging to* M_2*.

The following auxiliary proposition is required for the proof of Theorem 2.

Lemma 1. *Let the function* $u(x, t)$ *be a solution of the problem* (1_0), (11_0) *in the strip* $\{0 < t < T\}$, $T > 0$, *and satisfy the inequality* (12) *with some constants* $a > 0$ *and* $A > 0$. *Then* $u = 0$ *in the strip* $\{0 < t < T_1\}$, *where* $T_1 = \min\{T, 1/5a\}$.

Proof. Take an arbitrary $\varepsilon > 0$ and in $\{0 < t < T_1\}$ consider two functions

$$w_\pm(x, t) = \pm u(x, t) + \varepsilon \left(t + \frac{1}{(T_1 - t)^{n/2}} e^{\frac{|x|^2}{4(T_1 - t)}} \right).$$

Evidently, these functions belong to $C^{2,1}(0 < t < T_1) \cap C(0 \leqslant \leqslant t < T_1)$. Since $u|_{t=0} = 0$, for all $x \in R_n$

$$w_\pm(x, 0) = \varepsilon T_1^{-n/2} e^{\frac{|x|^2}{4T_1}} > 0. \qquad (13)$$

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* It can be shown that for any $\sigma > 2$ the solution of problem (1), (11) is not unique in M_σ (see Tikhonov, A. N. "Théoremès d'unicité pour l'équation de la chaleur", *Mat. Sb.* **42** (1935), 199-216).

Since $\mathscr{L}u = 0$ in $\{0 < t < T_1\}$, for all points $(x, t) \in \{0 < t < T_1\}$

$$\mathscr{L}(w_\pm) = \pm\mathscr{L}(u) + \varepsilon\mathscr{L}\left(t + (T_1 - t)^{-n/2} e^{\frac{|x|^2}{4(T_1-t)}}\right) = \varepsilon > 0. \quad (14)$$

Let (x^0, t^0) be any point in the strip $\{0 < t < T_1\}$. Take a large number $R > 0$ such that the point (x^0, t^0) lies inside the cylinder $\{|x| < R,\ 0 < t < T_1\}$ and the functions $w_\pm(x, t)$ are positive on the lateral surface $\{|x| = R,\ 0 < t < T_1\}$ of this cylinder:

$$w_\pm(x, t)\big|_{|x|=R} > 0, \qquad 0 < t < T_1 \quad (15)$$

(the latter property can be always fulfilled because $w_\pm\big|_{|x|=R} = \pm u\big|_{|x|=R} + \varepsilon\left(t + (T_1 - t)^{-n/2} e^{\frac{R^2}{4(T_1-t)}}\right) \geqslant -Ae^{aR^2} + \varepsilon T_1^{-n/2} e^{\frac{R^2}{4T_1}} \geqslant -Ae^{aR^2} + \varepsilon(5a)^{\frac{n}{2}} e^{\frac{5aR^2}{4}} \to +\infty$ as $R \to \infty$).

We shall now prove that if a function $w(x, t)$ belongs to $C^{2,1}(\{|x| < R,\ 0 < t < T_1\}) \cap C(\{|x| \leqslant R,\ 0 \leqslant t < T_1\})$ and satisfies the conditions

$$w(x, 0) \geqslant 0 \quad \text{for} \quad |x| \leqslant R, \quad (13')$$

$$\mathscr{L}w(x, t) > 0 \quad \text{in} \quad \{|x| < R,\ 0 < t < T_1\} \quad (14')$$

and

$$w\big|_{|x|=R} \geqslant 0 \quad \text{for} \quad 0 \leqslant t < T_1, \quad (15')$$

then

$$w(x, t) \geqslant 0 \quad \text{for all } (x, t) \in \{|x| < R,\ 0 < t < T_1\}. \quad (16')$$

Suppose, on the contrary, that there is a point (x', t') in $\{|x| < R,\ 0 < t < T_1\}$ such that $w(x', t') < 0$. Let (x'', t'') be the point in $\{|x| \leqslant R,\ 0 \leqslant t \leqslant t'\}$ where the function $w(x, t)$ ($w(x, t) \in C(\{|x| \leqslant R,\ 0 \leqslant t \leqslant \iota'\})$) attains its minimum, that is,

$$w(x'', t'') = \min_{\{|x| \leqslant R,\ 0 \leqslant t \leqslant t'\}} w(x, t) \leqslant w(x', t') < 0.$$

By (13′) and (15′), $(x'', t'') \in \{|x| < R,\ 0 < t \leqslant t'\}$. If $(x'', t'') \in \{|x| < R,\ 0 < t < t'\}$, then $\frac{\partial w(x'', t'')}{\partial t} = 0$ and $\frac{\partial^2 w(x'', t'')}{\partial x_i^2} \geqslant 0$, $i = 1, \ldots, n$, whence $\mathscr{L}w(x'', t'') \leqslant 0$ which contradicts (14′). If $(x'', t'') \in \{|x| < R,\ t = t'\}$, then $\frac{\partial w(x'', t'')}{\partial t} \leqslant 0$ and $\frac{\partial^2 w(x'', t'')}{\partial x_i^2} \geqslant 0$, $i = 1, \ldots, n$, whence $\mathscr{L}w(x'', t'') \leqslant 0$, again contradicting (14′). Thus inequality (16′) is proved.

Since, by (13)-(15), the functions $w_\pm(x, t)$ satisfy the conditions (13′)-(15′), for all $(x, t) \in \{|x| < R,\ 0 < t < T_1\}$ the inequalities $w_\pm(x, t) \geqslant 0$ hold thereby implying $w_\pm(x^0, t^0) \geqslant 0$, that is,

$$|u(x^0, t^0)| \leqslant \varepsilon \left(t^0 + \frac{1}{(T_1 - t^0)^{n/2}} e^{\frac{|x^0|^2}{4(T_1 - t^0)}} \right).$$

Since $\varepsilon > 0$ and the point (x^0, t^0) are arbitrary, it follows from this last inequality that $u(x, t) \equiv 0$ in $\{0 < t < T_1\}$. ▮

Proof of Theorem 2. Let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of the problem (1), (11) in $\{0 < t < T\}$ that belong to M_2. Then their difference $u = u_1 - u_2$ is a solution of the problem (1_0) and (11_0) in $\{0 < t < T\}$ and for all $(x, t) \in \{0 \leqslant t < T\}$ satisfies the inequality (12) with certain constants $A > 0$ and $a > 0$. By Lemma 1, $u(x, t) = 0$ in $\{0 < t < T_1\}$, where $T_1 = \min\left(T, \frac{1}{5a}\right)$. If $T_1 = T$, the theorem is proved.

Let $T_1 = \frac{1}{5a} < T$. Then the continuity of the function $u(x, t)$ in $\{0 < t < T\}$ implies $u\Big|_{t=\frac{1}{5a}} = 0$. Therefore the function $v(x, t) = u\left(x, t + \frac{1}{5a}\right)$ is a solution of the problem (1_0), (11_0) in the strip $\left\{0 < t < T - \frac{1}{5a}\right\}$ and in this strip satisfies inequality (12). According to Lemma 1, $v(x, t) = 0$ in $\{0 < t < T_2\}$, where $T_2 = \min\left\{T - \frac{1}{5a}, \frac{1}{5a}\right\}$. From this it follows that $u(x, t) = 0$ in $\left\{0 < t < T_2 + \frac{1}{5a}\right\}$. If $T_2 + \frac{1}{5a} < T$ $\left(\text{then } T_2 = \frac{1}{5a}\right)$, then repeating this argument we find that $u(x, t) = 0$ in $\left\{0 < t < 2 \cdot \frac{1}{5a} + T_3\right\}$, where $T_3 = \min\left(T - \frac{2}{5a}, \frac{1}{5a}\right)$, and so on. After a finite number of steps, we see that $u = 0$ in $\{0 < t < T\}$. ▮

We shall now prove the existence theorem regarding the solution of the Cauchy problem (1), (11). The results of the foregoing subsection imply that if the solution, bounded in $\{0 < t < T\}$, of the problem (1), (11) exists with function $f(x, t)$ bounded in $\{0 < t < T\}$, then it has the form

$$u(x, t) = \int_{R_n} U(x - \xi, t)\, \varphi(\xi)\, d\xi + \int_0^t \int_{R_n} U(x - \xi, t - \tau)\, f(\xi, \tau)\, d\xi\, d\tau. \qquad (17)$$

Therefore the proof of the existence of a solution naturally reduces to determining conditions on the functions φ and f such that the function $u(x, t)$ given by formula (17) is a classical solution of the problem (1), (11).

Let $B(R_n)$ and $B(0<t<T)$ denote Banach spaces of functions that are continuous and bounded in R_n or in the strip $\{0<t<T\}$, respectively, with norm $\|\varphi\|_{B(R_n)}=\sup\limits_{x\in R_n}|\varphi(x)|$ and $\|f\|_{B(0<t<T)}=$ $=\sup\limits_{(x,\, t)\in\{0<t<T\}}|f(x, t)|$.

Theorem 3. *If* $\varphi(x)$ *belongs to* $B(R_n)$ *and the functions* $f(x, t)$ *and* $f_{x_i}(x, t)$, $i=1, \dots, n$, *belong to* $B(0<t<T)$, *then the problem* (1), (11) *has a classical solution* $u(x, t)$. *This solution belongs to* $B(0<t<T)$ *and is given by formula* (17); *further,*

$$\|u\|_{B(0<t<T)}\leqslant\|\varphi\|_{B(R_n)}+T\|f\|_{B(0<t<T)}. \tag{18}$$

Theorem 3 is an immediate consequence of the following two auxiliary assertions.

Lemma 2. *If* $\varphi\in B(R_n)$, *then the function*

$$u_1(x, t)=\int\limits_{R_n} U(x-\xi, t)\,\varphi(\xi)\,d\xi \tag{19}$$

is a classical solution of the Cauchy problem (1_0), (11) *in the half-space* $\{t>0\}$; *moreover, for all* $x\in R_n$, $t>0$ *the inequalities*

$$\inf_{x\in R_n}\varphi(x)\leqslant u_1(x, t)\leqslant\sup_{x\in R_n}\varphi(x) \tag{20}$$

hold.

Lemma 3. *If* f *and* f_{x_i}, $i=1, \dots, n$, *belong to* $B(0<t<T)$, *then the function*

$$u_2(x, t)=\int\limits_0^t d\tau\int\limits_{R_n} U(x-\xi, t-\tau)\,f(\xi, \tau)\,d\xi, \qquad (x, t)\in\{0<t<T\}, \tag{21}$$

belongs to $B(0<t<T)$ *and is a classical solution of the Cauchy problem* (1), (11_0) *in the strip* $\{0<t<T\}$; *and*

$$\|u_2\|_{B(0<t<T)}\leqslant T\|f\|_{B(0<t<T)}. \tag{22}$$

Proof of Lemma 2. For the proof of Lemma 2, it suffices to establish that the function $u_1(x, t)$ has the following properties:

(a) $u_1\in C^{2,\,1}(t>0)$ and $\mathcal{L}u_1=0$ in $\{t>0\}$,

(b) for the function u_1 the inequalities (20) hold,

(c) the function u_1 belongs to the space $C(t\geqslant 0)$ and satisfies the initial condition (11).

Take arbitrary numbers δ and T_1, $0<\delta<T_1$. Since for all $x\in R_n$, $\xi\in R_n$, $t\in[\delta, T_1]$ and for any $\alpha=(\alpha_1, \dots, \alpha_n)$, $\alpha_i\geqslant 0$,

$i = 1, \dots, n$, and $\beta \geqslant 0$

$$\left|\frac{\partial^\beta}{\partial t^\beta} D_x^\alpha U(x-\xi, t)\right| \leqslant C_{\alpha\beta}(1+|\xi-x|)^{|\alpha|+2\beta} e^{-\frac{|x-\xi|^2}{4T_1}},$$

where $C_{\alpha\beta} = C_{\alpha\beta}(\delta)$ are some positive constants, the function $u_1(x, t) \in C^\infty$ $(\delta < t < T_1)$, and

$$\frac{\partial^\beta}{\partial t^\beta} D_x^\alpha u_1(x, t) = \int\limits_{R_n} \frac{\partial^\beta}{\partial t^\beta} D_x^\alpha U(x-\xi, t)\cdot\varphi(\xi)\, d\xi.$$

In the strip $\{\delta < t < T_1\}$ the function $U(x-\xi, t)$ satisfies Eq. (1_0) (with respect to (x, t)), therefore the function $u_1(x, t)$ satisfies Eq. (1_0) in this strip. Consequently, since the numbers $\delta > 0$ and $T_1 > \delta$ are arbitrary, the function $u_1(x, t)$ has Property (a).

Since the function U is nonnegative, according to (2) for all $x \in R_n$ and $t > 0$ we have the inequalities

$$u_1(x, t) \leqslant \int\limits_{R_n} U(x-\xi, t)\Big(\sup_{\xi\in R_n} \varphi(\xi)\Big)\, d\xi = \sup_{x\in R_n} \varphi(x),$$

$$u_1(x, t) \geqslant \int\limits_{R_n} U(x-\xi, t)\Big(\inf_{\xi\in R_n} \varphi(\xi)\Big)\, d\xi = \inf_{x\in R_n} \varphi(x).$$

This establishes Property (b).

We now establish Property (c). Taking an arbitrary point $x^0 \in R_n$, we shall show that $\lim\limits_{\substack{(x,\, t)\to(x^0,\, 0)\\ (x,\, t)\in\{t>0\}}} u_1(x, t) = \varphi(x^0)$. By (2), we have for any $(x, t) \in \{t > 0\}$ and for any $\delta > 0$

$$\begin{aligned} u_1(x, t) - \varphi(x^0) &= \int\limits_{R_n} U(x-\xi, t)\,(\varphi(\xi) - \varphi(x^0))\, d\xi \\ &= \int\limits_{|\xi - x^0|\leqslant\delta} U(x-\xi, t)\,(\varphi(\xi) - \varphi(x^0))\, d\xi \\ &\quad + \int\limits_{|\xi - x^0|>\delta} U(x-\xi, t)(\varphi(\xi) - \varphi(x^0))\, d\xi = I_{1,\,\delta} + I_{2,\,\delta}. \end{aligned} \tag{23}$$

Since the function φ is continuous at the point x^0, for an $\varepsilon > 0$ we can find a $\delta > 0$ (this δ we take in (23)) such that $|\varphi(\xi) - \varphi(x^0)| < \varepsilon/2$ whenever $|\xi - x^0| < \delta$. Therefore

$$|I_{1,\,\delta}| \leqslant \frac{\varepsilon}{2} \int\limits_{|x^0-\xi|\leqslant\delta} U(x-\xi, t)\, d\xi \leqslant \frac{\varepsilon}{2} \int\limits_{R_n} U(x-\xi, t)\, d\xi = \varepsilon/2. \tag{24}$$

Let $|x-x^0|<\delta/2$, then for $|\xi-x^0|>\delta$ we have $|x-\xi|= =|x-x^0+x^0-\xi|\geqslant|x^0-\xi|-|x-x^0|>\delta-\frac{\delta}{2}=\delta/2$. Thus

$$|I_{2,\,\delta}|\leqslant\int\limits_{|x^0-\xi|>\delta}\frac{e^{-\frac{|x-\xi|^2}{8t}}e^{-\frac{|x-\xi|^2}{8t}}}{(2\sqrt{\pi t})^n}(|\varphi(\xi)|+|\varphi(x^0)|)\,d\xi$$

$$\leqslant\frac{2\,\|\varphi\|_{B(R_n)}}{(2\sqrt{\pi t})^n}e^{-\frac{\delta^2}{32t}}\int\limits_{|x^0-\xi|>\delta}e^{-\frac{|x-\xi|^2}{8t}}\,d\xi$$

$$\leqslant\frac{2\,\|\varphi\|_{B(R_n)}}{(2\sqrt{\pi t})^n}e^{-\frac{\delta^2}{32t}}\int\limits_{R_n}e^{-\frac{|x-\xi|^2}{8t}}\,d\xi=\text{const}\,e^{-\frac{\delta^2}{32t}}\leqslant\varepsilon/2 \qquad (25)$$

whenever $t\in(0,\,\delta_0)$ with sufficiently small δ_0. Thus from (23)-(25) we see that $|u_1(x,\,t)-\varphi(x^0)|<\varepsilon$ for all points $(x,\,t)$ of the half-space $\{t>0\}$ such that $|x-x^0|^2+t^2<\min(\delta_0^2,\,\delta^2/4)$. ∎

Proof of Lemma 3. We represent the function $u_2(x,\,t)$ (see (21)) in the form

$$u_2(x,\,t)=\frac{1}{\pi^{n/2}}\int\limits_0^t d\tau\int\limits_{R_n}e^{-|\xi|^2}f(x+2\xi\sqrt{t-\tau},\,\tau)\,d\xi, \qquad (26)$$

$$(x,\,t)\in\{0<t<T\}.$$

To prove the lemma it is enough to check that

(a) $u_2(x,\,t)\in C\,(0\leqslant t<T)$, $u_2(x,\,0)=0$,

(b) inequality (22) holds,

(c) $u_2(x,\,t)\in C^{2,\,1}\,(0<t<T)$ and $\mathcal{L}u_2=f$ in $\{0<t<T\}$.

Since the function $f\,(x,\,t)$ is bounded in $\{0<t<T\}$, it follows that $|e^{-|\xi|^2}f(x+2\xi\sqrt{t-\tau},\,\tau)|\leqslant e^{-|\xi|^2}\|f\|_{B(0<t<T)}$ and therefore (f is continuous) the function

$$g(x,\,t,\,\tau)=\frac{1}{\pi^{n/2}}\int\limits_{R_n}e^{-|\xi|^2}f(x+2\xi\sqrt{t-\tau},\,\tau)\,d\xi$$

is continuous and bounded on the set $\{x\in R_n,\ 0<t<T,\ 0<\tau\leqslant t\}$ and $g(x,\,t,\,t)=f(x,\,t)$, $|g\,(x,\,t,\,\tau)|\leqslant\|f\|_{B(0<t<T)}$. Therefore the function $u_2(x,\,t)=\int\limits_0^t g(x,\,t,\,\tau)\,d\tau$ belongs to $C(0\leqslant t<T)$, $u_2|_{t=0}=0$ and $\|u_2\|_{B(0<t<T)}\leqslant T\,\|f\|_{B(0<t<T)}$. Properties (a) and (b) are proved.

Since the function $f(x,\,t)$ has continuous derivatives $f_{x_i}(x,\,t)$, $i=1,\,\ldots,\,n$, in $\{0<t<T\}$ and $|f|+|\nabla f|\leqslant\text{const}$ in $\{0<t<T\}$, the function $g(x,\,t,\,\tau)$ has continuous derivatives $g_{x_i}(x,\,t,\,\tau)$, $i=1,\,\ldots,\,n$, on the set $\{x\in R_n,\ 0<t<T,\ 0<$

$< \tau \leqslant t\}$. Consequently, the function $u_2(x, t)$ has continuous derivatives $u_{2x_i}(x, t)$, $i = 1, \ldots, n$, on $\{0 \leqslant t < T\}$, and

$$u_{2x_i}(x, t) = \frac{1}{\pi^{n/2}} \int_0^t d\tau \int_{R_n} e^{-|\xi|^2} f_{x_i}(x + 2\xi\sqrt{t-\tau}, \tau)\, d\xi$$

$$= \frac{1}{2\pi^{n/2}} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \int_{R_n} e^{-|\xi|^2} f_{\xi_i}(x + 2\xi\sqrt{t-\tau}, \tau)\, d\xi$$

$$= \frac{1}{\pi^{n/2}} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \int_{R_n} e^{-|\xi|^2} \xi_i f(x + 2\xi\sqrt{t-\tau}, \tau)\, d\tau, \quad i = 1, \ldots, n. \tag{27}$$

Since for all $j = 1, \ldots, n$

$$|e^{-|\xi|^2} \xi_j f_{x_j}(x + 2\xi\sqrt{t-\tau}, \tau)| \leqslant |\xi|\, e^{-|\xi|^2} \|f_{x_j}\|_{B(0<t<T)},$$

the functions $\int_{Rn} e^{-|\xi|^2} \xi_j f(x + 2\xi\sqrt{t-\tau}, \tau)\, d\xi$ have all first derivatives with respect to $x_1, \ldots, x_n$ which are continuous and bounded on the set $\{x \in R_n,\ 0 < t < T,\ 0 < \tau \leqslant t\}$. Then from (27) it follows that the function $u_2(x, t)$ has all derivatives up to second order with respect to $x_1, \ldots, x_n$ that are continuous in $\{0 \leqslant t < T\}$. Moreover,

$$\Delta u_2(x, t) = \frac{1}{\pi^{n/2}} \int_0^t \frac{d\tau}{\sqrt{t-\tau}} \int_{R_n} e^{-|\xi|^2} \sum_{i=1}^{n} \xi_i f_{x_i}(x + 2\xi\sqrt{t-\tau}, \tau)\, d\xi. \tag{28}$$

Further, for arbitrary points (x, t) and $(x, t + \Delta t)$, $\Delta t > 0$, lying in $\{0 < t < T\}$

$$\frac{u_2(x, t+\Delta t) - u_2(x, t)}{\Delta t} = \frac{1}{\Delta t} \int_t^{t+\Delta t} g(x, t+\Delta t, \tau)\, d\tau$$

$$+ \int_0^t \frac{g(x, t+\Delta t, \tau) - g(x, t, \tau)}{\Delta t}\, d\tau = I_1(\Delta t) + I_2(\Delta t). \tag{29}$$

In view of continuity of the function $g(x, t + \Delta t, \tau)$ in τ on the segment $[t, t + \Delta t]$, $I_1(\Delta t) = g(x, t + \Delta t, t + \theta\Delta t)$, where $\theta = \theta(x, t, \Delta t)$, $0 \leqslant \theta \leqslant 1$. Consequently,

$$\lim_{\Delta t \to +0} I_1(\Delta t) = g(x, t, t) = f(x, t). \tag{30}$$

Since in $\{0 < t < T\}$ the function f has continuous and bounded derivatives with respect to $x_1, \ldots, x_n$, the function $g(x, t, \tau)$ has a derivative with respect to t continuous in $\{x \in R_n,\ 0 < t < T,\ 0 < \tau < t\}$:

$$g_t(x, t, \tau) = \frac{1}{\pi^{n/2}\sqrt{t-\tau}} \int_{R_n} e^{-|\xi|^2} \sum_{i=1}^{n} \xi_i f_{x_i}(x + 2\xi\sqrt{t-\tau}, \tau)\, d\xi,$$

with $|g_t(x, t, \tau)| \leqslant \mathrm{const}/\sqrt{t-\tau}$. Then

$$\left|\frac{g(x, t+\Delta t, \tau) - g(x, t, \tau)}{\Delta t}\right| \leqslant \frac{1}{\Delta t}\int_t^{t+\Delta t} |g_{t'}(x, t', \tau)|\, dt'$$

$$\leqslant \frac{\mathrm{const}}{\Delta t}\int_t^{t+\Delta t} \frac{dt'}{\sqrt{t'-\tau}} \leqslant \mathrm{const}/\sqrt{t-\tau}.$$

Therefore (according to Lebesgue's theorem)

$$\lim_{\Delta t \to +0} I_2(\Delta t) = \frac{1}{\pi^{n/2}}\int_0^t \frac{d\tau}{\sqrt{t-\tau}} \int_{R_n} e^{-|\xi|^2} \sum_{i=1}^{n} \xi_i f_{x_i}(x + 2\xi\sqrt{t-\tau}, \tau)\, d\xi. \qquad (31)$$

From (28)-(31) it follows that

$$\lim_{\Delta t \to +0} \frac{u_2(x, t+\Delta t) - u_2(x, t)}{\Delta t} = f(x, t) + \Delta u_2(x, t). \qquad (32)$$

It is similarly proved that

$$\lim_{\Delta t \to -0} \frac{u_2(x, t+\Delta t) - u_2(x, t)}{\Delta t} = -\lim_{\Delta t \to -0} \frac{1}{\Delta t}\int_{t+\Delta t}^{t} g(x, t, \tau)\, d\tau$$

$$+ \lim_{\Delta t \to -0} \int_0^{t+\Delta t} \frac{g(x, t+\Delta t, \tau) - g(x, t, \tau)}{\Delta t}\, d\tau = f(x, t) + \Delta u_2(x, t). \qquad (32')$$

Therefore the function $u_2(x, t)$ has continuous derivative, equal to $f + \Delta u_2$, with respect to t in $\{0 < t < T\}$.

Theorem 3 establishes the existence of a classical solution of the Cauchy problem (1), (11) for any bounded $\varphi \in C(R_n)$ and any bounded $f \in C(0 < t < T)$ for which all first-order derivatives with respect to space variables are continuous and bounded in $\{0 < t < T\}$. Now the question arises: for the Cauchy problem (1), (11) to be solvable is it not sufficient to assume that the function f is only continuous and bounded? In fact, the condition that f has (bounded) derivatives with respect to space variables is more than what is necessary: it can be proved that for the solvability of the problem (1), (11) it suffices to assume that the function $f(x, t)$ (continuous and bounded) satisfies Hölder's condition with respect to space variables, that is, for any point (x, t) in $\{0 < t < T\}$ there exist constants $M > 0$, $\alpha > 0$ (depending on this point) such that $|f(x', t) - f(x, t)| \leqslant M |x' - x|^{\alpha}$ for all $x' \in R_n$. However, if the function f is only continuous (and bounded) in $\{0 < t < T\}$, then, as illustrated by the following example, the problem (1), (11) may not have a (classical) solution.

Let $\zeta = \zeta(|x|)$ be an arbitrary function infinitely differentiable in R_n, which is equal to 1 for $|x| < 1/2$ and to zero for $|x| > \frac{3}{4}$. We examine the following Cauchy problem

$$\mathscr{L}u = u_t - \Delta u = f_0(x), \tag{33}$$

$$u|_{t=0} = \varphi_0(x), \tag{34}$$

where

$$\begin{aligned} f_0(x) = & -\frac{x_1^2 - x_2^2}{2|x|^2}\zeta(|x|)\left((n+2)(-\ln|x|)^{-1/2} + \frac{1}{2}(-\ln|x|)^{-3/2}\right) \\ & + \frac{x_1^2 - x_2^2}{|x|}\zeta'(|x|)\left((n+3)(-\ln|x|)^{1/2} - (-\ln|x|)^{-1/2}\right) \\ & = (x_1^2 - x_2^2)\zeta''(|x|)(-\ln|x|)^{1/2}, \end{aligned}$$

and

$$\varphi_0(x) = (x_1^2 - x_2^2)\zeta(|x|)(-\ln|x|)^{1/2}.$$

The function $f_0(x) \in C(R_n) \cap C^{\infty}(|x| > 0)$ is equal to zero for $|x| > 3/4$, and therefore is bounded in R_n. The initial function $\varphi_0(x) \in C^1(R_n) \cap C^{\infty}(|x| > 0)$ is equal to zero when $|x| > 3/4$, and is therefore bounded in R_n. It can be directly verified (compare with the similar example for Poisson's equation, Chap. IV, Sec. 3.3) that the bounded function $u(x, t) \equiv \varphi_0(x)$ (it does not depend on t) satisfies Eq. (33) when $|x| > 0$. Moreover, the function $u(x, t)$ obviously satisfies the initial condition (34).

Nevertheless, the function $u(x, t) \equiv \varphi_0(x)$ does not belong to $C^{2,1}(0 < t < T)$ for any $T > 0$, because, for instance, $\lim\limits_{|x| \to 0} u_{x_1 x_1}(x, t) = \infty$. Consequently, this function is not a solution of the problem (33), (34).

Let us show that the problem (33), (34) has no solution in any strip $\{0 < t < T\}$. Suppose, on the contrary, that there exists a solution $v(x, t)$ of the problem (33), (34) in the strip $\{0 < t < T\}$ for a certain $T > 0$. Then the function $w(x, t) = u(x, t) -$

$-v(x, t) = \varphi_0(x) - v(x, t) \in C^{2,1}(\{|x| > 0, 0 < t < T\})$ and satisfies the homogeneous heat equation (1_0) on the set $\{|x| > 0, 0 < t < T\}$. Moreover, $w(x, t) \in C^1(T/2 \leqslant t < T)$, since $\varphi_0 \in C^1(R_n)$. Thus $w(x, t) \in C^{2,1}(\{0 < |x| \leqslant 1, T/2 \leqslant t < T\}) \cap C^1(\{|x| \leqslant 1, T/2 \leqslant t < T\})$ and $w_t - \Delta w = 0$ for all points (x, t) in $\{0 < |x| < 1, T/2 < t < T\}$.

We shall show that the function $w(x, t)$ must belong to $C^{2,1}(\{|x| < 1, T/2 < t < T\})$ which is impossible, because the function $v \in C^{2,1}(\{|x| < 1, T/2 < t < T\})$ and the function $u(x, t) = \varphi_0(x) \notin C^{2,1}(\{|x| < 1, T/2 < t < T\})$.

So, let $w(x, t) \in C^{2,1}(\{0 < |x| \leqslant 1, T/2 \leqslant t < T\}) \cap C^1(\{|x| \leqslant 1, T/2 \leqslant t < T\})$ and $\mathcal{L}w = 0$ in $\{0 < |x| < 1, T/2 < t < T\}$. We shall show that $w(x, t) \in C^{2,1}(\{|x| < 1, T/2 < t < T\})$. The proof of this fact repeats in a certain sense the arguments used in Subsec. 1 while establishing representation (9).

Take an arbitrary point (ξ, τ) in $\{0 < |x| < 1, T/2 < t < T\}$ and an arbitrary $\varepsilon \in (0, \tau - T/2)$. On the set $\{0 < |x| < 1, T/2 < t < \tau\}$ we have the equality

$$(w(x, t)\, U(\xi - x, \tau - t))_t + \sum_{i=1}^{n} (wU_{x_i} - w_{x_i}U)_{x_i}$$

$$= U(\xi - x, \tau - t)\, \mathcal{L}w(x, t) - w(x, t)\, \mathcal{L}^*_{x,\, t} U(\xi - x, \tau - t) = 0.$$

Integrating this equality over $\{\delta < |x| < 1, T/2 < t < \tau - \varepsilon\}$, where δ is an arbitrary number from the interval $(0, |\xi|)$, and applying Ostrogradskii's formula, we obtain

$$\int\limits_{\delta < |x| < 1} w(x, \tau - \varepsilon)\, U(\xi - x, \varepsilon)\, dx$$

$$= \int\limits_{\delta < |x| < 1} w(x, T/2)\, U(\xi - x, \tau - T/2)\, dx$$

$$- \int\limits_{T/2}^{\tau - \varepsilon} dt \int\limits_{|x| = 1} \left(w(x, t)\, \frac{\partial U(\xi - x, \tau - t)}{\partial n_x} \right.$$

$$\left. - \frac{\partial w(x, t)}{\partial n}\, U(\xi - x, \tau - t) \right) dS_x - \int\limits_{T/2}^{\tau - \varepsilon} dt \int\limits_{|x| = \delta} \left(w(x, t) \right.$$

$$\left. \times \frac{\partial U(\xi - x, \tau - t)}{\partial n_x} - \frac{\partial w(x, t)}{\partial n}\, U(\xi - x, \tau - t) \right) dS_x$$

$$= I_{1,\, \delta} + I_{2,\, \varepsilon} + I_{3,\, \varepsilon,\, \delta}. \qquad (35)$$

In (35) we pass to the limit first as $\varepsilon \to 0$ and then as $\delta \to 0$. We take an arbitrary δ_0, $0 < \delta_0 < \min (1 - |\xi|, |\xi| - \delta)$. Then

$$\int\limits_{\delta<|x|<1} w(x, \tau-\varepsilon(U(\xi-x, \varepsilon)\, dx$$

$$= \int\limits_{|x-\xi|<\delta_0} w(x, \tau-\varepsilon) U(\xi-x, \varepsilon)\, dx$$

$$+ \int\limits_{\{\delta<|x|<1\}\cap\{|x-\xi|\geqslant\delta_0\}} w(x, \tau-\varepsilon)\, U(\xi-x, \varepsilon)\, dx.$$

Since on the set $\{\delta<|x|<1\} \cap \{\delta_0\leqslant |x-\xi|\}$

$$|w(x, \tau-\varepsilon)\, U(\xi-x, \varepsilon)| \leqslant \max |w(x, t)| \exp\left(-\frac{\delta_0^2}{4\varepsilon}\right) \Big/ (2\sqrt{\pi\varepsilon})^n,$$

it follows that as $\varepsilon \to 0$ $\int\limits_{\{\delta<|x|<1\}\cap\{|x-\xi|\geqslant\delta_0\}} w(x, \tau-\varepsilon)\, U(\xi-x, \varepsilon) \times$ $\times\, dx \to 0$. Therefore

$$\lim_{\varepsilon\to 0} \int\limits_{\delta<|x|<1} w(x, \tau-\varepsilon)\, U(\xi-x, \varepsilon)\, dx$$

$$= \lim_{\varepsilon\to 0} \int\limits_{|x-\xi|<\delta_0} w(x, \tau-\varepsilon)\, U(\xi-x, \varepsilon)\, dx$$

$$= \lim_{\varepsilon\to 0} \frac{1}{\pi^{n/2}} \int\limits_{|\eta|<\frac{\delta_0}{2\sqrt{\varepsilon}}} w(\xi+2\eta\sqrt{\varepsilon}, \tau-\varepsilon)\, e^{-|\eta|^2}\, d\eta = w(\xi, \tau),$$

and consequently

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} \int\limits_{\delta<|x|<1} w(x, \tau-\varepsilon)\, U(\xi-x, \varepsilon)\, dx = w(\xi, \tau). \tag{36}$$

Further, it is obvious that

$$\lim_{\delta\to 0} I_{1,\,\delta} = \int\limits_{|x|<1} w(x, T/2)\, U(\xi-x, \tau-T/2)\, dx, \tag{37}$$

$$\lim_{\varepsilon\to 0} I_{2,\,\varepsilon} = \int\limits_{T/2}^{\tau} dt \int\limits_{|x|<1} \left(w(x, t)\, \frac{\partial U(\xi-x, \tau-t)}{\partial n_x} - \frac{\partial w(x, t)}{\partial n} U(\xi-x, \tau-t) \right) dS_x, \tag{38}$$

and since $w \in C^1(\{|x|\leqslant 1, T/2\leqslant t\leqslant\tau\})$, it follows that

$$\lim_{\delta\to 0} \lim_{\varepsilon\to 0} I_{3,\,\varepsilon,\,\delta} = 0. \tag{39}$$

The relations (35)-(39) imply that for any point (x, t) in $\{0<|x|<1,\ T/2<t<T\}$ we have the representation

$$w(x, t) = \int\limits_{|\xi|<1} w(\xi, T/2)\, U(x-\xi, t-T/2)\, d\xi$$

$$-\int\limits_{T/2}^{t} d\tau \int\limits_{|\xi|=1} \left(w(\xi, \tau)\, \frac{\partial U(x-\xi, t-\tau)}{\partial n_\xi} - \frac{\partial w(\xi, \tau)}{\partial n_\xi}\, U(x-\xi, t-\tau)\right) dS_\xi,$$

from which it immediately follows that w belongs to $C^\infty(\{|x|<1,\ T/2<t<T\})$ and, more so, to $C^{2,1}(\{|x|<1,\ T/2<t<T)\}$. This proves the assertion.

§ 2. MIXED PROBLEMS

1. Uniqueness of Solution. Let D be a bounded region in the n-dimensional space R_n ($x = (x_1, \ldots, x_n)$ is a point of this space). As in the case of the mixed problems for hyperbolic equations, in the $(n+1)$-dimensional space $R_{n+1} = R_n\ \{-\infty < t < +\infty\}$ we consider a bounded cylinder $Q_T = \{x \in D,\ 0 < t < T\}$ of height $T > 0$, and let Γ_T denote the lateral surface of this cylinder: $\Gamma_T = \{x \in \partial D,\ 0 < t < T\}$ and D_τ, $\tau \in [0, T]$, the set $\{x \in D,\ t = \tau\}$; in particular, $D_0 = \{x \in D,\ t = 0\}$ is the base of the cylinder Q_T and $D_T = \{x \in D,\ t = T\}$ its top.

In the cylinder Q_T with some $T > 0$, we examine the parabolic equation

$$\mathcal{L}u \equiv u_t - \operatorname{div}(k(x)\, \nabla u) + a(x)\, u = f(x, t), \qquad (1)$$

where $k(x) \in C^1(\bar{Q}_T)$, $a(x) \in C(\bar{Q}_T)$, $k(x) \geqslant k_0 = \text{const} > 0$.

A function $u(x, t)$ belonging to the space $C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T \cup \bar{D}_0)$* and satisfying Eq. (1) in Q_T and the initial condition

$$u|_{t=0} = \varphi(x) \qquad (2)$$

on D_0 as well as the boundary condition

$$u|_{\Gamma_T} = \chi$$

on Γ_T is called a *classical solution of the first mixed problem for Eq.* (1).

A function $u(x, t)$ belonging to the space $C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T \cup \bar{D}_0) \cap C^{1,0}(Q_T \cup \Gamma_T)$ and satisfying Eq. (1) in Q_T, the initial condition (2) on D_0 and on Γ_T the boundary condition

$$\left(\frac{\partial u}{\partial n} + \sigma(x)\, u\right)\Big|_{\Gamma_T} = \chi,$$

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* For the definition of the spaces $C^{p,q}$ see Sec. 7.1, Chap. III.

where $\sigma(x)$ is a function continuous on Γ_T, is called a *classical solution of the third mixed problem for Eq.* (1).

If $\sigma \equiv 0$, then the third mixed problem is known as the *second mixed problem.*

Since the case of nonhomogeneous boundary conditions is easily reduced to that of homogeneous boundary conditions, in the sequel we shall investigate homogeneous boundary conditions

$$u\,|_{\Gamma_T} = 0 \tag{3}$$

and

$$\left(\frac{\partial u}{\partial n} + \sigma(x)\,u\right)\Big|_{\Gamma_T} = 0. \tag{4}$$

We shall assume that the coefficient $a(x)$ in Eq. (1) is nonnegative in Q_T and the function $\sigma(x)$ in the boundary condition (4) is nonnegative on Γ_T.

Lemma 1. *Let* $f(x, t) \in L_2(Q_T)$ *and let* $u(x, t)$ *be a classical solution of the third (second) mixed problem* (1), (2), (4) *or a classical solution, belonging to* $C^{1,0}(Q_T \cup \Gamma_T)$, *of the first mixed problem* (1)-(3). *Then* $u(x, t) \in H^{1,0}(Q_T)$*.

Proof. Let us take arbitrary $\tau \in (0, T)$ and $\varepsilon \in (0, \tau)$ and after multiplying Eq. (1) by u integrate it over the cylinder $Q_{\varepsilon,\tau} = \{x \in D,\ \varepsilon < t < \tau\}$. Since in Q_T $uu_t = \frac{1}{2}(u^2)_t$, $u \operatorname{div}(k \nabla u) = \operatorname{div}(ku \nabla u) - k\,|\nabla u|^2$ and $\frac{1}{2}(u^2)_t - \operatorname{div}(ku \nabla u) = fu - au^2 - k\,|\nabla u|^2 \in L_1(Q_{\varepsilon,\tau})$, according to Ostrogradskii's formula we have

$$\frac{1}{2}\int_{D_\tau} u^2\,dx - \frac{1}{2}\int_{D_\varepsilon} u^2\,dx + \int_{Q_{\varepsilon,\tau}} k\,|\nabla u|^2\,dx\,dt + \int_{Q_{\varepsilon,\tau}} au^2\,dx\,dt - \int_{\Gamma_{\varepsilon,\tau}} ku\frac{\partial u}{\partial n}\,dS\,dt = \int_{Q_{\varepsilon,\tau}} fu\,dx\,dt,$$

where $\Gamma_{\varepsilon,\tau} = \{x \in \partial D,\ \varepsilon < t < \tau\}$, whence, when $u(x, t)$ is a solution of the first mixed problem,

$$\frac{1}{2}\int_{D_\tau} u^2\,dx - \frac{1}{2}\int_{D_\varepsilon} u^2\,dx + \int_{Q_{\varepsilon,\tau}} k\,|\nabla u|^2\,dx\,dt + \int_{Q_{\varepsilon,\tau}} au^2\,dx\,dt = \int_{Q_{\varepsilon,\tau}} fu\,dx\,dt$$

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* The spaces $H^{r,0}(Q_T)$ were introduced in Sec. 7.2, Chap. III. The properties of the elements of this space are considered in the same section.

and, when $u(x, t)$ is a solution of the third (second) mixed problem,

$$\frac{1}{2}\int\limits_{D_\tau} u^2\,dx-\frac{1}{2}\int\limits_{D_\varepsilon} u^2\,dx+\int\limits_{Q_{\varepsilon,\tau}} k|\nabla u|^2\,dx\,dt$$
$$+\int\limits_{Q_{\varepsilon,\tau}} au^2\,dx\,dt+\int\limits_{\Gamma_{\varepsilon,\tau}} k\sigma u^2\,dS\,dt=\int\limits_{Q_{\varepsilon,\tau}} fu\,dx\,d\tau.$$

Consequently,

$$\frac{1}{2}\int\limits_{D_\tau} u^2\,dx+k_0\int\limits_{Q_{\varepsilon,\tau}} |\nabla u|^2\,dx\,dt\leqslant\frac{1}{2}\int\limits_{D_\tau} u^2\,dx$$
$$+\int\limits_{Q_{\varepsilon,\tau}} k(x)\,|\nabla u|^2\,dx\,dt\leqslant\frac{1}{2}\int\limits_{D_\varepsilon} u^2\,dx$$
$$+\int\limits_{Q_{\varepsilon,\tau}} |f|\,|u|\,dx\,dt\leqslant\frac{1}{2}\int\limits_{D_\varepsilon} u^2\,dx+\|u\|_{L_2(Q_{\varepsilon,\tau})}\|f\|_{L_2(Q_T)}.$$

Passing to the limit in this inequality as $\varepsilon\to 0$, we obtain

$$\frac{1}{2}\int\limits_{D_\tau} u^2\,dx\leqslant\frac{1}{2}\|\varphi\|^2_{L_2(D)}+\|u\|_{L_2(Q_\tau)}\|f\|_{L_2(Q_T)}\tag{5}$$

and

$$k_0\int\limits_{Q_\tau} |\nabla u|^2\,dx\,dt\leqslant\frac{1}{2}\|\varphi\|^2_{L_2(D)}+\|u\|_{L_2(Q_\tau)}\|f\|_{L_2(Q_T)}.\tag{6}$$

Let us take an arbitrary $t\in(0, T)$ and integrate inequality (5) with respect to $\tau\in(0, t)$:

$$\int\limits_0^t\int\limits_{D_\tau} u^2\,dx\,d\tau\leqslant T\|\varphi\|^2_{L_2(D)}+2T\|u\|_{L_2(Q_t)}\|f\|_{L_2(Q_T)}$$
$$\leqslant T\|\varphi\|^2_{L_2(D)}+2T^2\|f\|^2_{L_2(Q_T)}+\frac{1}{2}\|u\|^2_{L_2(Q_t)},$$

whence

$$\|u\|^2_{L_2(Q_t)}\leqslant 2T\|\varphi\|^2_{L_2(D)}+4T^2\|f\|^2_{L_2(Q_T)}=C_0^2$$

for any $t\in(0, T)$. Consequently, $u\in L_2(Q_T)$ and

$$\|u\|_{L_2(Q_T)}\leqslant C_0.\tag{7}$$

Then from (6) we have

$$\|\,|\nabla u|\,\|^2_{L_2(Q_\tau)}\leqslant\frac{1}{2k_0}\|\varphi\|^2_{L_2(D)}+\frac{C_0}{k_0}\|f\|_{L_2(Q_T)}$$

for any $\tau\in(0, T)$. Accordingly, $|\nabla u|\in L_2(Q_T)$. ■

Remark. From inequalities (5) and (7) it readily follows that the classical solution of the third (second) mixed problem (1), (2), (4) and the classical solution, belonging to $C^{1,0}(Q_T \cup \Gamma_T)$, of the first mixed problem (1)-(3) have the estimate

$$\|u\|_{L_2(D_\tau)} \leqslant C_1, \qquad \tau \in (0, T), \tag{8}$$

where the constant C_1 depends only on T, $\|\varphi\|_{L_2(D)}$ and $\|f\|_{L_2(Q_T)}$.

Let u be a classical solution of the third (second) mixed problem (1), (2), (4) or a classical solution, belonging to $C^{1,0}(Q_T \cup \Gamma_T)$, of the first mixed problem (1)-(3), and let $f(x, t) \in L_2(Q_T)$. Multiply (1) by an arbitrary function $v(x, t) \in C^1(\overline{Q}_T)$ such that

$$v|_{D_T} = 0 \tag{9}$$

and integrate the resulting relation over the cylinder $Q_{\varepsilon,\tau}$, where τ is an arbitrary number from $(0, T)$ and ε an arbitrary number from $(0, \tau)$. According to Ostrogradskii's formula, we obtain

$$\int\limits_{Q_{\varepsilon,\tau}} (-uv_t + k\nabla u \nabla v + auv)\, dx\, dt - \int\limits_{\Gamma_{\varepsilon,\tau}} kv \frac{\partial u}{\partial n}\, dS\, dt + \int\limits_{D_\tau} uv\, dx = \int\limits_{D_\varepsilon} uv\, dx + \int\limits_{Q_{\varepsilon,\tau}} fv\, dx\, dt. \tag{10}$$

If u is a solution of the first mixed problem, then we assume additionally that

$$v|_{\Gamma_T} = 0. \tag{11}$$

In this case identity (10) has the form

$$\int\limits_{Q_{\varepsilon,\tau}} (-uv_t + k\nabla u \nabla v + auv)\, dx\, dt + \int\limits_{D_\tau} uv\, dx = \int\limits_{D_\varepsilon} uv\, dx + \int\limits_{Q_{\varepsilon,\tau}} fv\, dx\, dt. \tag{10'}$$

If u is a solution of the third (second) mixed problem, identity (6) has the form

$$\int\limits_{Q_{\varepsilon,\tau}} (-uv_t + k\nabla u \nabla v + auv)\, dx\, dt + \int\limits_{\Gamma_{\varepsilon,\tau}} k\sigma uv\, dS\, dt + \int\limits_{D_\tau} uv\, dx = \int\limits_{D_\varepsilon} uv\, dx + \int\limits_{Q_{\varepsilon,\tau}} fv\, dx\, dt. \tag{10''}$$

By virtue of Lemma 1, $u \in H^{1,0}(Q_T)$ and therefore (see Sec. 7, Chap. III) $u|_{\Gamma_T} \in L_2(\Gamma_T)$. Using (8) and (9), we pass to the limit in identities (10′) and (10″) as $\varepsilon \to 0$ and $\tau \to T$. This yields the following assertions.

A classical solution $u(x, t)$ *of the first mixed problem which belongs to* $C^{1,0}(Q_T \cup \Gamma_T)$ *satisfies the integral identity*

$$\int\limits_{Q_T} (-uv_t + k\nabla u\nabla v + auv)\,dx\,dt = \int\limits_{D_0} \varphi v\,dx + \int\limits_{Q_T} fv\,dx\,dt \tag{12}$$

for all $v \in C^1(\overline{Q}_T)$ satisfying conditions (9) and (11), and consequently *for all* $v \in H^1(Q_T)$ *satisfying conditions* (9) *and* (11).

A classical solution $u(x, t)$ *of the third* (*second if* $\sigma = 0$) *mixed problem satisfies the integral identity*

$$\int\limits_{Q_T} (-uv_t + k\nabla u\nabla v + auv)\,dx\,dt + \int\limits_{\Gamma_T} k\sigma uv\,dS\,dt = \int\limits_{D_0} \varphi v\,dx + \int\limits_{Q_T} fv\,dx\,dt \tag{13}$$

for all $v \in C^1(\overline{Q}_T)$ satisfying condition (9), and consequently *for all* $v \in H^1(Q_T)$ *satisfying condition* (9).

With the aid of the above identities, we introduce the notions of generalized solutions of the mixed problems under discussion. We shall assume that $f(x, t) \in L_2(Q_T)$ and $\varphi(x) \in L_2(D)$.

A function $u(x, t)$ belonging to the space $H^{1,0}(Q_T)$ is called a *generalized solution of the first mixed problem* (1)-(3) if it satisfies the boundary condition (3) and the identity (12) for all $v(x, t)$ in $H^1(Q_T)$ that obey conditions (9) and (11).

A function $u(x, t)$ belonging to the space $H^{1,0}(Q_T)$ is called a *generalized solution of the third* (*second if* $\sigma = 0$) *mixed problem* (1), (2), (4) if it satisfies the identity (13) for all $v(x, t)$ in $H^1(Q_T)$ that obey condition (9).

Together with the classical and generalized solutions of the mixed problems, we may also introduce the notion of an a.e. solution.

A function $u(x, t)$ is called an *a.e. solution of the first mixed problem* (1)-(3) *or the third* (*second if* $\sigma = 0$) *mixed problem* (1), (2), (4) if it belongs to the space $H^{2,1}(Q_T)$, satisfies Eq. (1) for almost all $(x, t) \in Q_T$ and satisfies the initial condition (2) and one of the boundary conditions (3) or (4), respectively.

It was shown above that a classical solution of the third (second) mixed problem (1), (2), (4) and a classical solution, belonging to $C^{1,0}(Q_T \cup \Gamma_T)$, of the first mixed problem (1)-(3) are generalized solutions of the corresponding mixed problems. It can be similarly proved that an a.e. solution of the first, second or third mixed problem is a generalized solution of the corresponding problem. It is also easy to establish that if a generalized solution of the first mixed problem (1)-(3) or third (second) mixed problem (1), (2), (4) belongs to $H^{2,1}(Q_T)$, then it is an a.e. solution of the same problem. If, however, the generalized solution of the first mixed problem (1)-(3) belongs to $C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T \cup \bar{D}_0)$ and that of the third (second) problem (1), (2), (4) belongs to $C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T \cup \bar{D}_0) \cap C^{1,0}(Q_T \cup \Gamma_T)$, then it is a classical solution (compare Sec. 2.1, Chap. V, where corresponding statements have been proved regarding solutions of the mixed problems for a hyperbolic equation).

We further note that the generalized solution, like the classical and a.e. solutions, of the mixed problem for a parabolic equation has the following property: if $u(x, t)$ is a generalized solution of the mixed problem (1)-(3) or the problem (1), (2), (4) in the cylinder Q_T, then it is a generalized solution of the corresponding problem in the cylinder $Q_{T'}$ for any T', $0 < T' < T$. The proof of this assertion is also completely analogous to that of the corresponding assertion regarding solutions of the mixed problems for a hyperbolic equation.

We shall now establish uniqueness theorems for solutions of the mixed problems.

Theorem 1. *The first mixed problem* (1)-(3) *cannot have more than one generalized solution.*

The third (*second*) *mixed problem* (1), (2), (4) *cannot have more than one generalized solution.*

Proof. This theorem is proved along the same lines as the corresponding theorem regarding generalized solutions of the mixed problems for a hyperbolic equation (Theorem 1, Sec. 2.1, Chap. V).

Let $u_1(x, t)$ and $u_2(x, t)$ be two generalized solutions of the problem (1)-(3) or the problem (1), (2), (4). Then the function $u = u_1 - u_2$ is a generalized solution of the corresponding problem with $f = 0$ and $\varphi = 0$. We must show that $u = 0$ in Q_T.

In Q_T consider the function

$$v(x, t) = \int_t^T u(x, \theta)\, d\theta.$$

It is directly verified that in Q_T the function v has generalized derivatives

$$v_t = -u,$$

$$v_{x_i} = \int_t^T u_{x_i}(x, \theta)\, d\theta, \qquad i = 1, \ldots, n.$$

Since the functions v, v_t and v_{x_i}, $i = 1, \ldots, n$, evidently belong to $L_2(Q_T)$, we see that $v \in H^1(Q_T)$. Moreover, $v|_{D_T} = 0$, $v|_{\Gamma_T} = \int_t^T u|_{\Gamma_T}\, d\theta$, and, in particular, if u is a generalized solution of the first mixed problem, then $v|_{\Gamma_T} = 0$. We substitute v in the identity (12) if u is a solution of the problem (1)-(3) or in (13) if u is a solution of the problem (1), (2), (4). Then, in the case of the first mixed problem, we obtain the identity

$$\int_{Q_T} \Big(u^2(x, t) + k\nabla u(x, t)\cdot \int_t^T \nabla u(x, \theta)\, d\theta - av(x, t)\, v_t(x, t)\Big) dx = dt0 \tag{14}$$

and, in the case of the third (second) mixed problem, the identity

$$\int_{Q_T} \Big(u^2(x, t) + k(x)\, \nabla u(x, t)\cdot \int_t^T \nabla u(x, \theta)\, d\theta - avv_t \Big)\, dx\, dt + \int_{\Gamma_T} k\sigma u(x, t) \int_t^T u(x, \theta)\, d\theta\, dS\, dt = 0. \tag{14'}$$

Since (see the proof of Theorem 1, Sec. 2.1, Chap. V)

$$\int_{Q_T} k\nabla u(x, t) \int_t^T \nabla u(x, \theta)\, d\theta\, dx\, dt = \frac{1}{2} \int_D k \left| \int_0^T \nabla u(x, t)\, dt \right|^2 dx \geqslant 0,$$

$$\int_{\Gamma_T} k\sigma u(x, t) \int_t^T u(x, \theta)\, d\theta\, dS\, dt = \frac{1}{2} \int_{\partial D} k\sigma \left(\int_0^T u(x, t)\, dt \right)^2 dS \geqslant 0$$

and

$$\int_{Q_T} avv_t\, dx\, dt = -\frac{1}{2} \int_{D_0} av^2\, dx \leqslant 0,$$

from (14) and (14′) we have

$$\int_{Q_T} u^2(x, t)\, dx\, dt \leqslant 0$$

from which it follows that $u = 0$ in Q_T. ∎

Since an a.e. solution of the mixed problem (1)-(3) or the mixed problem (1), (2), (4) is also a generalized solution of the corresponding problem, Theorem 1 has the following corollary.

Corollary 1. *The first mixed problem* (1)-(3) *cannot have more than one a.e. solution.*

The third (second) mixed problem (1), (2), (4) *cannot have more than one a.e. solution.*

Theorem 1 also implies the following statement.

Corollary 2. *The third (second) mixed problem* (1), (2), (4) *cannot have more than one classical solution.*

Indeed, let u_1 and u_2 be two classical solutions of the problem (1), (2), (4). Then their difference is a classical solution of the problem (1), (2), (4) with $\varphi = 0$ and $f = 0 \in L_2(Q_T)$. Consequently, $u_1 - u_2$ is a generalized solution and, in view of Theorem 1, is equal to zero.

We shall now prove the uniqueness theorem for the classical solution of the first mixed problem.

Theorem 2. *The first mixed problem* (1)-(3) *cannot have more than one classical solution.*

Proof. Let u_1 and u_2 be two classical solutions of the first mixed problem (1)-(3) in the cylinder Q_T. Then the function $u = u_1 - u_2$ belongs to $C^{2,1}(Q_T) \cap C(Q_T \cup \Gamma_T \cup \bar{D}_0)$ and satisfies the homogeneous equation

$$\mathcal{L}u = u_t - \operatorname{div}(k \nabla u) + au = 0 \tag{1_0}$$

in Q_T together with the boundary condition (3) on Γ_T and the homogeneous initial condition

$$u|_{t=0} = 0 \tag{2_0}$$

on D_0. We shall show that $u(x, t)$ vanishes in Q_T.

Suppose that there is a point $(x^0, t^0) \in Q_T$ such that $u(x^0, t^0) \neq 0$. We assume that $u(x^0, t^0) > 0$ (if $u(x^0, t^0) < 0$, then instead of the function u we consider the function $-u$ for which (1_0), (2_0) and (3) are fulfilled and $-u(x^0, t^0) > 0$).

Let M denote $u(x^0, t^0)$, and consider the function

$$v(x, t) = u(x, t) - \frac{M}{2t^0}(t - t^0).$$

First note that

$$\mathcal{L}v = -\frac{M}{2t^0} < 0 \text{ for all } (x, t) \in Q_T. \tag{15}$$

Since $v \in C(\bar{Q}_{t^0})$, there is a point (x^1, t^1) in $\bar{Q}_{t^0}$ where the function $v(x, t)$ attains its maximum value; moreover, since $v(x^0, t^0) = u(x^0, t^0) = M$, $v(x^1, t^1) \geqslant v(x^0, t^0) = M$.

The point (x^1, t^1) cannot lie in the set $\bar{\Gamma}_{t^0} \cup D_0$, since $v|_{\Gamma_T} = u|_{\Gamma_T} - \frac{M}{2t^0}(t - t^0) = \frac{M}{2t^0}(t^0 - t) \leqslant \frac{M}{2}$ and $v|_{D_0} = u|_{D_0} + \frac{M}{2} = \frac{M}{2}$. Consequently, the point (x^1, t^1) must lie in the set $Q_{t^0} \cup D_{t^0}$. Let it belong to Q_{t^0}. Then $v_t(x^1, t^1) = 0$, $v_{x_i}(x^1, t^1) = 0$ and $v_{x_ix_i}(x^1, t^1) \leqslant 0$, $i = 1, \ldots, n$. That is, $\mathcal{L}v(x^1, t^1) = v_t(x^1, t^1) - k(x^1)\Delta v(x^1, t^1) - \nabla k(x^1)\nabla v(x^1, t^1) + a(x^1)v \times (x^1, t^1) \geqslant 0$ and this contradicts (15). If, however, $(x^1, t^1) \in D_{t^0}$, then $v_t(x^1, t^1) \geqslant 0$, $v_{x_i}(x^1, t^1) = 0$ and $v_{x_ix_i}(x^1, t^1) \leqslant 0$, $i = 1, \ldots, n$. That is, again $\mathcal{L}v(x^1, t^1) \geqslant 0$. ■

2. Existence of a Generalized Solution. We shall now establish the existence of the solutions of the problems (1)-(3) and (1), (2), (4). For this, as in the case of hyperbolic equations, we shall apply the Fourier method.

Let $v(x)$ be a generalized eigenfunction of the first boundary-value problem

$$\operatorname{div}(k(x)\nabla v) - av = \lambda v, \qquad x \in D \tag{16}$$
$$v|_{\partial D} = 0$$

or the third (second if $\sigma = 0$) boundary-value problem

$$\operatorname{div}(k(x)\nabla v) - av = \lambda v, \qquad x \in D,$$
$$\left(\frac{\partial v}{\partial n} + \sigma(x) v\right)\Big|_{\partial D} = 0 \tag{17}$$

(λ is the corresponding eigenvalue). This means that in the case of the first boundary-value problem v belongs to $\overset{\circ}{H}{}^1(D)$ and satisfies the integral identity

$$\int_D (k\nabla v\nabla\eta + av\eta)\,dx + \lambda\int_D v\eta\,dx = 0$$

for all $\eta \in \overset{\circ}{H}{}^1(D)$, while in the case of the third (second) boundary-value problem $v \in H^1(Q)$ and satisfies the integral identity

$$\int_D (k\nabla v\nabla\eta + av\eta)\,dx + \int_{\partial D} k\sigma v\eta\,dS + \lambda\int_D v\eta\,dx = 0$$

for all $\eta \in H^1(D)$.

Consider an orthonormal system $v_1, v_2, \ldots$ in $L_2(D)$ which consists of all generalized eigenfunctions of the problem (16) or the problem (17), respectively, and let $\lambda_1, \lambda_2, \ldots$ denote the sequence of the corresponding eigenvalues which is, as usual, considered nonincreasing and in which each eigenvalue is repeated according to its multiplicity. As shown in Sec. 1, Chap. IV, the system $v_1, v_2, \ldots$ is an orthonormal basis for $L_2(D)$ and $\lambda_k \to -\infty$ as $k \to \infty$. In the case of the first, third when $\sigma \not\equiv 0$ on ∂D and second when $a \not\equiv 0$ in D boundary-value problems (note that $a(x) \geqslant 0$ in D and $\sigma(x) \geqslant$ $\geqslant 0$ on ∂D) the first eigenvalue $\lambda_1 < 0$, that is, $0 > \lambda_1 \geqslant \lambda_2 \geqslant \ldots$. If $a \equiv 0$ in D, then $0 = \lambda_1 > \lambda_2 \geqslant \lambda_3 \geqslant \ldots$ in the case of the second boundary-value problem.

Suppose that the initial function φ in (2) belongs to $L_2(D)$ and that the function $f \in L_2(Q_T)$. According to Fubini's theorem, $f(x, t) \in L_2(D_t)$ for almost all $t \in (0, T)$. We expand the functions φ and $f(x, t)$ for almost all $t \in (0, T)$ in Fourier series in terms of the system $v_1, v_2, \ldots$ of generalized eigenfunctions of problem (16) if the problem in question is (1), (2), (3) or of problem (17) if the problem in question is (1), (2), (4):

$$\begin{aligned} \varphi(x) &= \sum_{k=1}^{\infty} \varphi_k v_k(x), \\ f(x, t) &= \sum_{k=1}^{\infty} f_k(t)\, v_k(x) \end{aligned} \tag{18}$$

where

$$\varphi_k = (\varphi, v_k)_{L_2(D)}, \qquad f_k(t) = (f(x, t), v_k(x))_{L_2(D)}, \tag{19}$$

and the functions $f_k(t)$ belong to $L_2(0, T)$. By the Parseval-Steklov equality,

$$\sum_{k=1}^{\infty} \varphi_k^2 = \| \varphi \|_{L_2(D)}^2 \tag{20}$$

and for almost all $t \in (0, T)$

$$\sum_{k=1}^{\infty} f_k^2(t) = \int_D f^2(x, t)\, dx,$$

whence

$$\sum_{k=1}^{\infty} \int_0^T f_k^2(t)\, dt = \int_{Q_T} f^2(x, t)\, dx\, dt. \tag{20'}$$

For any $k = 1, 2, \ldots$ consider the function

$$U_k(t) = \varphi_k e^{\lambda_k t} + \int_0^t f_k(\tau)\, e^{\lambda_k (t-\tau)}\, d\tau \tag{21}$$

belonging to $H^1(0, T)$ and satisfying the equation

$$U_k' - \lambda_k U_k = f_k \tag{22}$$

a.e. on $(0, T)$ and ($H^1(0, T) \subset C([0, T])$) the condition

$$U_k(0) = \varphi_k. \tag{22'}$$

It is easy to check (as in the case of a hyperbolic equation) that the function

$$u_k(x, t) = U_k(t)\, v_k(x)$$

is a generalized solution of the first mixed problem, if $v_k(x)$ is the eigenfunction of the problem (16), or of the third (second), if $v_k(x)$ is the eigenfunction of the problem (17), mixed problem for the equation

$$u_t - \operatorname{div}(k \nabla u) + au = f_k(t)\, v_k(x)$$

with the initial condition

$$u|_{t=0} = \varphi_k v_k(x).$$

Consequently, if we take the partial sums $\sum_{k=1}^{N} \varphi_k v_k(x)$ and $\sum_{k=1}^{N} f_k(t)\, v_k(x)$ of series (18) as the initial function in (2) and the right-hand side of Eq. (1), then a generalized solution of the problem (1)-(3) or the problem (1), (2), (4), respectively, will be the function

$$S_N(x, t) = \sum_{k=1}^{N} U_k(t)\, v_k(x).$$

In particular, in the case of the first mixed problem $S_N(x, t)$ satisfies the integral identity

$$\int_{Q_T} (-S_N v_t + k\nabla S_N \cdot \nabla v + aS_N v)\, dx\, dt$$
$$= \int_{D_0} \sum_{k=1}^{N} \varphi_k v_k(x)\, v(x, 0)\, dx + \int_{Q_T} \sum_{k=1}^{N} f_k(t)\, v_k(x)\, v(x, t)\, dx\, dt \qquad (23)$$

for all v in $H^1(Q_T)$ that obey conditions (9) and (11), and in the case of the third (second) mixed problem the integral identity

$$\int_{Q_T} (-S_N v_t + k\nabla S_N \cdot \nabla v + aS_N v)\, dx\, dt + \int_{\Gamma_T} k\sigma S_N v\, dS\, dt$$
$$= \int_{D_0} \sum_{k=1}^{N} \varphi_k v_k(x)\, v(x, 0)\, dx + \int_{Q_T} \sum_{k=1}^{N} f_k(t)\, v_k(x)\, v(x, t)\, dx\, dt \qquad (23')$$

for all v in $H^1(Q_T)$ such that condition (9) is fulfilled.

Let us show that the generalized solution of the problem (1)-(3) or the problem (1), (2), (4) is given by the series

$$u(x, t) = \sum_{k=1}^{\infty} U_k(t)\, v_k(x), \qquad (24)$$

where in the case of the problem (1)-(3) $v_k(x)$, $k = 1, 2, \ldots$, are eigenfunctions of the problem (16) and in the case of the problem (1), (2), (4) $v_k(x)$, $k = 1, 2, \ldots$, are eigenfunctions of the problem (17).

Theorem 3. *If $f \in L_2(Q_T)$ and $\varphi \in L_2(D)$, then each of the mixed problems* (1), (2), (3) *or* (1), (2), (4) *has a generalized solution u. This solution is represented by a convergent series* (24) *in $H^{1,0}(Q_T)$. More-*

over,

$$\|u\|_{H^{1,0}(Q_T)} \leqslant C\,(\|\varphi\|_{L_2(D)} + \|f\|_{L_2(Q_T)}), \tag{25}$$

where the constant $C > 0$ does not depend on φ or f.

Proof. From formula (21) it follows that, for all $t \in [0, T]$,

$$|U_k(t)| \leqslant |\varphi_k|\, e^{\lambda_k t} + \int_0^t |f_k(\tau)|\, e^{\lambda_k (t-\tau)}\, d\tau$$

$$\leqslant |\varphi_k|\, e^{\lambda_k t} + \frac{\|f_k\|_{L_2(0,T)}}{\sqrt{2|\lambda_k|}} \quad \text{when } k > 1$$

and

$$|U_1(t)| \leqslant |\varphi_1| + C_1 \|f_1\|_{L_2(0,T)},$$

where $C_1 = \sqrt{T}$ in the case of the second mixed problem with $a \equiv 0$ and in the remaining cases $C_1 = 1/\sqrt{2|\lambda_1|}$. Therefore for all $t \in [0, T]$

$$U_k^2(t) \leqslant 2\varphi_k^2 e^{2\lambda_k t} + \frac{1}{|\lambda_k|} \|f_k\|^2_{L_2(0,T)} \quad \text{when } k > 1 \tag{26}$$

and

$$U_1^2(t) \leqslant 2\varphi_1^2 + 2C_1^2 \|f_1\|^2_{L_2(0,T)}. \tag{26'}$$

We consider the partial sum $S_N(x, t)$ of the series (24). For each $t \in [0, T]$ it belongs to the space $\overset{\circ}{H}{}^1(D_t)$ in the case of the first mixed problem or to the space $H^1(D_t)$ in the case of the third (second) mixed problem.

In investigating the problem (1)-(3) it is convenient to introduce in the space $\overset{\circ}{H}{}^1(D_t)$ a scalar product

$$\int_{D_t} (k \nabla u \nabla v + auv)\, dx.$$

For the problem (1), (2), (4) we introduce in the space $H^1(D_t)$ a scalar product

$$\int_{D_t} (k \nabla u \nabla v + auv)\, dx + \int_{\partial D_t} k\sigma uv\, dS$$

if either $a \not\equiv 0$ in D or $\sigma \not\equiv 0$ on ∂D and the scalar product

$$\int_{D_t} (k \nabla u \nabla v + uv)\, dx$$

if $a \equiv 0$ in D and $\sigma \equiv 0$ on ∂D. Since in the case of the first and third, $\sigma \not\equiv 0$, mixed problem and in the case of the second mixed problem with $a \not\equiv 0$ the system of functions $v_1/\sqrt{-\lambda_1}$, $v_2/\sqrt{-\lambda_2}$, ... are orthonormal in the corresponding scalar products, while in the case of the second mixed problem with $a \equiv 0$ the system of functions $v_1/\sqrt{1-\lambda_1}$, $v_2/\sqrt{1-\lambda_2}$, ... is orthonormal, we have for all $t \in [0, T]$ and any M and N, $1 \leqslant M < N$, in view of (26),

$$\| S_N(x, t) - S_M(x, t) \|^2_{H^1(D_t)} = \Big\| \sum_{k=M+1}^{N} U_k(t)\, v_k(x) \Big\|^2_{H^1(D_t)}$$

$$\leqslant \sum_{k=M+1}^{N} U_k^2(t) \,|\lambda_k| \leqslant \sum_{k=M+1}^{N} \Big(2e^{2\lambda_k t} \varphi_k^2 \,|\lambda_k| + \int_0^T f_k^2(t)\, dt \Big)$$

in the case of the first mixed problem and in the case of the second or third mixed problems if either $a \not\equiv 0$ in D or $\sigma(x) \not\equiv 0$ on ∂D, while

$$\| S_N(x, t) - S_M(x, t) \|^2_{H^1(D_t)} = \sum_{k=M+1}^{N} U_k^2(t)\,(1 + |\lambda_k|)$$

$$\leqslant \sum_{k=M+1}^{N} \Big[2e^{2\lambda_k t} \varphi_k^2 \,(1 + |\lambda_k|) + \frac{1+|\lambda_k|}{|\lambda_k|} \int_0^T f_k^2(t)\, dt \Big]$$

$$\leqslant 2\, \frac{1+|\lambda_2|}{|\lambda_2|} \sum_{k=M+1}^{N} \Big[e^{2\lambda_k t} (1 + |\lambda_k|)\, \varphi_k^2 + \int_0^T f_k^2(t)\, dt \Big]$$

if $a \equiv 0$ in D and $\sigma \equiv 0$ on ∂D. That is, in both cases

$$\| S_N(x, t) - S_M(x, t) \|^2_{H^1(D_t)}$$

$$\leqslant C_1 \sum_{k=M+1}^{N} \Big(\varphi_k^2 e^{2\lambda_k t} (1 + |\lambda_k|) + \int_0^T f_k^2(t)\, dt \Big). \qquad (27)$$

Together with this inequality, we also have, in view of (26′), the inequality

$$\| S_N(x, t) \|^2_{H^1(D_t)} = \Big\| U_1 v_1 + \sum_{k=2}^{N} U_k v_k \Big\|^2_{H^1(D_t)}$$

$$\leqslant C_2 \sum_{k=1}^{N} \Big(\varphi_k^2 e^{2\lambda_k t} (1 + |\lambda_k|) + \int_0^T f_k^2(t)\, dt \Big) \qquad (28)$$

valid for all $t \in [0, T]$ and any $N \geqslant 1$. Integrating the inequalities (27) and (28) with respect to $t \in (0, T)$, we obtain

$$\|S_N - S_M\|^2_{H^{1,0}(Q_T)} \leqslant C_3 \sum_{k=M+1}^{N} \left(\varphi_k^2 + \int_0^T f_k^2(t)\,dt\right), \tag{29}$$

$$\|S_N\|^2_{H^{1,0}(Q_T)} \leqslant C_4 \sum_{k=1}^{N} \left(\varphi_k^2 + \int_0^T f_k^2(t)\,dt\right). \tag{30}$$

According to (20) and (20′), the series with general term $\varphi_k^2 + \int_0^T f_k^2(t)\,dt$ converges. Therefore it follows from (29) that the series (24) converges in $H^{1,0}(Q_T)$, and consequently its sum $u(x, t)$ belongs to $H^{1,0}(Q_T)$ and satisfies the boundary condition (3) in the case of the first mixed problem. Passing to the limit as $N \to \infty$ in the identity (23) in the case of the first problem or in (23′) in the case of the third (second) problem, we find that the function $u(x, t)$ satisfies the identity (12) or the identity (13), respectively. Hence $u(x, t)$ is a generalized solution. Inequality (25) follows from (30) if we pass in it to the limit as $N \to \infty$ and use the identities (20) and (20′). ■

Note that the existence of the generalized solutions of the above mixed problems can be established, as in the case of a hyperbolic equation, by the Galerkin method.

3. Smoothness of Generalized Solutions of Mixed Problems. Existence of an A. E. Solution and the Classical Solution. In investigating the smoothness of generalized solutions we shall confine our discussion to the first and second (in the boundary condition (4) $\sigma \equiv 0$) mixed problems for a particular case of Eq. (1), the heat equation (in (1) $k \equiv 1$, $a \equiv 0$), though analogous results can be obtained by the same method in the general case if the coefficients and the function σ are sufficiently smooth.

Let $u(x, t)$ be a generalized solution of the first or second mixed problem for the heat equation

$$u_t - \Delta u = f, \tag{31}$$

$$u|_{t=0} = \varphi \tag{32}$$

and either

$$u|_{\Gamma_T} = 0 \tag{33}$$

in the case of the first mixed problem or

$$\left.\frac{\partial u}{\partial n}\right|_{\Gamma_T} = 0 \tag{34}$$

in the case of the second mixed problem.

Let us recall (see Sec. 2.4, Chap. IV) that if the boundary ∂D of the region D belongs to the class C^r for some $r \geqslant 1$, then the generalized eigenfunctions $v_k(x)$, $k = 1, 2, \ldots$, of the first and second boundary-value problems for the Laplace operator belong to the spaces $H^r_{\mathscr{D}}(D)$ and $H^r_{\mathscr{N}}(D)$, respectively, that is, they belong to $H^r(D)$ and on ∂D satisfy the boundary conditions

$$v_k\,|_{\partial D} = \ldots = \Delta^{\left[\frac{r-1}{2}\right]} v_k\,|_{\partial D} = 0, \qquad k = 1, 2, \ldots$$

in the case of the first boundary-value problem, while in the case of the second boundary-value problem for $r > 1$ the boundary conditions

$$\left.\frac{\partial v_k}{\partial n}\right|_{\partial D} = \ldots = \frac{\partial}{\partial n}\Delta^{\left[\frac{r}{2}\right]-1} v_k\,|_{\partial D} = 0, \qquad k = 1, 2, \ldots;$$

when $r = 1$, $H^r_{\mathscr{N}}(D) = H^1_{\mathscr{N}}(D) = H^1(\mathscr{D})$.

Let $\widetilde{H}^{2l,\,l}_{\mathscr{D}}(Q_T)$ denote a subspace of the space $H^{2l,\,l}(Q_T)$, with integer $l \geqslant 1$ (see Sec. 7.2, Chap. III), that functions f in $H^{2l,\,l}(Q_T)$ such that

$$f\,|_{\Gamma_T} = \ldots = \Delta^{l-1} f\,|_{\Gamma_T} = 0;$$

when $l = 0$, by $\widetilde{H}^{2l,\,l}_{\mathscr{D}}(Q_T)$ we shall mean the space $L_2(Q_T)$: $\widetilde{H}^{0,\,0}_{\mathscr{D}}(Q_T) = H^{0,\,0}(Q_T) = L_2(Q_T)$.

With integer $l \geqslant 1$, we let $\widetilde{H}^{2l,\,l}_{\mathscr{N}}(Q_T)$ denote a subspace of the space $H^{2l,\,l}(Q_T)$ that contains all functions f in $H^{2l,\,l}(Q_T)$ such that

$$\left.\frac{\partial f}{\partial n}\right|_{\Gamma_T} = \ldots = \frac{\partial}{\partial n}\Delta^{l-1} f\,|_{\Gamma_T} = 0;$$

when $l = 0$, by $\widetilde{H}^{2l,\,l}_{\mathscr{N}}(Q_T)$ we shall mean the space $L_2(Q_T)$: $\widetilde{H}^{0,\,0}_{\mathscr{N}}(Q_T) = L_2(Q_T)$.

The following statement is valid.

Theorem 4. *Let* $\partial D \in C^{2s}$ *for some* $s \geqslant 1$, *and let* $\varphi \in H^{2s-1}_{\mathscr{D}}(D)$, $f \in \widetilde{H}^{2(s-1),\,(s-1)}_{\mathscr{D}}(Q_T)$ *in the case of the first mixed problem* (31)-(33) *while* $\varphi \in H^{2s-1}_{\mathscr{N}}(D)$, $f \in \widetilde{H}^{2(s-1),\,(s-1)}_{\mathscr{N}}(Q_T)$ *in the case of the second mixed problem* (31), (32), (34). *Then the generalized solution* $u(x, t)$ *of each of these problems belongs to the space* $H^{2s,\,s}(Q_T)$ *and the series* (24) *converges to it in the space* $H^{2s,\,s}(Q_T)$. *Moreover,*

$$\|u\|_{H^{2s,\,s}(Q_T)} \leqslant C\left(\|\varphi\|_{H^{2s-1}(D)} + \|f\|_{H^{2(s-1),\,(s-1)}(Q_T)}\right), \tag{35}$$

where the positive constant C *does not depend on* φ *or* f.

Note that the hypothesis of Theorem 4 assumes the fulfillment of, besides the smoothness of the given functions, the conditions

$$\varphi\,|_{\partial D} = \ldots = \Delta^{s-1}\varphi\,|_{\partial D} = 0$$

and

$$f\,|_{\Gamma_T} = \ldots = \Delta^{s-2} f\,|_{\Gamma_T} = 0$$

in the case of the first mixed problem and the conditions

$$\frac{\partial\varphi}{\partial n}\Big|_{\partial D} = \ldots = \frac{\partial}{\partial n}\,\Delta^{s-2}\varphi\,|_{\partial D} = 0$$

as well as

$$\frac{\partial f}{\partial n}\Big|_{\Gamma_T} = \ldots = \frac{\partial}{\partial n}\,\Delta^{s-2} f\,|_{\Gamma_T} = 0$$

in the case of the second mixed problem. These conditions are necessary for the validity of Theorem 4 regarding convergence, in $H^{2s,\,s}(Q_T)$, of series (24) to the generalized solution of the corresponding mixed problem. However, if we are interested only in smoothness of the generalized solution (and not in the convergence to it of the Fourier series) then, as in the case of hyperbolic equations (see Theorem 3′, Sec. 2.4, Chap. V), these conditions can be very much weakened; they can be replaced, as in the case of hyperbolic equations, by compatible conditions on φ and f on ∂D_0.

Proof of Theorem 4. According to Lemma 2, Sec. 2.4, Chap. V, the functions $f_k(t)$, $k = 1, 2, \ldots$, defined by (19) belong to the space $H^{s-1}(0, T)$ (and therefore to $C^{s-2}([0, T])$ for $s \geqslant 2$). Consequently, the functions $U_k(t)$, $k = 1, 2, \ldots$, defined by (21) and satisfying Eqs. (22) on $(0, T)$ belong to the space $H^s(0, T)$ and therefore to $C^{s-1}([0, T])$. Hence, by the properties of the eigenfunctions $v_k(x)$, the partial sums $S_N(x, t) = \sum\limits_{k=1}^{N} U_k(t)\, v_k(x)$ of the series (24) belong to the space $\widetilde{H}^{2s,\,s}_{\mathscr{D}}(Q_T)$ and for all $t \in [0, T]$ to the space $H^{2s}_{\mathscr{D}}(D_t)$ in the case of the first mixed problem or in the case of the second mixed problem they belong to the space $\widetilde{H}^{2s,\,s}_{\mathscr{N}}(Q_T)$ and for all $t \in [0, T]$ to the space $H^{2s}_{\mathscr{N}}(D_t)$. Moreover, for $p = 1, \ldots, s$ the functions $\frac{\partial^p S_N}{\partial t^p}$ belong to the space $H^{2(s-p),\,s-p}(Q_T)$ and for all $t \in [0, T]$ to the space $H^{2s}_{\mathscr{D}}(D_t)$ in the case of the first mixed problem or, in the case of the second mixed problem, to the space $H^{2s}_{\mathscr{N}}(D_t)$. Therefore, in view of Lemma 3, Sec. 2.5, Chap. IV, and orthogonality of eigenfunctions $v_k(x)$ in $L_2(D_t)$, for all $t \in [0, T]$, any $p = 0, \ldots, s$ and any M and N, $1 \leqslant M < N$, we have the inequalities

$$\left\| \frac{\partial^p S_N}{\partial t^p} - \frac{\partial^p S_M}{\partial t^p} \right\|^2_{H^{2(s-p)}(D_t)} \leqslant C_1 \left\| \Delta^{s-p} \frac{\partial^p}{\partial t^p} (S_N - S_M) \right\|^2_{L_2(D_t)}$$

$$= C_1 \left\| \sum_{k=M+1}^{N} |\lambda_k|^{s-p} \frac{d^p U_k}{dt^p} v_k(x) \right\|^2_{L_2(D_t)}$$

$$= C_1 \sum_{k=M+1}^{N} |\lambda_k|^{2(s-p)} \left(\frac{d^p U_k}{dt^p} \right)^2. \qquad (36)$$

Similarly, for all $t \in [0, T]$, any $p = 0, \ldots, s$ and any $N \geqslant 1$

$$\left\| \frac{\partial^p S_N}{\partial t^p} \right\|^2_{H^{2(s-p)}(D_t)} \leqslant C_1 \sum_{k=1}^{N} |\lambda_k|^{2(s-p)} \left(\frac{d^p U_k}{dt^p} \right)^2$$

for the first mixed problem ($\lambda_1 \neq 0$) and

$$\left\| \frac{\partial^p S_N}{\partial t^p} \right\|^2_{H^{2(s-p)}(D_t)} = \left\| \frac{\partial^p (U_1 v_1)}{\partial t^p} + \frac{\partial^p (S_N - S_1)}{\partial t^p} \right\|^2_{H^{2(s-p)}(D_t)}$$

$$\leqslant 2 \left(\frac{d^p U_1}{dt^p} \right)^2 \left\| \frac{1}{\sqrt{|D|}} \right\|^2_{H^{2(s-p)}(D_t)} + 2 \left\| \frac{\partial^p (S_N - S_1)}{\partial t^p} \right\|^2_{H^{2(s-p)}(D_t)}$$

$$\leqslant C_2 \left(\left(\frac{d^p U_1}{dt^p} \right)^2 + \sum_{k=2}^{N} |\lambda_k|^{2(s-p)} \left(\frac{d^p U_k}{dt^p} \right)^2 \right)$$

for the second mixed problem ($\lambda_1 = 0$). Thus, in both cases, for all $t \in [0, T]$, $p = 0, \ldots, s$, $N \geqslant 1$

$$\left\| \frac{\partial^p S_N}{\partial t^p} \right\|^2_{H^{2(s-p)}(D_t)} \leqslant C_3 \left(\left(\frac{d^p U_1}{dt^p} \right)^2 + \sum_{k=1}^{N} |\lambda_k|^{2(s-p)} \left(\frac{d^p U_k}{dt^p} \right)^2 \right). \qquad (37)$$

Integrating inequalities (36) with respect to $t \in (0, T)$ and summing over p, $p = 0, \ldots, s$, we obtain

$$\| S_N - S_M \|^2_{H^{2s,\, s}(Q_T)} \leqslant C_1 \sum_{p=0}^{s} \sum_{k=M+1}^{N} |\lambda_k|^{2(s-p)} \left\| \frac{d^p U_k}{dt^p} \right\|^2_{L_2(0,\, T)}. \qquad (38)$$

Analogously from inequalities (37) we obtain

$$\| S_N \|^2_{H^{2s,\, s}(Q_T)} \leqslant C_3 \sum_{p=0}^{s} \left(\left\| \frac{d^p U_1}{dt^p} \right\|^2_{L_2(0,\, T)} \right.$$

$$\left. + \sum_{k=1}^{N} |\lambda_k|^{2(s-p)} \left\| \frac{d^p U_k}{dt^p} \right\|^2_{L_2(0,\, T)} \right). \qquad (39)$$

Next we use the following lemma whose proof will be given later.

Lemma 2. *Let* $\partial D \in C^{2q+2}$ *for some* $q \geqslant 0$ *and let* $\varphi \in H^{2q+1}_{\mathscr{D}}(D)$, $f \in \widetilde{H}^{2q,q}_{\mathscr{D}}(Q_T)$ *in the case of the first mixed problem* (31)-(33) *while in the case of the second mixed problem* (31), (32), (34) $\varphi \in H^{2q+1}_{\mathscr{N}}(D)$, $f \in \widetilde{H}^{2q,q}_{\mathscr{N}}(Q_T)$. *Then for any* p, $0 \leqslant p \leqslant q+1$,

$$\sum_{k=1}^{\infty} |\lambda_k|^{2(q+1-p)} \left\| \frac{d^pU_k}{dt^p} \right\|^2_{L_2(0,T)} \leqslant C\left(\|\varphi\|^2_{H^{2q+1}(D)} + \|f\|^2_{H^{2q,q}(Q_T)}\right), \tag{40}$$

where the positive constant C *does not depend on* φ *or* f.

In view of this lemma (with $q = s - 1$), from inequalities (38) it follows that the series (24) converges in $H^{2s,s}(Q_T)$. Therefore the generalized solutions of the problems (31)-(33) and (31), (32), (34) belong to the space $H^{2s,s}(Q_T)$ (and even to $\widetilde{H}^{2s,s}_{\mathscr{D}}(Q_T)$ or to $\widetilde{H}^{2s,s}_{\mathscr{N}}(Q_T)$, respectively). Passing to the limit as $N \to \infty$ in (39) and taking into account (40) and the obvious inequalities $\left\| \frac{d^pU_1}{dt^p} \right\|^2_{L_2(0,T)} \leqslant$ $\leqslant \text{const}\left(\|\varphi\|^2_{L_2(D)} + \|f\|^2_{H^{2(s-1),(s-1)}(Q_T)}\right)$, $p = 0, \ldots, s$, we obtain the inequality (35). $\blacksquare$

Since a generalized solution of the mixed problem that belongs to the space $H^{2,1}(Q_T)$ is an a.e. solution, Theorem 4 with $s = 1$ implies the following result.

Corollary. *Let* $\partial D \in C^2$, $f \in L_2(Q_T)$, *and let* $\varphi \in \overset{\circ}{H}{}^1(D)$ *in the case of the first mixed problem* (31)-(33) *while* $\varphi \in H^1(D)$ *in the case of the second mixed problem* (31), (32), (34). *Then the series* (24) *converges in* $H^{2,1}(Q_T)$ *and its sum is an a.e. solution of the problem* (31)-(33), *or of the problem* (31), (32), (34), *respectively. Furthermore,*

$$\|u\|_{H^{2,1}(Q_T)} \leqslant C\left(\|\varphi\|_{H^1(D)} + \|f\|_{L_2(Q_T)}\right),$$

where the positive constant C *does not depend on* φ *or* f.

Before establishing Lemma 2, which we used in the proof of Theorem 4, we shall prove the following auxiliary assertions.

Lemma 3. *If* $f(x, t) \in H^{r,0}(Q_T)$, $r \geqslant 1$, *and* $g(t) \in L_2(0, T)$, *then the function*

$$h(x) = \int_0^T f(x, t)\, g(t)\, dt$$

belongs to $H^r(D)$, *and for any* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| \leqslant r$,

$$D^{\alpha}_x h(x) = \int_0^T D^{\alpha}_x f(x, t)\, g(t)\, dt. \tag{41}$$

If, moreover, $f|_{\Gamma_T}=0$, *then* $h|_{\partial D}=0$ *and if* $\left.\frac{\partial f}{\partial n}\right|_{\Gamma_T}=0$ *for* $r\geqslant 2$, *then* $\left.\frac{\partial h}{\partial n}\right|_{\partial D}=0$.

Proof. First note that the fact that f belongs to $L_2(Q_T)$ implies $h\in L_2(D)$. Indeed, since $f(x, t)\, g(t)\in L_1(Q_T)$, by Fubini's theorem $h\in L_1(D)$ and since, apart from this, $h^2(x)\leqslant\int\limits_0^T f^2(x, t)\,dt\times\times\|g\|^2_{L_2(0,T)}$, we find that $h\in L_2(D)$.

Hence the function h and the functions

$$h_\alpha(x)=\int\limits_0^T D_x^\alpha f(x, t)\,g(t)\,dt,\qquad \alpha=(\alpha_1, \ldots, \alpha_n),\qquad |\alpha|\leqslant r,$$

belong to $L_2(D)$.

Take an arbitrary function $\eta(x)\in\dot{C}^r(\bar{D})$. Evidently, $g(t)\,\eta(x)\in\in H^{r,0}(Q_T)$, so for any α, $|\alpha|\leqslant r$,

$$\int\limits_D h_\alpha(x)\,\eta(x)\,dx=\int\limits_{Q_T} D_x^\alpha f(x, t)\cdot\eta(x)\,g(t)\,dx\,dt$$

$$=(-1)^{|\alpha|}\int\limits_{Q_T} f(x, t)\cdot D_x^\alpha\,\eta(x)\cdot g(t)\,dx\,dt=(-1)^{|\alpha|}\int\limits_D h(x)\,D_x^\alpha\eta(x)\,dx.$$

Consequently, h has generalized derivatives $D_x^\alpha h=h_\alpha$, $|\alpha|\leqslant r$, which belong to $L_2(D)$, that is, $h\in H^r(D)$.

If $f|_{\Gamma_T}=0$, then for any function $\eta(x)\in C^1(\bar{D})$ and any $i==1, 2, \ldots, n$

$$\int\limits_D h_{x_i}\eta\,dx=\int\limits_{Q_T} f_{x_i}(x, t)\,\eta(x)\,g(t)\,dx\,dt$$

$$=-\int\limits_{Q_T} f(x, t)\,\eta_{x_i}(x)\,g(t)\,dx\,dt=-\int\limits_D h\eta_{x_i}\,dx.$$

On the other hand, since $h\in H^1(D)$, for arbitrary $\eta\in C^1(\bar{D})$

$$\int\limits_D h_{x_i}\eta\,dx=\int\limits_{\partial D} h\eta n_i\,dS-\int\limits_D h\eta_{x_i}\,dx,$$

where $n_i(x)$ are components of outward normal vector to ∂D at the point x. Consequently, for any $\eta(x)\in C^1(\partial D)$,

$$\int\limits_{\partial D} h\eta n_i\,dS=0,\qquad i=1, \ldots, n,$$

which implies that $h|_{\partial D} = 0$ (compare with the proof of Lemma 4, Sec. 2.4, Chap. V).

If $r \geqslant 2$ and $\left.\frac{\partial f}{\partial n}\right|_{\Gamma_T} = 0$, then (compare with the proof of Lemma 4, Sec. 2.4, Chap. V) for any function $\eta \in C^2(\overline{D})$

$$\int_D \Delta h(x)\cdot\eta\,(x)dx = \int_{Q_T} \Delta f(x, t)\cdot\eta(x)\, g(t)\,dx\,dt$$

$$= -\int_{Q_T} \nabla f(x, t)\cdot\nabla\eta(x)\, g(t)\,dx\,dt = -\int_D \nabla h\cdot\nabla\eta\,dx.$$

On the other hand, since $h \in H^2(D)$,

$$\int_D \Delta h\cdot\eta\,dx = \int_{\partial D} \frac{\partial h}{\partial n}\eta\,dS - \int_D \nabla h\cdot\nabla\eta\,dx$$

for any $\eta \in C^1(\overline{D})$. Therefore, for any $\eta \in C^1(\partial D)$,

$$\int_{\partial D} \frac{\partial h}{\partial n}\eta\,dS = 0,$$

that is, $\left.\frac{\partial h}{\partial n}\right|_{\partial D} = 0$. ∎

Corollary. *Let the function* $g(t) \in L_2(0, T)$ *and let the function* $f(x, t)$ *belong to the space* $\widetilde{H}^{2r,r}_{\mathscr{D}}(Q_T)$ *or to the space* $\widetilde{H}^{2r,r}_{\mathscr{N}}(Q_T)$ *for some* $r \geqslant 0$. *Then the function* $h(x)$ *belongs to the space* $H^{2r}_{\mathscr{D}}(D)$ *or to the space* $H^{2r}_{\mathscr{N}}(D)$, *respectively; and for any* $\alpha = (\alpha_1, \ldots, \alpha_n)$, $|\alpha| \leqslant 2r$, *formula* (41) *holds.*

Lemma 4. *Let* $\partial D \in C^2$. *If for some* $q \geqslant 0$ *the function* $f(x, t) \in \widetilde{H}^{2q,q}_{\mathscr{D}}(Q_T)$, *then for any* p, $p = 0, \ldots, q$, $\frac{\partial^p f}{\partial t^p} \in \widetilde{H}^{2(q-p),q-p}_{\mathscr{D}}(Q_T)$. *If* $f \in \widetilde{H}^{2q,q}_{\mathscr{N}}(Q_T)$, *then for any* p, $p = 0, \ldots, q$, $\frac{\partial^p f}{\partial t^p} \in \widetilde{H}^{2(q-p),q-p}_{\mathscr{N}}(Q_T)$.

Proof. The conclusions of this lemma for $q = 0$ and $q = 1$ are evident. When $q \geqslant 2$, the first statement is a direct consequence of the following assertion established in the proof of Lemma 4, Sec. 2.4, Chap. V: *if* $G \in H^2(Q_T)$ *and* $G|_{\Gamma_T} = 0$, *then* $G_t|_{\Gamma_T} = 0$. The second statement of the lemma follows, obviously, from the next assertion: *if* $G \in H^{4,2}(Q_T)$ *and* $\left.\frac{\partial G}{\partial n}\right|_{\Gamma_T} = 0$, *then* $\left.\frac{\partial G_t}{\partial n}\right|_{\Gamma_T} = 0$. Note that this is proved exactly in the same way as the similar statement in Lemma 4, Sec. 2.4, Chap. V. Indeed, since $\left.\frac{\partial G}{\partial n}\right|_{\Gamma_T} = 0$, for any $\eta \in C^2(\overline{Q}_T)$, $\eta\,|_{D_0} = \eta\,|_{D_T} = 0$, we have

$$\int_{Q_T} \Delta G_t \cdot \eta\, dx\, dt = -\int_{Q_T} \Delta G \cdot \eta_t\, dx\, dt = \int_{Q_T} \nabla G \cdot \nabla \eta_t\, dx\, dt$$

$$= -\int_{Q_T} \nabla G_t \cdot \nabla \eta\, dx\, dt.$$

On the other hand,

$$\int_{Q_T} \Delta G_t \cdot \eta\, dx\, dt = \int_{\Gamma_T} \frac{\partial G_t}{\partial n} \cdot \eta\, dS\, dt - \int_{Q_T} \nabla G_t \cdot \nabla \eta\, dx\, dt.$$

Hence

$$\int_{\Gamma_T} \frac{\partial G_t}{\partial n} \cdot \eta\, dS\, dt = 0$$

for any $\eta \in C^2(\overline{\Gamma}_T)$, $\eta\,|_{\partial D_0} = \eta\,|_{\partial D_T} = 0$. Consequently, $\left.\frac{\partial G_t}{\partial n}\right|_{\Gamma_T} = 0$. ∎

Lemma 5. *Let* $\partial D \in C^2$, *and let* $f(x, t) \in \widetilde{H}^{2q,\,q}_{\mathscr{D}}(Q_T)$ *or* $f(x, t) \in \widetilde{H}^{2q,\,q}_{\mathscr{N}}(Q_T)$ *for some* $q \geqslant 0$. *Then for any* p, $p = 0, \ldots, q$,

$$\sum_{k=1}^{\infty} |\lambda_k|^{2(q-p)} \left\| \frac{d^p f_k}{dt^p} \right\|^2_{L_2(0,\,T)} \leqslant C \,\| f \|^2_{H^{2q,\,q}(Q_T)}, \tag{42}$$

where the positive constant C *does not depend on* f.

Proof. According to Lemma 2, Sec. 2.4, Chap. V, for any p, $0 \leqslant p \leqslant q$, $\frac{d^p f_k(t)}{dt^p} = \int_D \frac{\partial^p f(x, t)}{\partial t^p} v_k(x)\, dx$, therefore

$$|\lambda_k|^{2(q-p)} \int_0^T \left(\frac{d^p f_k(t)}{dt^p} \right)^2 dt$$

$$= |\lambda_k|^{2(q-p)} \int_D \left(\int_0^T \frac{\partial^p f(x, t)}{\partial t^p} \frac{d^p f_k(t)}{dt^p}\, dt \right) v_k(x)\, dx$$

$$= \lambda_k^{q-p} \int_D \left(\int_0^T \frac{\partial^p f(x, t)}{\partial t^p} \frac{d^p f_k(t)}{\partial t^p}\, dt \right) \Delta^{q-p} v_k(x)\, dx.$$

According to Lemma 4, the function $\frac{\partial^p f(x, t)}{\partial t^p}$ belongs to $\widetilde{H}^{2(q-p),\,q-p}_{\mathscr{D}}(Q_T)$ or to $\widetilde{H}^{2(q-p),\,q-p}_{\mathscr{N}}(Q_T)$, respectively; this means, in view of Corollary to Lemma 3, that the function $\int_0^T \frac{\partial^p f(x, t)}{\partial t^p} \frac{d^p f_k(t)}{dt^p}\, dt$ belongs to $H^{2(q-p)}_{\mathscr{D}}(D)$ or, respectively, to

$H^{2(q-p)}_{\mathscr{N}}(D)$. Hence

$$|\lambda_k|^{2(q-p)} \int\limits_0^T \left(\frac{d^p f_k(t)}{dt^p}\right)^2 dt = \lambda_k^{q-p} \int\limits_D \Delta^{q-p}\left(\int\limits_0^T \frac{\partial^p f}{\partial t^p}\,\frac{d^p f_k}{dt^p}\,dt\right)\cdot v_k\,dx$$

$$= \lambda_k^{q-p}\int\limits_{Q_T} \Delta^{q-p}\,\frac{\partial^p f(x,\, t)}{\partial t^p}\,\frac{d^p f_k(t)}{dt^p}\, v_k(x)\,dx\,dt$$

$$= \lambda_k^{q-p}\int\limits_D\int\limits_0^T\left(\Delta_y^{q-p}\,\frac{\partial^p f(y,\, t)}{\partial t^p}\right)\left(\int\limits_D \frac{\partial^p f(x,\, t)}{\partial t^p}\, v_k(x)\,dx\right) v_k(y)\,dy\,dt$$

$$= \lambda_k^{q-p}\int\limits_D\left(\int\limits_0^T \frac{\partial^p f(x,\, t)}{\partial t^p}\, g_k^{(p)}(t)\,dt\right)v_k(x)\,dx$$

$$= \int\limits_D\left(\int\limits_0^T \frac{\partial^p f(x,\, t)}{\partial t^p}\, g_k^{(p)}(t)\,dt\right)\Delta^{q-p}v_k(x)\,dx, \qquad (43)$$

where, by Lemma 2, Sec. 2.4, Chap. V, the function $g_k^{(p)}(t) = \int\limits_D \Delta^{q-p}\,\frac{\partial^p f(x,\, t)}{\partial t^p}\cdot v_k(x)\,dx$ belongs to $L_2(0,\, T)$. The function $\Delta^{q-p}\,\frac{\partial^p f(x,\, t)}{\partial t^p} \in L_2(Q_T)$, therefore for almost all $t \in (0,\, T)$ $\Delta^{q-p}\,\frac{\partial^p f(x,\, t)}{\partial t^p} \in L_2(D_t)$ and for almost all $t \in (0,\, T)$ $\sum\limits_{k=1}^{\infty} (g_k^{(p)}(t))^2 = \left\|\Delta^{q-p}\,\frac{\partial^p f}{\partial t^p}\right\|^2_{L_2(D_t)}$. Consequently,

$$\sum_{k=1}^{\infty} \|g_k^{(p)}\|^2_{L_2(0,\, T)} = \left\|\Delta^{q-p}\,\frac{\partial^p f}{\partial t^p}\right\|^2_{L_2(Q_T)} \leqslant \text{const}\, \|f\|^2_{H^{2q,\, q}(Q_T)}. \qquad (44)$$

Since, by Lemma 4 and Corollary to Lemma 3, the function $\int\limits_0^T \frac{\partial^p f(x,\, t)}{\partial t^p}\, g_k^{(p)}(t)\,dt$ belongs to $H^{2(q-p)}_{\mathscr{D}}(D)$ or to $H^{2(q-p)}_{\mathscr{N}}(D)$, from (43) we have

$$|\lambda_k|^{2(q-p)}\int\limits_0^T \left(\frac{d^p f_k(t)}{dt^p}\right)^2 dt$$

$$= \int\limits_D \Delta^{q-p}\left(\int\limits_0^T \frac{\partial^p f(x,\, t)}{\partial t^p}\, g_k^{(p)}(t)\,dt\right) v_k(x)\,dx = \int\limits_0^T (g_k^{(p)}(t)^2\,dt,$$

which immediately yields (42) if account is taken of (44). ∎

Proof of Lemma 2. Since the function $f \in H^{2q,q}(Q_T) \subset H^q(Q_T)$, the functions $f_k(t)$, $k = 1, 2, \ldots$, belong to $H^q(0, T)$ (Lemma 2, Sec. 2.4, Chap. V). Therefore, according to (21) and (22), the functions $U_k(t)$, $k = 1, 2, \ldots$, belong to $H^{q+1}(0, T)$. It follows from (22) that for any p, $1 \leqslant p \leqslant q + 1$,

$$\frac{d^pU_k}{dt^p} = \lambda_k^p U_k + \sum_{r=0}^{p-1} \lambda_k^{p-r-1} \frac{d^r f_k}{dt^r}, \qquad t \in (0,\ T).$$

Accordingly, for the proof of (40) it suffices, in view of the inequality (42) of Lemma 5, to establish that

$$\sum_{k=1}^{\infty} |\lambda_k|^{2(q+1)} \|U_k\|^2_{L_2(0,\,T)} \leqslant \text{const}\,(\|\varphi\|^2_{H^{2q+1}(D)} + \|f\|^2_{H^{2q,\,q}(Q_T)}). \quad (45)$$

We multiply (22) by U_k and integrate the resulting identity with respect to $t \in (0, T)$. Using (22′), we obtain

$$\frac{1}{2} U_k^2(T) - \frac{1}{2}\varphi_k^2 - \lambda_k \int_0^T U_k^2(t)\,dt = \int_0^T f_k(t)\,U_k(t)\,dt,$$

from which ($\lambda_k \leqslant 0$) we have the inequality

$$|\lambda_k|\,\|U_k\|^2_{L_2(0,\,T)} \leqslant \frac{1}{2}\varphi_k^2 + \|f_k\|_{L_2(0,\,T)}\|U_k\|_{L_2(0,\,T)}$$

and hence the inequality

$$\begin{aligned} |\lambda_k|^{2q+2}\|U_k\|^2_{L_2(0,\,T)} &\leqslant \frac{1}{2}\varphi_k^2(\lambda_k)^{2q+1} \\ &+ (|\lambda_k|^q\|f_k\|_{L_2(0,\,T)})(|\lambda_k|^{q+1}\|U_k\|_{L_2(0,\,T)}) \leqslant \frac{1}{2}\varphi_k^2|\lambda_k|^{2q+1} \\ &+ \frac{1}{2}|\lambda_k|^{2q}\|f_k\|^2_{L_2(0,\,T)} + \frac{1}{2}|\lambda_k|^{2q+2}\|U_k\|^2_{L_2(0,\,T)}. \end{aligned}$$

Thus

$$|\lambda_k|^{2q+2}\|U_k\|^2_{L_2(0,\,T)} \leqslant \varphi_k^2|\lambda_k|^{2q+1} + |\lambda_k|^{2q}\|f_k\|^2_{L_2(0,\,T)},$$

and consequently inequality (45) follows from the inequality (42) (with $p = 0$) and the inequality (Theorem 8, Sec. 2.5, Chap. IV)

$$\sum_{k=1}^{\infty} \varphi_k^2|\lambda_k|^{2q+1} \leqslant \text{const}\,\|\varphi\|^2_{H^{2q+1}(D)}. \quad \blacksquare$$

Now we shall prove the existence theorem regarding the classical solutions of the problems (31)-(33) and (31), (32), (34).

Note that if $f \in H^{2,1}(Q_T)$, then the functions $U_k(t)$, $k = 1, 2, \ldots$, defined by (21) belong to the space $H^2(0, T)$, and consequently to the space $C^1([0, T])$.

If $\partial D \in C^{[\frac{n}{2}]+3}$, then, by Theorem 7, Sec. 2.4, Chap. IV, the eigenfunctions $v_k(x)$ of the first or second boundary-value problem for the Laplace operator in D belong to the space $H^{[\frac{n}{2}]+3}(D)$, and consequently (Theorem 3, Sec. 6.2, Chap. III) also to the space $C^2(\ddot{D})$. Then the partial sums S_N of the series (24) belong to the space $C^{2,1}(\overline{Q}_T)$.

Theorem 5. *Let* $\partial D \in C^{2s_0+1}$, *where* $2s_0+1 \geqslant \left[\frac{n}{2}\right]+3$, *and let* $\varphi \in H_{\mathscr{D}}^{2s_0+1}(D)$, $f \in \widetilde{H}_{\mathscr{D}}^{2s_0, s_0}(Q_T)$ *in the case of the first mixed problem* (31)-(33) *while* $\varphi \in H_{\mathscr{N}}^{2s_0+1}(D)$, $f \in \widetilde{H}_{\mathscr{N}}^{2s_0, s_0}(Q_T)$ *in the case of the second mixed problem* (31), (32), (34). *Then the series* (24) *converges in* $C^{2,1}(\overline{Q}_T)$ *and its sum is a classical solution of the first mixed problem* (31)-(33) *or correspondingly of the second mixed problem* (31), (32), (34). *Moreover,*

$$\|u\|_{C(\overline{Q}_T)} \leqslant C\left(\|\varphi\|_{H^{2s_0-1}(D)} + \|f\|_{H^{2(s_0-1),\, s_0-1}(Q_T)}\right), \tag{46}$$

where the positive constant C does not depend on φ *or* f.

Proof. We start by establishing necessary estimates for the functions $U_k(t)$ and its derivatives $U_k'(t)$, $k = 1, 2, \ldots$. By formula (21) we obtain

$$|U_k(t)| \leqslant |\varphi_k| + \frac{1}{\sqrt{2|\lambda_k|}} \|f_k\|_{L_2(0,T)} \text{ when } k > 1$$

and

$$|U_1(t)| \leqslant |\varphi_1| + C_1 \|f_1\|_{L_2(0,T)},$$

where $C_1 = 1/\sqrt{2|\lambda_1|}$ in the case of the first mixed problem and $C_1 = \sqrt{T}$ in the case of the second mixed problem. Then from (22) it follows that for all $t \in [0, T]$

$$|U_k'|(t)| \leqslant |\lambda_k||U_k| + |f_k| \leqslant |\lambda_k||\varphi_k| + |f_k| + \frac{\sqrt{|\lambda_k|}}{\sqrt{2}} \|f_k\|_{L_2(0,T)} \text{ when } k \geqslant 1.$$

Therefore for all $t \in [0, T]$

$$U_k^2(t) \leqslant 2|\varphi_k|^2 + \frac{1}{|\lambda_k|} \|f_k\|_{L_2(0,T)}^2, \qquad k > 1, \tag{47}$$

$$U_1^2(t) \leqslant 2\varphi_1^2 + 2C_1^2 \|f_1\|_{L_2(0,T)}^2, \tag{47'}$$

$$U_k'^2(t) \leqslant 3\lambda_k^2\varphi_k^2 + \frac{3}{2}|\lambda_k| \|f_k\|_{L_2(0,T)}^2 + 3|f_k|^2, \qquad k \geqslant 1 \tag{48}$$

We shall prove the following auxiliary assertion.

Lemma 6. *Let* $f(t)$ *be an arbitrary function in* $H^1(0, T)$ *and* ε *an arbitrary number from* $(0, T]$. *Then for all* $t \in [0, T]$ *the inequality*

$$f^2(t) \leqslant \frac{2}{\varepsilon} \| f \|^2_{L_2(0,\,T)} + 2\varepsilon \| f' \|^2_{L_2(0,\,T)} \tag{49}$$

holds.

Proof of Lemma 6. Let α denote the average value of f over the interval $(0, T)$:

$$\alpha = \frac{1}{T} \int\limits_0^T f(t)\,dt,$$

and on $[0, T]$ consider the continuous function

$$f_\alpha(t) = f(t) - \alpha.$$

Since $\int\limits_0^T f_\alpha(t)\,dt = 0$, there exists a point $t^0 \in (0, T)$ such that $f_\alpha(t^0) = 0$. Therefore for all $t \in [0, T]$ and any $\varepsilon > 0$

$$f_\alpha^2(t) = 2 \int\limits_{t^0}^t f_\alpha(t)\, f'_\alpha(t)\,dt \leqslant \frac{1}{\varepsilon} \int\limits_0^T f_\alpha^2(t)\,dt + \varepsilon \int\limits_0^T f'^2(t)\,dt.$$

Consequently, for any $t \in [0, T]$ and any ε, $0 < \varepsilon \leqslant T$, we have

$$\begin{aligned} f^2(t) - 2\alpha f(t) + \alpha^2 &\leqslant \frac{1}{\varepsilon} \left(\int\limits_0^T f^2(\tau)\,d\tau - 2\alpha \int\limits_0^T f(\tau)\,d\tau + \alpha^2 T \right) \\ &+ \varepsilon \int\limits_0^T f'^2(\tau)\,d\tau = \frac{1}{\varepsilon} \int\limits_0^T f^2(\tau)\,d\tau + \varepsilon \int\limits_0^T f'^2(\tau)\,d\tau - \frac{\alpha^2 T}{\varepsilon} \\ &\leqslant \frac{1}{\varepsilon} \int\limits_0^T f^2(\tau)\,d\tau + \varepsilon \int\limits_0^T f'^2(\tau)\,d\tau - \alpha^2, \end{aligned}$$

from which follows the inequality

$$\begin{aligned} \frac{1}{\varepsilon} \| f \|^2_{L_2(0,\,T)} + \varepsilon \| f' \|^2_{L_2(0,\,T)} &\geqslant 2\alpha^2 - 2\alpha f(t) + f^2(t) \\ &= \left(\sqrt{2}\,\alpha - \frac{1}{\sqrt{2}} f(t) \right)^2 + \frac{f^2(t)}{2} \geqslant \frac{f^2(t)}{2} \end{aligned}$$

coinciding with inequality (49). This proves the lemma.

We consider the inequality (48) for k such that $| \lambda_k | \geqslant 1/T$; let k_0 denote the least of these k. Then, by Lemma 6, for all $k \geqslant k_0$ (recall that the sequence $| \lambda_k |$ is monotone nondecreasing)

$$f_k(t) |^2 \leqslant 2 | \lambda_k | \, \| f_k \|^2_{L_2(0,\,T)} + \frac{2}{| \lambda_k |} \| f'_k \|^2_{L_2(0,\,T)}$$

Substituting the last inequality in (48), for all $t \in [0, T]$ and $k \geqslant k_0$ we obtain

$$U_k'^2(t) \leqslant 3\lambda_k^2 \varphi_k^2 + \frac{15}{2} |\lambda_k| \, \| f_k \|^2_{L_2(0,T)} + \frac{6}{|\lambda_k|} \| f_k' \|^2_{L_2(0,T)}$$

$$\leqslant 8 \left(\lambda_k^2 \varphi_k^2 + |\lambda_k| \, \| f_k \|^2_{L_2(0,T)} + \frac{1}{|\lambda_k|} \| f_k' \|^2_{L_2(0,T)} \right). \qquad (50)$$

According to Theorem 3, Sec. 6.2, Chap. III, Lemma 3, Sec. 2.5, Chap. IV, and inequalities (47) and (50), we have for all $t \in [0, T]$ and all M and N, $k_0 \leqslant M < N$,

$$\| S_N - S_M \|^2_{C^2(\bar{D}_t)} + \left\| \frac{\partial}{\partial t} (S_N - S_M) \right\|^2_{C(\bar{D}_t)}$$

$$\leqslant C_1 \left(\| S_N - S_M \|^2_{H^{2s_0+1}(D_t)} + \left\| \frac{\partial}{\partial t} (S_N - S_M) \right\|^2_{H^{2s_0-1}(D_t)} \right)$$

$$\leqslant C_2 \Bigl(\Bigl\| \sum_{k=M+1}^{N} U_k(t) \Delta^{s_0} v_k(x) \Bigr\|^2_{H^1(D_t)}$$

$$+ \Bigl\| \sum_{k=M+1}^{N} U_k'(t) \Delta^{s_0-1} v_k(x) \Bigr\|^2_{H^1(D_t)}$$

$$\leqslant C_3 \sum_{k=M+1}^{N} \left(|\lambda_k|^{2s_0+1} U_k^2(t) + |\lambda_k|^{2s_0-1} U_k'^2(t) \right)$$

$$\leqslant C_4 \sum_{k=M+1}^{N} \left(\varphi_k^2 |\lambda_k|^{2s_0+1} + \lambda_k^{2s_0} \| f_k \|^2_{L_2(0,T)} + \lambda_k^{2s_0-2} \| f_k' \|^2_{L_2(0,T)} \right).$$

Consequently,

$$\| S_N - S_M \|^2_{C^{2,1}(\bar{Q}_T)} \leqslant C_5 \sum_{k=M+1}^{N} \Bigl(\varphi_k^2 |\lambda_k|^{2s_0+1}$$

$$+ |\lambda_k|^{2s_0} \| f_k \|^2_{L_2(0,T)} + \lambda_k^{2s_0-2} \| f_k' \|^2_{L_2(0,T)} \Bigr). \qquad (51)$$

Analogously, with the aid of (47′), we find that for all $t \in [0, T]$ and all $N \geqslant 1$ the inequalities

$$\| S_N \|^2_{C(\bar{D}_t)} \leqslant C_6 \| S_N \|^2_{H^{2s_0-1}(D_t)} \leqslant C_7 \Bigl(U_1^2(t) + \sum_{k=1}^{N} |\lambda_k|^{2s_0-1} U_k^2(t) \Bigr)$$

$$\leqslant C_8 \Bigl(\varphi_1^2 + \| f_1 \|^2_{L_2(0,T)} + \sum_{k=1}^{N} \bigl(\varphi_k^2 |\lambda_k|^{2s_0-1} + \lambda_k^{2s_0-2} \| f_k \|^2_{L_2(0,T)} \bigr) \Bigr)$$

hold and therefore also the inequalities

$$\| S_N \|^2_{C(\bar{Q}_T)} \leqslant C_9 \Bigl(\varphi_1^2 + \| f_1 \|^2_{L_2(0,T)}$$

$$+ \sum_{k=1}^{N} \bigl(\varphi_k^2 |\lambda_k|^{2s_0-1} + \lambda_k^{2s_0-2} \| f_k \|^2_{L_2(0,T)} \bigr) \Bigr). \qquad (52)$$

Since the function φ belongs to the space $H_{\mathscr{D}}^{2s_0+1}(D)$ in the case of the first mixed problem and to the space $H_{\mathscr{N}}^{2s_0+1}(D)$ in the case of the second mixed problem, the series $\sum\limits_{k=1}^{\infty} \varphi_k^2 \mid \lambda_k \mid^{2s_0+1}$ converges. Besides this, since φ belongs to $H_{\mathscr{D}}^{2s_0-1}(D)$ or correspondingly to $H_{\mathscr{N}}^{2s_0-1}(D)$, it follows (Theorem 8, Sec. 2.5, Chap. IV) that

$$\sum_{k=1}^{\infty} \varphi_k^2 \mid \lambda_k \mid^{2s_0-1} \leqslant \text{const} \| \varphi \|_{H^{2s_0-1}(D)}^2. \tag{53}$$

Since for the first mixed problem $f \in \widetilde{H}_{\mathscr{D}}^{2s_0, s_0}(Q_T)$ and for the second mixed problem $f \in \widetilde{H}_{\mathscr{N}}^{2s_0; s_0}(Q_T)$, the series

$$\sum_{k=1}^{\infty} \mid \lambda_k \mid^{2s_0} \| f_k \|_{L_2(0,T)}^2 \quad \text{and} \quad \sum_{k=1}^{\infty} \mid \lambda_k \mid^{2(s_0-1)} \| f_k' \|_{L_2(0,T)}^2$$

converge, in view of Lemma 5. Moreover, using the fact that f belongs to the space $\widetilde{H}_{\mathscr{D}}^{2(s_0-1), s_0-1}(Q_T)$ or, correspondingly, to the space $\widetilde{H}_{\mathscr{N}}^{2(s_0-1), s_0-1}(Q_T)$ and by inequalities (42) of Lemma 5, we have

$$\sum_{k=1}^{\infty} \mid \lambda_k \mid^{2(s_0-1)} \| f_k \|_{L_2(0,T)}^2 \leqslant \text{const} \| f \|_{H^{2(s_0-1), s_0-1}(Q_T)}^2. \tag{54}$$

Therefore from inequalities (51) it follows that the series (24) converges in $C^{2,1}(\overline{Q}_T)$ and its sum $u(x, t)$ belongs to $C^{2,1}(\overline{Q}_T)$ and consequently is a classical solution of the corresponding mixed problem. The estimate (46) follows from the inequalities (52)-(54). This completes the proof of the theorem.

PROBLEMS ON CHAPTER VI

1. Let D be a bounded region of the space R_n, $n > 2$, and x^0 a point in D. Let the function $u(x, t) \in C^{2,1}(\{x \in D \setminus x^0, 0 < t < T\})$, $T > 0$, satisfy the homogeneous heat equation in $\{x \in D \setminus x^0, 0 < t < T\}$, and let $u(x, t) \mid x - x^0 \mid^{n-2} \to 0$ as $x \to x^0$ uniformly in $t \in (0, T)$. Show that the function $u(x, t)$ can be redefined on the set $\{x = x^0, 0 < t < T\}$ so that the resulting function will belong to $C^\infty(\{x \in D, 0 < t < T\})$.

2. Let the function $u(x, t)$ belong to $C^{2,1}(t > 0)$ and be a solution of the homogeneous heat equation in the half-space $\{t > 0\}$, and let there be a function $A(x)$ such that $u(x, t) \to A(x)$, as $t \to \infty$, uniformly in $x \in \{\mid x \mid < R\}$ for any $R > 0$. Prove that the function $A(x)$ is harmonic in R_n.

3. Let the function $\varphi(x)$ belong to $C(R_n)$ and satisfy the inequality $\mid \varphi(x) \mid \leqslant Ce^{a|x|^2}$, where C and a are positive constants, for all $x \in R_n$. Prove that

the Cauchy problem

$$u_t - \Delta u = 0, \qquad x \in R_n, \qquad 0 < t < \frac{1}{4a}, \tag{1}$$
$$u\,|_{t=0} = \varphi(x)$$

has a solution $u(x, t)$ in the strip $\left\{x \in R_n, 0 < t < \frac{1}{4a}\right\}$. This solution is given by Poisson's formula and belongs to the uniqueness class B_2.

If the function $\varphi(x) \in C(R_n)$ and satisfies the condition: for any $\varepsilon > 0$ there exists a $C = C(\varepsilon) > 0$ such that

$$|\varphi(x)| < Ce^{\varepsilon|x|^2} \quad \text{for all} \quad x \in R_n, \tag{2}$$

then the result of Probl. 3 implies that in the half-space $\{t > 0\}$ there exists a solution of the Cauchy problem for the homogeneous heat equation with the initial function $\varphi\,(x)$ belonging to the uniqueness class B_2; moreover, this solution is given by Poisson's formula.

4. Suppose that the function $\varphi(x) \in C\,(R_n)$ and for any $\varepsilon > 0$ there is a constant $C = C(\varepsilon) > 0$ such that (2) holds. Let $u(x, t)$ (belonging to the class B_2) denote a solution of the Cauchy problem

$$\begin{aligned} u_t - \Delta u = 0, \qquad x \in R_n, \qquad t > 0,\\ u\,|_{t=0} = \varphi(x). \end{aligned} \tag{3}$$

Prove the following assertion. If there exists a function $A(x)$ such that for any $R > 0$ $\frac{n}{\sigma_n \rho^n} \int\limits_{|x-\xi|<\rho} u(\xi)\, d\xi \to A(x)$, as $\rho \to \infty$ (σ_n is the surface area of the unit sphere in R_n), uniformly with respect to $x \in \{|\,x\,| < R\}$, then $\lim\limits_{t\to\infty} u(x, t) = A(x)$ uniformly with respect to $x \in \{|\,x\,| < R\}$ (for any $R > 0$); moreover, $A(x)$ is a harmonic function.

5. Let $u(x, t)$ be a solution of the Cauchy problem (3) belonging to B_2, where $\varphi(x) \in B(R_n)$, and let $\lim\limits_{t\to\infty} u(0, t) = A$. Prove that then for any point $x \in R_n$ $\lim\limits_{t\to\infty} u(x, t) = A$.

6. Show that the solution $u(x, t)$ of the Cauchy problem (3) with $\varphi \in B(R_n)$ is an analytic function of (x, t) in the half-space $\{x \in R_n, t > 0\}$.

7. Show that, if $\partial D \in C^2$, the classical solution of the first mixed problem

$$u_t - \Delta u = 0, \quad (x, t) \in Q_T = \{D \times (0, T)\},$$
$$u\,|_{D_0} = \varphi(x),$$
$$u\,|_{\Gamma_T} = 0$$

is a generalized solution of this problem.

8. Prove existence and uniqueness theorems concerning generalized solutions of the first, second and third mixed problems for the parabolic equation (problems (1)-(3) and (1), (2), (4) from Sec. 2.1) without assuming that the functions $a(x)$ and $\sigma(x)$ are nonnegative.

9. Let the function $u(x, t)$ belong to $C^{2,1}(Q_T) \cap C(\bar{Q}_T)$, satisfy the homogeneous heat equation ($u_t - \Delta u = 0$) in Q_T and the homogeneous initial condition ($u|_{D_0} = 0$, D_0 is the base of the cylinder Q_T). Prove that then $u \in C^\infty\,(Q_T \cup D_0)$. Prove also that for any point (x, t) of the cylinder

$\{D' \times (0,\ T)\}$ where $D' \Subset D_0$, $\rho = \inf\limits_{\substack{x' \in \partial D' \\ x'' \in \partial D_0}} |\, x' - x'' \,| > 0$,

$$|\, D^{\alpha} u(x,\ t)| \leqslant C(\alpha,\ T)\frac{e^{-\frac{\rho^2}{8T}}}{\rho^{2\alpha_0+\alpha_1+\ldots+\alpha_n+2}} \|\, u \,\|_{C(\bar{Q}_T)},$$

where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $D^{\alpha} u = \dfrac{\partial^{\alpha_0+\ldots+\alpha_n} u}{\partial t^{\alpha_0} \ldots \partial x_n^{\alpha_n}}$ and $C(\alpha, T)$ is a positive constant depending only on the vector α and the number T.

10. Let the function $\varphi \in B(R_n)$ and D_i, $i = 1, 2, \ldots$, be a sequence of regions of the space R_n, $D_i \subset D_{i+1}$, $i = 1, 2, \ldots$, $\bigcup\limits_{i=1}^{\infty} D_i = R_n$. Let $u_i(x,\ t)$ be the solution of the equation $u_t - \Delta u = 0$ in $D_i \times (0,\ T)$ that is continuous in $\{\bar{D}_i \times [0,\ T]\}$ and satisfies the initial condition $u_i|_{D_i} = \varphi$. Suppose that $\|\, u_i \,\|_{C(\bar{D}_i \times [0,\ T])} \leqslant C$, where the positive constant C does not depend on i. Then the sequence u_i, $i = 1, 2, \ldots$, converges uniformly in $(x,\ t) \in \bar{D} \times [0,\ T]$, where D is an arbitrary bounded region in R_n, to the (bounded) solution of the Cauchy problem in the strip $\{x \in R_n,\ 0 < t < T\}$ for the homogeneous heat equation with the initial function φ. Prove this.

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